Talbot 2019 Talk 15

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4/8/2019

"And now for something completely different"

1 Positive Scalar Curvature

Let (M, g) be a Riemannian manifold of dimension n. In differential geometry one studies several notions of *curvature*. In these two talks we will be concerned with *scalar curvature*, which in some sense is the curvature with the least amount of information, since it assings to each point of M just a scalar in $\mathbb{R}$ instead of a being a tensor as for example the Riemannian curvature is.[1]

Definition. The *scalar curvature* $\mathrm{scal}_g \colon M \to \mathbb{R}$ is defined by

$$\frac{\operatorname{vol} B_\varepsilon(x, M)}{\operatorname{vol} B_\varepsilon(x, \mathbb{R}^d)} = 1 - \frac{\mathrm{scal}_g(x)}{6(n+2)}\varepsilon^2 + \mathcal{O}(\varepsilon^4)$$

Example. Let g be the round metric on S^n. Then $\mathrm{scal}_g \equiv n(n-1)$. We say that the round sphere has *positive scalar curvature* ($\mathrm{scal} > 0$, hereafter abbreviated as *psc*).

Remark. The above definition of scalar curvature gives some geometric understanding by comparing the volumes of small balls in M with reference balls in euclidian space. However it is not helpful for computations at all. In differential geometry one therefore defines it as a tensor contraction of the Riemannian curvature.

The computation in the example follows most easily via the fact, that the scalar curvature is $n(n-1)$ times the average of the sectional curvatures, which in the above case is $\frac{1}{r^2}$.

Why should a topologist be interested in scalar curvature?—Due to a result by Kazdan–Warner every smooth function on M with negative value at least one point of M arises as the scalar curvature of some Riemannian metric. The existence of psc metrics on the other side is a much more delicate matter – as it turns out there are topological obstructions to its existence![2] This existence question belongs to the realm of index theory, which we wont discuss here too much, since we will be taking the question one step further: We will define a *space of psc metrics* and study its homotopy type.

[1] well, scalar curvature is a $(0, 0)$-tensor, but you get the point ...

[2] namely the $\hat{A}$-genus and its various refinements

1.1 Spaces of Metrics

Let $W\colon M_0 \rightsquigarrow M_1$ a morphism in the cobordism category $\mathcal{C}^d$, i.e. a collared cobordism. As there is a suitable Fréchet-type C^∞-topology on the sections of the bundle $\mathrm{Sym}^2 T^*W \to W$, we can define the *space* of metrics as follows:

Definition. Let

$$\mathcal{R}(W) \subset \Gamma(W, \mathrm{Sym}^2 T^*W)$$

be the subspace of metrics that have "product form" near the boundary, i.e. $g = h_i + dt^2$ where h_i is some metric on M_i . Let $\mathcal{R}^+(W) \subset \mathcal{R}(W)$ be the subspace of psc metrics.

Note that this only makes sense, if working with collared cobordisms. Also note, that there is an obvious restriction map

$$\mathrm{res}\colon \mathcal{R}^+(W) \to \mathcal{R}^+(M_0) \times \mathcal{R}^+(M_1)$$

which allows us to define a space with prescribed boundary conditions as

$$\mathcal{R}^+(W)_{h_0,h_1} = \mathrm{res}^{-1}(h_0, h_1)$$

Remark. These spaces have the homotopy type of a CW complex, which can be shown using a result of Palais [Pal66]. However in the case of all metrics, the homotopy type is rather boring, since $\mathcal{R}(W)$ is contractible by taking convex combinations. This is a good reason to study $\mathcal{R}^+$ instead.

1.2 Spin structures

As in the major part of the results about psc in index theory we need to assume a bit of additional structure on our manifolds – a *spin structure*. Recall that

- $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal double cover,
- $\mathrm{Spin}(n) \subset C\ell(\mathbb{R}^n)^\times$, i.e. Spin may be concretely defined as subgroup of the group of invertibles in the Clifford algebra of $\mathbb{R}^n$.

Via the covering map $\theta\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ we get a tangential structure (see first talks) and thereby the notion of *spin structure*. Here is an equivalent definition: Let (M, g) be oriented, i.e. equipped with a $\mathrm{SO}(n)$-structure. A *spin structure* is an "equivariant"[3] lift of the oriented orthonormal frame bundle $P_{\mathrm{SO}(n)}(TM)$ to a principal $\mathrm{Spin}(n)$-bundle denoted by $P_{\mathrm{Spin}(n)}$.

Theorem. *M admits a spin structure if and only if the second Stiefel–Whitney class*

$$w_2 \in H^2(M; \mathbb{Z}/2)$$

vanishes.

