13 Kreck's analysis of MCGs

Speaker: Shruthi

Let M^{2k} manifold which is $(k-1)$-connected, and M is almost parallelizable, i.e. $M \setminus \{p\}$ is parallelizable, and stably parallelizable. (Take $W = W_g = \#_g S^k \times S^k$). If $k \geq 3$, $H_k(M)$ represented by embedded spheres.

Let $\operatorname{Diff}^+(M)$ = orientation-preserving diffeos of M, and $\operatorname{Diff}^+(M \operatorname{rel} D^{2k})$ diffeos fixing $D^{2k} \subset M$. $\pi_0 \operatorname{Diff}^+(M)$ = isotopy classes of $\operatorname{Diff}^+(M)$ and $\tilde{\pi}_0 \operatorname{Diff}^+(M)$ 'pseudo-isotopy classes'

Definition. f_0, f_1 are *pseudo-isotopic* if $\exists F : M \times [0,1] \to M \times [0,1]$ diffeomorphism such that $F(-,0) = f_0, F(-,1) = f_1$.

Fact. (Cerf) When M simply-connected, $\pi_0 = \tilde{\pi}_0$.

Definition.

$$\begin{aligned} G : \pi_0 \operatorname{Diff}^+(M) &\to \operatorname{Aut}(H_k(M)) \\ [f] &\mapsto f_* \end{aligned}$$

$\operatorname{Aut}(H_k(M))$ = automorphisms of H_k that preserve intersection form and a map α where

$$\begin{aligned} \alpha : H_k(M) &\to \pi_{k-1} SO(k) \\ x &\mapsto \text{classifying map of normal bundle} \end{aligned}$$

Denote $\ker G =: \pi_0 S \operatorname{Diff}^+(M)$

Theorem. *Have a SES*

$$0 \to \pi_0 S \operatorname{Diff}^+(M \operatorname{rel} D^{2k}) \to \pi_0 \operatorname{Diff}^+(M \operatorname{rel} D^{2k}) \xrightarrow{G} \operatorname{Aut}(H_k(M))$$

Remark. RHS is an arithmetic group.

Proof. Need to show G surjective.

Remove B^{2k} another disk from M disjoint from D^{2k} to get N. Then N is a "handle body." $N = D^{2k} \left(\cup_{S^{k-1} \times D^k} D^k \times D^k \right)^{2g}$.

$N \simeq$ wedge of $2g$ spheres S^k. $H_k(N) = \mathbb{Z}^{2g}$. Take generators α, β. Use a Dehn twist[11] to map $\alpha \mapsto \alpha + \beta$, $\beta \mapsto \beta$. Any automorphism of $H_k(N)$ gives rise to a diffeomorphism f of N.

We want a diffeomorphism of M. Note that $f|_{\partial N = S^{2k-1}}$ is a diffeomorphism of S^{2k-1}. □

Definition. The *inertia group* $I(M)$ of M is the subgroup $I(M) \subseteq \mathcal{O}_{2k+1}$ homotopy spheres up to h-cobordism. $\Sigma^{2k} \in I(M)$ if $M \# \Sigma \cong M$ diffeomorphic.

Glue B^{2k} along $f|_{S^{2k-1}}$ to give $M' = M \# \Sigma$. Want $M' \cong M$.

Theorem. *(Kosinski) Suppose M $k-1$-connected and $2k$-dimensional π-manifold (stably parallelizable). Then $I(M) = 0$.*

Therefore we can extend f to M.

Now we want to understand the kernel $= \pi_0 S \operatorname{Diff}^+(M \operatorname{rel} D^{2n})$ of the SES.

[11] uses (almost) parallelizability of M ($\implies$ parallelizability of N)

1. Exhibit a map $\chi : \pi_0 S\,\mathrm{Diff}^+(M) \to \mathrm{Hom}(H_k(M), S\pi_k SO(k))$ where $S : \pi_k SO(k) \to \pi_k SO(k+1)$
   Suppose $[f] \in \pi_0 S\,\mathrm{Diff}^+(M)$. If $x \in H_k(M)$, $f|_{x(S^n)} = id_{S^k}$, $df| : \nu(S^k) \to \nu(S^k)$. Fact: all spheres are stably parallelizable. Since $\nu(S^k) \oplus \varepsilon = \varepsilon^{k+1}$, $df \oplus id : \varepsilon^{k+1} \to \varepsilon^{k+1}$ an automorphism, therefore an element of $\pi_k SO(k+1)$ and can be seen as an element $\chi(f)$ coming from $\pi_k SO(k)$[12]
2. Now we're going to exhibit a map $\gamma : \mathcal{O}_{2k+1} \to \pi_0 S\,\mathrm{Diff}(M \operatorname{rel} D^{2k})$. Given $\Sigma \in \mathcal{O}_{2k+1}$, $\Sigma = D^{2k+1} \cup_F D^{2k+1}$ for some $F : S^{2k+1} \to S^{2k+1}$. Choose F such that F fixes a disk D^{2k}_-. Define $f : M \to M$ via $f|_{M\setminus D^{2k}} = id$, $f|_{D^{2k}} = F|_{D^{2k}_+}$.

Theorem. *Have SES*

$$0 \to \mathcal{O}_{2k+1} \to \pi_0 S\,\mathrm{Diff}^+(M \operatorname{rel} D^{2k}) \xrightarrow{\chi} \mathrm{Hom}(H_k(M), S\pi_k SO(k)) \to 0$$

Remark. LHS is finite (Kervaire, Milnor) and RHS is finitely generated abelian (we know both of the inputs to Hom), and **??** helps us understand $\pi_0\,\mathrm{Diff}^+(M)$.

Proof. of 13

Surjectivity of RHS: take out a disc to get a 'handlebody,' twist based on given element of Hom, and 'reattach.'

Double composite is zero: Recall that doing the surgery doesn't do anything (?) to the middle homology of M.

ker $\subseteq$ im (middle). Given $f \in \ker\chi$, $f|_{N=M\setminus B^{2k}} \simeq id$ (isotopic?), $f|_{B^{2k}}$ gives a homotopy sphere in the previous sense.

To show injectivity, find an inverse $\sigma : \pi_0 S\,\mathrm{Diff}^+(M \operatorname{rel} D^{2k}) \to \mathcal{O}_{2k+1}$. $f \in \mathrm{Diff}^+(M)$, let M_f be the mapping torus. Have a nontrivial fundamental group and [something having to do with the homology]. Doing surgery on $S^1 \times D^{2k}$ (glue in $S^0 \times D^{2k+1}$) kills π_1.

Take $2g$ embeddings of $(S^k \times D^{k+1}) \subset M \times (0,1) \subset M_f$ representing a basis of $H_k(M)$ and perturb so images are disjoint. Do surgery on them to get $\sigma(f)$ a homotopy sphere.

Where do we use that $f \in \ker\chi$?

Surjectivity of σ: If $f \in \pi_0 S\,\mathrm{Diff}$ is the image of Σ a homology (homotopy?) sphere. $M_f = M \times S^1 \# \Sigma$. Doing surgery gives back Σ. □

[12] measures twisting of normal bundle.