Talbot 2019 Talk 14 - Putting together the pieces

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We'll sketch the proof of the following theorem as it appears in [Kup17].

Theorem. *For* $d \neq 4, 5, 7$ $\pi_i B \operatorname{Diff}_\partial(D^d)$ *are finitely generated for all* i*.*

1 Some history

The group $\operatorname{Diff}_\partial(D^d)$ is the most fundamental diffeomorphism group. In the 50's, Smale conjectured that there is a homotopy equivalence

$$SO(d+1) \overset{\sim}{\hookrightarrow} \operatorname{Diff}^+(S^d)$$

where $\operatorname{Diff}^+(S^d)$ is the orientation preserving diffeomorphisms of S^d.

- $d = 1$ elementary
- $d = 2$ proved by Smale
- $d = 3$ proved by Hatcher
- $d = 4$ disproved by Watanabe
- $d \geq 5$ wrong (even on the level of connected components!) from Kervaire-Milnor

As spaces, there is a homotopy equivalence $\operatorname{Diff}^+(S^d) \simeq SO(d+1) \times \operatorname{Diff}_\partial(D^d)$ and

$$\pi_0 \operatorname{Diff}_\partial(D^d) = \Theta_{d+1}/h\text{-cob}$$

where Θ_{d+1} is the group of exotic structures on spheres S^{d+1} up to diffeomorphism. Since Θ_{d+1} is non-trivial in most dimensions, $\operatorname{Diff}^+(S^d)$ is almost never connected $\implies$ Smale's conjecture fails horribly.

(Farrell-Hsiáng) showed that

$$\pi_i \operatorname{Diff}_\partial(D^d) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } d \text{ is even} \\ K_{i+2}(\mathbb{Z}) \otimes \mathbb{Q} & \text{if } d \text{ is odd} \end{cases}$$

for $i < d/6 - 7$.

We also know that

$$K_j(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if if } j \neq 1 \bmod 4 \\ \mathbb{Q} & \text{otherwise} \end{cases}$$

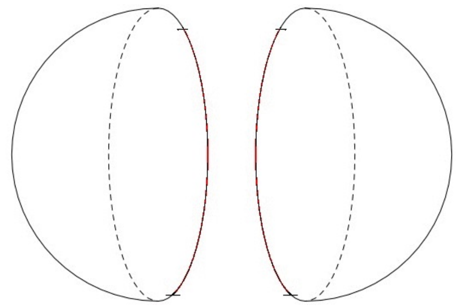

Figure 1: Forming an exotic S^{d+1} by gluing $d+1$-dimensional discs along identity on one half of the boundary and some element of $\pi_0 \operatorname{Diff}_\partial D^d$ along other half.

2 Several lemmas

We will only be considering the even dimensional case today. Perhaps later on we'll talk about the odd dimensions later but it is harder to do.[1]

Basic strategy For the manifold $M = W_{g,1} = \#_g S^n \times S^n \setminus \mathring{D}^{2n}$, consider the delooped Weiss fiber sequence

$$B \operatorname{Diff}_\partial(M) \to B \operatorname{Emb}^{\cong}_{1/2\partial}(M, M) \to B(B \operatorname{Diff}_\partial(D^d), \natural)$$

and analyze $B \operatorname{Diff}_\partial(M)$ by GRW's machinery, $B \operatorname{Emb}^{\cong}_{1/2\partial}(M, M)$ by embedding calculus.

We'll begin by defining three finiteness classes of spaces.

Definition. For each of these classes, we require π_0 to be finite. Then for each connected component, a space X is in

Fin if $\pi_1 X$ finite and $\pi_i X$ finitely generated for $i \geq 2$,

Hfin if for any A which is a finitely generated $\pi_1 X$ module that is finitely generated as an abelian group $\implies H_*(X; A)$ is finitely generated in each degree,

Π**fin** if $B\Pi_1(X)$ is in **Hfin** and the groups $\pi_i X$ are finitely generated for $i \geq 2$.

Fact. We have the following inclusions

$$\mathbf{Hfin} \cap \{\pi_1 \text{ finite}\} = \mathbf{Fin} \subsetneq \Pi\mathbf{fin} \subsetneq \mathbf{Hfin}$$

Furthermore, these inclusions are strict, as witnessed by the spaces $S^2, S^1, S^1 \vee S^2$ respectively.