A spin structure is needed in order to define the following objects, which play the central rôle in index theory:

[3]with respect to $\theta\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$

Definition. The *spinor bundle* is defined as

$$\mathfrak{S}_M \coloneqq P_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} C\ell(\mathbb{R}^n)$$

The *Dirac operator* is a differential operator $\not{\mathfrak{D}} = \not{\mathfrak{D}}_g \colon \Gamma(\mathfrak{S}_M) \to \Gamma(\mathfrak{S}_M)$ defined by

$$\not{\mathfrak{D}}(\sigma) = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \sigma$$

where $\cdot$ denotes Clifford multiplication, ∇ is the induced covariant derivative on the spin bundle and e_i an orthonormal basis of $T_x M$.

On the space of smooth sections $\Gamma(\mathfrak{S}_M)$ there is an inner product given by

$$\langle \sigma_1, \sigma_2 \rangle \coloneqq \int_M g_x(\sigma_1(x), \sigma_2(x))\, dx$$

The relevance of the Dirac operator to scalar curvature comes from the following famous formula (which involves an adjoint with respect to the inner product, we just defined):

Theorem (Schrödinger–Lichnerowicz–Weitzenböck, [LM89, II. Thm. 8.8])**.**

$$\not{\mathfrak{D}}_g^2 = \nabla^* \nabla + \frac{\mathrm{scal}_g}{4}$$

Using some advanced functional analysis this implies the following corollary if M is a closed[4] manifold, such that $\not{\mathfrak{D}}_g$ becomes essentially self-adjoint. Roughly speaking, the above formula together with g being psc ensures, that the spectrum of $\not{\mathfrak{D}}_g$ does not contain a neighbourhood of zero, since $\nabla^*\nabla$ is a positive operator. This allows for modifications of $\not{\mathfrak{D}}_g$ via the functional calculus for unbounded operators.

Corollary. *If (M, g) is closed and g a psc metric, then $\not{\mathfrak{D}}_g$ is invertible (in a certain sense).*

Glossing over some technical details we have the following

"Fact". $\not{\mathfrak{D}}_g$ is a Fredholm operator, i.e. the *index*

$$\dim \ker \not{\mathfrak{D}}_g - \dim \operatorname{coker} \not{\mathfrak{D}}_g$$

is a well defined integer.

The importance of Fredholm operators[5] for topology can be witnessed in the following

Theorem (Atiyah, Jänich)**.** *Let* $\mathrm{Fred}(\mathcal{H})$ *be the space of Fredholm operators on a infinite dimensional, separable Hilbert space $\mathcal{H}$, then* $\mathrm{Fred}(\mathcal{H})$ *is a classifying space for topological KO-theory:*

$$[X, \mathrm{Fred}(\mathcal{H})] \cong KO(X)$$

For $X = *$ this reduces to the classical Fredholm index, as $KO(*) \cong \mathbb{Z}$.

Generalising the above "fact", the spinor bundle $\mathfrak{S}_M$ has a certain $C\ell(\mathbb{R}^n)$-bundle structure and $\not{\mathfrak{D}}$ satisfies some relations with it. Now we let Fred^n be the space of all Fredholm operators fulfilling those relations. We then have the following generalisation of the previous theorem:

Theorem (Atiyah–Singer, [AS69])**.** *The space* Fred^n *classifies higher KO-groups.*

[4] or, more generally, if M satisfies a certain completeness property with respect to $\not{\mathfrak{D}}_g$

[5] which are a standard subject of every functional analysis course

1.3 Hitchin's secondary index invariant

In the last section we saw, how index theory and psc metrics interact with each other. This leads to the following naïve attempt to understand $\mathcal{R}^+(M)$: Simply apply the index map to the Dirac operator associated to a psc metric. However, as noted in the last section, if g is a psc metric, $\mathfrak{D}_g$ is invertible and therefore has index zero — we need a better invariant!

Instead we take the "index difference" of two psc metrics $g_0, g_1 \in \mathcal{R}^+(M)$: Consider the path $(1-t)\cdot g_0 + t\cdot g_1$ lying in $\mathcal{R}(M)$. The associated path of Dirac operators, has its two ends in $G^n \subset \mathrm{Fred}^n$, where G^n is the subset of invertible operators. Now, due to a famous result by Kuiper, we have $G^n \simeq *$. Hence we get a map

$$\mathcal{R}^+(M) \times \mathcal{R}^+(M) \longrightarrow \Omega_{G^n,G^n}\,\mathrm{Fred}^n \simeq \Omega\,\mathrm{Fred}^n$$

Fixing g_0 we get an element $\mathrm{inddiff}_{g_0} \in [\mathcal{R}^+(M), \Omega\,\mathrm{Fred}^n] \cong KO^{-n-1}(\mathcal{R}^+(M))$.

