Talbot 2019 Talk 12 - Weiss Fiber Sequence

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References: [Kup17], [Wei15], [ER17].
Overview of Talks 11-14:

$$\begin{array}{c} \text{Embedding calculus + Weiss fibration + Homological stability} \\ \Big\downarrow{\scriptstyle \text{magic}} \\ \text{finiteness results about } \mathrm{Diff}_\partial(D^n) \end{array}$$

NOTATION: D^n are discs with boundary, $\mathrm{Diff}_\partial(-)$ = diffeomorphisms that are the identity on a neighborhood of the boundary (this is so that we don't have to worry about corners).

There are a lot of superscripts and subscripts to keep track of. We have the following general rule: in the notation $\mathrm{Emb}_*^\bullet$: the superscripts $*$ will denote certain connected components, and the subscripts $\bullet$ will denote subspaces which are fixed.

1 Weiss fiber sequence

If we want to study diffeomorphisms of the disc, the natural place to start is $M = \#_g S^n \times S^n \setminus D^{2n}$.

Let M^n be a compact connected smooth manifold with $\partial M \equiv S^{n-1}$ (this is the only property we need for this lecture).

We can decompose the boundary of M as

$$\partial M = \partial_{-1/2} M \cup \partial_{1/2} M$$

where $\partial_{\pm 1/2} M \cong D^{n-1}$.

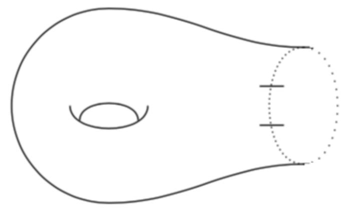

Figure 1: The manifold M with the boundary divided into two parts $\partial_{-1/2}M \cup \partial_{1/2}M$. Figure from [Kup17].

Definition. Define $\mathrm{Emb}_\partial(M, M)$ to be the space of embeddings which fix the boundary, which is the same as $\mathrm{Diff}_\partial(M)$.

Definition. The space $\mathrm{Emb}_{1/2\partial}(M, M)$ consists of embeddings $M \to M$ that restrict to the identity on a neighborhood of $\partial_{1/2}M$.[1]

There is a natural map $\mathrm{Diff}_\partial(M) \overset{i}{\hookrightarrow} \mathrm{Emb}_{1/2\partial}(M, M)$.

Definition. Define $\mathrm{Emb}^{\cong}_{1/2\partial}(M, M)$ to be the union of the connected components of $\mathrm{Emb}_{1/2\partial}(M, M)$ which intersect the image of i.

Theorem 1 (Weiss fiber sequence)**.** *The homotopy fiber of* $\mathrm{Diff}_\partial(M) \overset{i}{\hookrightarrow} \mathrm{Emb}^{\cong}_{1/2\partial}(M, M)$ *is* $\mathrm{Diff}_\partial(D^n)$.

Theorem 2 (Weiss fiber sequence)**.** *This fiber sequence can be delooped to give a fiber sequence*

$$B\,\mathrm{Diff}_\partial(M) \to B\,\mathrm{Emb}^{\cong}_{1/2\partial}(M, M) \to B\left(B\,\mathrm{Diff}_\partial(D^n)\right)$$

The original motivation for studying this fiber sequence was to study certain exotic pontrjagin classes coming from manifold bundles, [Wei15].

"Proof" of Theorem 1. We will start by defining a map $\mathrm{Emb}^{\cong}_{1/2\partial}(M, M) \to B\,\mathrm{Diff}_\partial(D^n)$.

WLOG suppose $M \subset \mathbb{R}^\infty$. Pick a submanifold M' of M obtained from M by 'pushing' the negative part of the boundary in the interior of M. This gives us a weak equivalence

$$\mathrm{Emb}^{\cong}_{1/2\partial}(M, M) \to \mathrm{Emb}^{\cong}_{1/2\partial}(M, M')$$

using an isotopy from $M \to M'$.

Let e be an embedding in $\mathrm{Emb}^{\cong}_{1/2\partial}(M, M)$, so that it lies in the connected component that can be isotoped to the boundary. Using an h-cobordism argument[2], $M \setminus e(M)$ is diffeomorphic to D^n up to smoothing of corners.

For each e, we have the pair

$$(\overline{M \setminus e(M)}, \partial(\overline{M \setminus e(M)}) \xrightarrow{e,id} S^{n-1})$$

[1] $\partial_{-1/2}M$ might get mapped to the interior.

[2] Let us assume for now $n \geq 5$.

where the first element in the pair is a subset of $\mathbb{R}^\infty$ diffeomorphic to D^n[3]) and the second element is an explicit diffeomorphism. This is an element in $B\,\mathrm{Diff}_\partial(D^n)$. So, we have constructed the desired map $\mathrm{Emb}^{\cong}_{1/2\partial}(M,M) \to B\,\mathrm{Diff}_\partial(D^n)$.