Remark (Reason for considering the delooped version of the Weiss fiber sequence)*.* If we have fiber sequence $F \to E \to B$ with B, $E \in \mathbf{Hfin}$, it does not imply that $F \in \mathbf{Hfin}$. An easy example is as follows: take classifying groups of the short exact sequence $0 \to F_\infty \to F_2 \to F_1 \to 0$ where $F_n =$ free (nonabelian) group on n generators. Then both BF_1 and BF_2 are in **Hfin** but the same is not true for BF_∞.

[1] Proved both proofs can be found in [Kup17].

Lemma.

$$\{\textit{finite-type CW complexes}\} \subset \mathbf{Hfin}$$

Definition. We say that two discrete groups G, H *differ by finite groups* if there exists a zig-zag of group homomorphisms

$$G = G_0 \leftarrow G_1 \rightarrow \cdots \rightarrow G_r = H$$

where all arrows have finite kernel and cokernel. One can show that for two such groups,

$$BG \in \mathbf{Hfin} \iff BH \in \mathbf{Hfin}.$$

Suppose $p : E \to B$ is a Serre fibration. Let $A \subseteq B$ be a subcomplex such that there exists a (partial) section $s_A : A \to E$ to the fibration $p^{-1}(A) \to A$.

Definition. With the notation as above, define

$$\Gamma(E, B; A) = \{s : E \to B \text{ section } \mid s|_A = s_A\}$$

Lemma. *Let B be a finite, connected CW complex. Let $F \to E \xrightarrow{p} B$ be a Serre fibration such that $\pi_i(F)$ are finitely generated all i. Let A be a non-empty CW-subcomplex of B with a partial section $s_A : A \to E$. Then,*

$$\pi_i \Gamma(E, B; A)$$

is finitely generated for all i.

Proof idea: The proof is by induction on the cells in $B \setminus A$. Let D^d be a cell in $B \setminus A$. There is a fibration

$$\Gamma(E, B; A \cup D^d) \to \Gamma(E, B; A) \to \Gamma(E|_{D^d}, D^d; S^d)$$

There is a natural homotopy equivalence $\Gamma(E|_{D^d}, D^d; S^d) \simeq \Omega^d F$, all of whose homotopy groups are finitely generated. We are done by induction and the long exact sequence of homotopy groups for fibrations. □

3 Proof for d even

We want to prove the following.

Theorem.

$$B\,\mathrm{Emb}^{\cong}_{1/2\partial}(M, M) \in \mathbf{Hfin}$$

We will separate the proof into two independent steps: $\pi_1, \pi_i (i \geq 2)$.

3.1 π_1

Look at the following sequence of homotopy groups induced by the Weiss fiber sequence

$$\pi_1 B\operatorname{Diff}(D^{2n}) \to \pi_1 B\operatorname{Diff}_\partial(M) \to \pi_1 B\operatorname{Emb}^{\cong}_{1/2\partial}(M,M) \to 0$$

which is equivalent to,

$$\pi_0 \operatorname{Diff}(D^{2n}) \to \pi_0 \operatorname{Diff}_\partial(M) \to \pi_0 \operatorname{Emb}^{\cong}_{1/2\partial}(M,M) \to 0$$

The group $\pi_0 \operatorname{Diff}(D^{2n})$, which equals Θ_{2n+1}, is known to be finite.

We need to show that $B\pi_0 \operatorname{Diff}_\partial(M) \in \mathbf{HFin}$. This was shown in the last talk (this is an arithmetic group[2].) (Serre, Borel): arithmetic groups are virtually (has finite index subroup satisfying) of finite type. Therefore, $B\operatorname{Emb}^{\cong}_{1/2\partial}(M,M) \in \mathbf{Hfin}$.

3.2 π_i for $i > 2$

Now need to look at $B\operatorname{Emb}^{\cong id}_{1/2\partial}(M,M)$. This is where embedding calculus comes in. To be able to apply embedding calculus we need to get rid of 1/2, because this is not quite the setting where embedding calculus is usually applied.

Consider $M^* = M \setminus \frac{1}{2}\partial M$ (which is no longer compact). $(M, \frac{1}{2}\partial M) \simeq (M^*, \partial M^*)$ is an isotopy equivalence and embedding calculus can be applied to the embedding space $\operatorname{Emb}_\partial(M^*, M^*)$.