Disclaimer *This doesn't strictly make sense!*—Here's why: The spinor bundle varies with the metric along the path of metrics and therefore the Hilbert space structure does as well. So instead of a single Hilbert space one has to look at Hilbert *bundles* and *families of operators* acting on them.[6] The theory for this is worked out in detail in [Ebe16] in a manner, that is tailored to the application here ([BER17] only contains the bare facts). The actual definition can however be seen as a technically sound version of the conceptual idea sketched above.

Remark. This bundle generalisation is not straightforward! For example it turns out, that requiring every operator T_x of a family of operators to be Fredholm, is *not* the right definition of a "Fredholm family"! (this pointwise definition is too weak) Furthermore we should remark, that the language for this (specifically the model for KO-theory) has evolved since [BER17], see [Ebe18; Ebe19].

In particular it should be noted, that $\mathcal{R}^+(M)$ is not locally compact and therefore the classical picture for $KO(\mathcal{R}^+(M))$ using isomorphism classes of vector bundles does not suffice.

After fixing those technical problems and properly working on a cobordism W^d, i.e. a non-closed manifold instead of M closed), we get an element

$$\mathrm{inddiff}_{g_0} \in KO^{-d-1}(\mathcal{R}^+(W)_h, g_0)$$

which corresponds to a unique homotopy class of maps

$$\mathrm{inddiff}_{g_0} \colon \mathcal{R}^+(W)_h \longrightarrow (\Omega^{\infty+d+1}\mathsf{KO}, *)$$

(where KO denotes the real topological K-theory spectrum) Applying π_k yields

$$A_k(W,g_0)\colon \pi_k(\mathcal{R}^+(W)_h, g_0) \longrightarrow KO_{\underbrace{k+d+1}_{=:m}}(*) = \begin{cases} \mathbb{Z} & m\equiv 0 \mod 4 \qquad \text{Case (1)} \\ \mathbb{Z}/2 & m \equiv 1,2 \mod 8 \qquad \text{Case (2)} \\ 0 & \text{else} \end{cases}$$

As the right side is pretty well-known, the expected statement concerning the homotopy type of the space of psc metrics is now, that it is at least as complicated as the right side, in fact:

Theorem ([BER17, Thm. A])**.** *W spin manifold of dimension $d \geq 6$, $h \in \mathcal{R}^+(\partial W)$, $g_0 \in \mathcal{R}^+(W)_h$. In case*

[6] this also explains, why the $\mathcal{H}$ was tacitly omitted right after its introduction ...

Figure 1: The torpedo metric g^2_{tor} on D^2.

(1) $A_k(W, g_0)$ *is rationally surjective*
(2) $A_k(W, g_0)$ *is surjective*

Proving this theorem will be the main objective this talk and the subsequent one. The proof strategy is roughly as follows: One precomposes the above map by a map from $\Omega^{\infty+1} MT\,\mathrm{Spin}(2n)$, such that the composition is (the infinite loop-space version of) a well-known map from index theory. In general: To get some understanding of a space, have a known map factor through it!

Remark. The way we prove Theorem A here differs slightly from the treatment in [BER17], where instead a certain Theorem B, which implies Theorem A, may be considered the main theorem; cf. [BER17, Rmk. 1.2.6].

1.4 The Cobordism Theorem

Theorem (Chernysh, Ebert–Frenck [EF18])**.** *The restriction map*

$$\mathrm{res}\colon \mathcal{R}^+(W) \longrightarrow \mathcal{R}^+(\partial W)$$

is a quasifibration, i.e. the fibre inclusion $\mathcal{R}^+(W)_n = \mathrm{res}^{-1}(h) \hookrightarrow \mathrm{hofib}_h(\mathrm{res})$ *is a homotopy equivalence.*

Definition. Our preferred metric on D^n is the *torpedo metric* g^n_{tor}, which is defined as the lower hemisphere metric near the origin and the product metric on a collar of the boundary $\partial D^n = S^{n-1}$. See fig. 1 for a picture in the case $n = 2$.

Let $\phi\colon X^{k-1} \times D^{d-k+1} \to W^d$ be an embedding disjoint to the collar and $g_X \in \mathcal{R}(X)$ such that $g_X + g^{d-k+1}_{\mathrm{tor}}$ has psc.

Definition. $\mathcal{R}^+(W; \phi, g_X)_h \subseteq \mathcal{R}^+(W)_h$ is the subspace of metrics g such that $\phi^* g = g_X + g^{d-k+1}_{tor}$. This is the space of psc metrics *which are standard near* X.

Theorem (Cobordism Theorem; Chernysh, [EF18])**.** *If* $d - k + 1 \geq 3$*, then the inclusion*

$$\mathcal{R}^+(W; \phi, g_X)_h \hookrightarrow \mathcal{R}^+(W)_h$$

is a homotopy equivalence.