It remains to show that the following is a homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{Diff}_\partial(M) & \longrightarrow & E\,\mathrm{Diff}_\partial(D^n) \\ \downarrow & & \downarrow \\ \mathrm{Emb}^{\cong}_{1/2\partial}(M,M') & \longrightarrow & B\,\mathrm{Diff}_\partial(D^n) \end{array}$$

This is a standard isotopy extension argument. □

Remark. Why do we expect delooping? In the Weiss fiber sequence, the first space is a topological group and the second is a topological monoid

$$\mathrm{Diff}_\partial(M) \to \mathrm{Emb}^{\cong}_{1/2\partial}(M,M) \to B\,\mathrm{Diff}_\partial(D^n).$$

1. Smoothing theory tells us that

$$\begin{aligned} B\,\mathrm{Diff}_\partial(D^n) &\simeq \Omega^n_0 PL(n)/O(n) \\ &\simeq \Omega^n_0 Top(n)/O(n) \text{ if } n \neq 4. \end{aligned}$$

2. (Fun exercise) One can explicitly show that $B\,\mathrm{Diff}_\partial(D^n)$ is an E_n-algebra, which in conjuction with May's recognition principle suggests delooping.

Proof strategy : if a group G acts on a space X, then

$$X \to EG \times_G X \to BG$$

is a fiber sequence. Analogously, we will construct Moore monoids $BM \simeq B\,\mathrm{Diff}_\partial(M)$, $BD \simeq B\,\mathrm{Diff}_\partial(D^n)$

$$BM \to BM \,/\!\!/\, BD \to * \,/\!\!/\, BD$$

The hardest part of the proof is to show $BM \,/\!\!/\, BD \simeq B\,\mathrm{Emb}^{\cong}_{1/2\partial}(M,M)$

2 Moore monoids for $B\,\mathrm{Diff}_\partial(D^n)$

We will use the identification

$$\mathrm{Diff}_\partial(D^n) \simeq \mathrm{Diff}_\partial(D^{n-1} \times [0,1]).$$

This gives us a special direction to glue diffeomorphisms.

Definition. Define

$$\begin{aligned} D_\infty &= D^{n-1} \times [0,\infty) \\ D_t &= D^{n-1} \times [0,t] \subset D_\infty \end{aligned}$$

[3] *There exists* a diffeomorphism but the diffeomorphism is not part of the data.

Definition. A *Moore monoid* for $B\operatorname{Diff}_\partial(D^n)$ is defined to be

$$\mathbb{D} = \{(t, \varphi) \mid t \in [0, \infty), \varphi \in \operatorname{Diff}_\partial(D_\propto), \operatorname{supp}(\varphi) \subset D_t\}$$

where $\operatorname{supp}(\varphi) = \overline{\{x \mid \varphi(x) \neq x\}}$. This is a monoid by concatenation

$$\begin{aligned} \sqcup : \mathbb{D} \times \mathbb{D} &\to \mathbb{D}, \\ (t, \varphi_t), (s, \varphi_s) &\mapsto (t + s, \varphi_t \sqcup \varphi_s). \end{aligned}$$

Definition. Define the semi-simplicial space[4]

$$N_p\mathbb{D} = \{(t, \varphi_1, \dots, \varphi_p) \mid \varphi_i \in \operatorname{Diff}_\partial(D_\propto) \cup \operatorname{supp}(\varphi_i) \subset D_t\}$$

where boundary maps are given by composition of diffeomorphisms. $N_\bullet\mathbb{D}$ is a monoid in simplicial spaces, where the monoid structure is given by levelwise concatenation. [5]

Definition. Finally, define

$$B\mathbb{D} = \|N_\bullet D\|$$

where $\| - \|$ is the thick geometric realization.

Proposition 1. *The following inclusion of monoids is a weak homotopy equivalence.*

$$\begin{aligned} \operatorname{Diff}_\partial(D^{n-1} \times [0, 1]) &\to \mathbb{D} \\ \varphi &\mapsto (1, \varphi) \end{aligned}$$

Proof. This is an 'isotopy extension' argument: $[0, \infty) \xrightarrow{\text{for all } t} [0, 1) \hookrightarrow [0, 1]$ defines a deformation retraction from latter onto former. □

3 'Moore monoid' for $B\operatorname{Diff}_\partial M$

We will costruct a space $B\mathbb{M}$ which will be a module over $B\mathbb{D}$.