Step 1. Consider embedding calculus tower for M^*. We want to study the basepoint component of $\operatorname{Emb}_\partial(M^*,M^*)_0 \simeq \operatorname{Emb}^{\cong id}_{1/2\partial}(M,M)$.[3] Consider the Taylor tower for the right-hand side.

$$\begin{array}{ccccc}
\operatorname{Emb}_\partial(M^*,M^*)_0 & \longrightarrow & T_k \operatorname{Emb}_\partial(M^*,M^*) & & \\
 & & \downarrow & & \\
 & & T_{k-1} \operatorname{Emb}_\partial(M^*,M^*) & & \\
 & & \vdots & & \\
 & & T_1 \operatorname{Emb}_\partial(M^*,M^*) & = & \operatorname{Imm}(M^*,M^*) \\
 & & \downarrow & & \\
 & & T_0 \operatorname{Emb}_\partial(M^*,M^*) & = & \operatorname{Map}(M^*,M^*)
\end{array}$$

Base case: The space $\operatorname{Map}(M^*,M^*)$ is in $\mathbf{Hfin}$. The Smale-Hirsch theorem implies that $\operatorname{Imm}(M^*,M^*) = \Gamma(\operatorname{Iso}(TM^*), M^*, \partial M^*) \in \mathbf{HFin}$.

Inductive step: We know that the k^{th} homogeneous layer $L_k \operatorname{Emb}_\partial(M^*,M^*) = \operatorname{hofib}(T_k \operatorname{Emb}_\partial(M^*,M^*) \to T_{k-1}\operatorname{Emb}_\partial(M^*,M^*))$ is given by compactly supported functions

$$\Gamma\left(E, \binom{M}{k}/\Sigma_n,\ \text{nbhd of (missing) fat diagonal and boundary}\right).$$

[2]Recall: $\Gamma = \pi_0 \operatorname{Diff}_\partial(M)$ an arithmetic group if $\Gamma \subset G \subset GL_n(\mathbb{Q})$ where G is $\mathbb{Q}$-algebraic and $\Gamma \cap GL_n(\mathbb{Q})$ has finite index in both Γ and $G \cap GL_n(\mathbb{Q})$

[3]This is proven by a hands-on geometric argument.

The fibration $E \to \binom{M}{k}/\Sigma_k$ has fibers given by the total homotopy fiber of the cubical diagram

$$\operatorname{Emb}(I, M)$$

where I varies over subsets of $\{1, 2, \ldots, k\}$. Using the Fulton-MacPherson model for configuration spaces, this implies that the base and fiber are in **HFin** which then shows that the layers $T_k \operatorname{Emb}_\partial(M^*, M^*)$ are in **HFin**.

Finally, the Goodwillie-Klein-Weiss analyticity result in [GW99] applies here, since the handle dimension of $W_{g,1} = n \leq 2n - 3$ and $\dim(W_{g,1}) = 2n$ which implies that $\operatorname{Emb}_\partial(M^*, M^*) \simeq T_\infty \operatorname{Emb}_\partial(M^*, M^*)$ is in **HFin**.

It then remains to show that $B \operatorname{Diff}_\partial(W_{g,1})$ is in **HFin**. For this look at

$$\begin{array}{ccccc} B\operatorname{Diff}_\partial(W_{g,1}) & \longrightarrow & B\operatorname{Emb}^{\cong}_{1/2\partial}(M, M) & \longrightarrow & B(B\operatorname{Diff}_\partial(D^{2n}), \natural) \\ \downarrow {\scriptstyle H_*\text{-iso in range of degrees } \leq \frac{g-3}{2}} & & & & \\ \Omega^\infty_0 MT\theta & & & & \end{array}$$

where we are choosing θ to be $W_{g,1} \to BSO(2n)\langle n\rangle \xrightarrow{\theta} BSO(2n)$.

$\implies B\operatorname{Diff}_\partial(D^{2n}) \in$ **HFin**. π_1 finite implies that $\in$ **Fin**, all π_* are finitely generated, thereby finishing the proof.

□

Remark (by Oscar). There is a missing step in the above proof: we need to say something about finitely generated π_1-modules, we have only shown H_* is iso for $\mathbb{Z}$-coefficients.

4 Odd-dimensional case

The odd dimensional case is treated similarly, but the manifolds W_g have to be replaced by $H_g = \#_g(D^{n+1} \times S^n)$, and the results by GRW get replaced by a theorem due to Botvinnik and Perlmutter in [BP17]. As their results only hold in odd dimensions greater than 7, the dimensions 5 and 7 have to be excluded in Kuper's main theorem.

References

[BP17] Boris Botvinnik and Nathan Perlmutter. Stable moduli spaces of high-dimensional handlebodies. *Journal of Topology*, 10(1):101–163, 2017.

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