Remark. Unfortunately only the hard calculations needed for the Cobordism Theorem actually show, why the torpedo metric is the right metric. The following theorem at least let's us glimpse its good properties.

Theorem. *Let* $W\colon M_0 \rightsquigarrow M_1$ *compact d-dimensional cobordism and* $\phi\colon S^{k-1} \times D^{k-k+1} \hookrightarrow \mathrm{int}(W)$ *an embedding. Let* W' *be the result of doing surgery along* ϕ*. If* $3 \leq k \leq d-2$ *there is a homotopy equivalence*

$$SE\colon \mathcal{R}^+(W)_{h_0,h_1} \xrightarrow{\sim} \mathcal{R}^+(W')_{h_0,h_1}$$

Proof. As the surgery takes place in the interior of W, the boundary of W and the metric on it are not affected, so we may assume, that W is closed.

The result of the surgery $W' = W \setminus \phi(S^{k-1} \times \mathrm{int}(D^{d-k+1})) \cup_{S^{k-1}\times D^{d-k+1}} (D^k \times S^{d-k})$ has a canonical embedding $\phi' \colon D^k \times S^{d-k} \hookrightarrow W'$. As the spheres alway carry the round metric $g_\circ$ the construction of the torpedo metric implies the isomorphism in the following diagram:

$$\begin{array}{ccc} \mathcal{R}^+(W, \phi; g_\circ^{k-1})_{h_0,h_1} & \xrightarrow{\cong} & \mathcal{R}^+(W', \phi'; g_{tor}^{k-1})_{h_0,h_1} \\ \big\downarrow & & \big\downarrow \\ \mathcal{R}^+(W)_{h_0,h_1} & & \mathcal{R}^+(W')_{h_0,h_1} \end{array}$$

By Chernysh's Theorem the two vertical arrows are equivalences. □

From this theorem one can derive the cobordism invariance of the space $\mathcal{R}^+(W)$ for closed, simply-connected spin manifolds of dimension at least five (this is a rather classical result due to Gromov and Lawson).

1.5 Reduction of Theorem A to the case $d = 6$

The promised reduction of the main theorem to the case $d = 6$ hinges on the following theorem:

Theorem ([BER17, Thm. 3.6.1])**.** *Let W^d be compact spin with $\partial W = M$ and $h_0 \in \mathcal{R}^+(M), g_0 \in \mathcal{R}^+(W)_{h_0}$. Under $\mathcal{R}^+(W) \xrightarrow{\sim} \mathrm{hofib}_{h_0}(\mathrm{res})$ the fiber transport gives a homotopy class of a map T such that the following diagram is weakly homotopy commutative*

$$\begin{array}{ccc} \Omega\mathcal{R}^+(M) & \xrightarrow{\quad T \quad} & \mathcal{R}^+(W)_h \\ & {\scriptstyle \Omega\mathrm{inddiff}_{h_0}} \searrow \quad \swarrow {\scriptstyle \mathrm{inddiff}_{g_0}} & \\ & \Omega^{\infty+d+1}\mathsf{KO} & \end{array}$$

The proof of this theorem needs quite a few deep statements, so we will only say the following: The hard part is a certain identification of $\mathrm{inddiff}_h$, which needs a version of the so called *spectral flow theorem* proven in [Ebe16], also an additivity theorem for index classes in KO-theory (see [BER17, Thm. 3.4.2]) and the quasifibration theorem above is used here. We refer to the original source for further explanations.

Now to the actual reduction: The first step is to invoke a "detection theorem" [BER17, Prop. 3.4.14] by which it suffices to consider $W = D^d$. Now let $d > 6$ and assume that Theorem A holds for $d - 1$. Writing $k = 4s - (d - 1) + 1$ this means, that

$$A_k(\partial W, h_0) \colon \pi_k(\mathcal{R}^+(\partial W), h_0) \longrightarrow KO_{4s}(*) = \mathbb{Z}$$

is rationally surjective. Analogously we get the surjectivity statement in the other case of Theorem A. This allows us to pass from S^6 to D^7. For the next step of the induction one has to derive Theorem A for S^7 from knowing it for D^7. But since S^7 minus a small disc is D^7 we can look at the space of psc metrics which are standard near this disc, say $\mathcal{R}^+(S^7; \phi, g_{\mathrm{tor}})$, which is the same space as $\mathcal{R}^+(D^7)$. Now the cobordism theorem implies, that $\mathcal{R}^+(S^7; \phi, g_{\mathrm{tor}})$ is equivalent to $\mathcal{R}^+(S^7)$, which completes the induction step.

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