Definition. Define

$$\begin{aligned} M_\propto &:= M \cup D_\propto \\ M_t &:= M \cup D_t. \end{aligned}$$

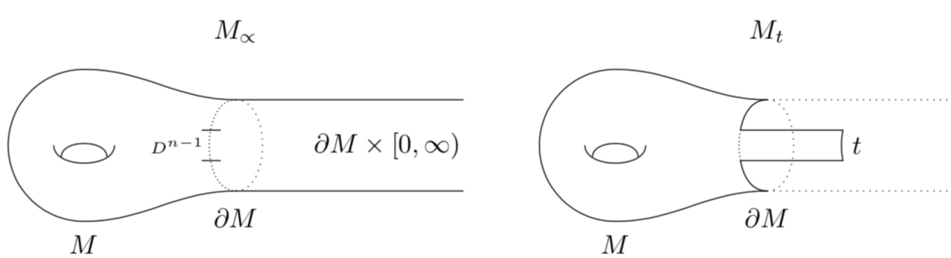

Figure 2: The manifold $M_t \subseteq M_\propto$. Figure from [Kup17].

[4] 'fake nerve'

[5] This is a monoid as concatenation commutes with composition.

Definition. A *'Moore monoid'* for $B\,\mathrm{Diff}_\partial(M)$ is defined to be

$$\mathbb{M} = \{(t, \varphi) \mid t \in [0, \infty), \varphi \in \mathrm{Diff}_\partial(M_\infty), \mathrm{supp}(\varphi) \subset M_t\}.$$

This is a module over $\mathbb{D}$ by concatenation

$$\begin{gathered} \sqcup : \mathbb{M} \times \mathbb{D} \to \mathbb{M}, \\ (t, \varphi_t), (s, \varphi_s) \mapsto (t + s, \varphi_t \sqcup \varphi_s). \end{gathered}$$

Definition. Define the semi-simplicial space[6]

$$N_p\mathbb{M} = \{(t, \varphi_1, \ldots, \varphi_p) \mid \varphi_i \in \mathrm{Diff}_\partial(M_\infty) \cup \mathrm{supp}(\varphi_i) \subset M_t\}$$

where boundary maps are given by composition of diffeomorphisms. $N_\bullet\mathbb{M}$ is a module in simplicial spaces over $N_\bullet\mathbb{D}$.

Definition. Finally, define

$$B\mathbb{M} = \|N_\bullet M\|$$

We now have the setup of a group $B\mathbb{D}$ acting on a space $B\mathbb{M}$. By formal properties of simplicial sets, we have a fiber sequence (since BD is connected)

$$\begin{aligned} B\mathbb{M} \to & B_\bullet(B\mathbb{M}, B\mathbb{D}, *) \to B_\bullet(*, B\mathbb{D}, *) \\ \text{written equivalently as, } B\mathbb{M} \to & \quad B\mathbb{M} /\!\!/ B\mathbb{D} \to \quad * /\!\!/ B\mathbb{D} \end{aligned}$$

where $B_\bullet(-, B\mathbb{D}, -)$ is the two sided bar construction.

By Proposition 1 we have levelwise equivalences between simplicial spaces, giving us equivalences on thick geometric realizations [ER17].

$$\begin{aligned} BM &\simeq B\,\mathrm{Diff}_\partial(M) \\ BD &\simeq B\,\mathrm{Diff}_\partial(D^n) \end{aligned}$$

The final hard theorem[7] in [Kup17] shows $BM /\!\!/ BD \simeq B\,\mathrm{Emb}^{\cong}_{1/2\partial}(M, M)$.

Idea behind the proof of the hard theorem We wish to show that $B\mathbb{M} /\!\!/ B\mathbb{D} \simeq B\,\mathrm{Emb}^{\cong}_{1/2\partial}(M, M)$. If the action of $B\mathbb{D}$ on $B\mathbb{M}$ were free then we would have

$$\begin{aligned} B\mathbb{M} /\!\!/ B\mathbb{D} &= B(\mathbb{M}/\mathbb{D}) \\ &= B\,\mathrm{Emb}^{\cong}_{1/2\partial}(M, M) \end{aligned}$$

However, this is not true. Much of the work in the proof goes in replacing the $B\mathbb{M}$ and $B\mathbb{D}$ so that the action of the latter on the former is free, and then showing that the homotopy type is unchanged when we make this replacement.

[6]'fake nerve'

[7]Read the details if you want to be amazed by isotopy extension theorems.

References

[ER17] Johannes Ebert and Oscar Randal-Williams. Semi-simplicial spaces. *arXiv e-prints*, May 2017.

[Kup17] Alexander Kupers. Some finiteness results for groups of automorphisms of manifolds. *Geom. Topol.*, 2017.

[Wei15] Michael S. Weiss. Dalian notes on rational Pontryagin classes. *arXiv e-prints*, Jul 2015.