Talbot 2019 Talk 11 - Embeddings Calculus

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April 11, 2019

In calculus, we understand smooth functions by taking (Taylor) polynomial approximations to them. Analogously, in *embeddings calculus* or more generally in *functor calculus* we try to approximate functors on a certain nice category via 'polynomials.'

0 Introduction

In *embeddings calculus* we work with functors of the form

$$F : \mathcal{O}(M)^{op} \to \mathbf{Top},$$

where $\mathcal{O}(M)$ is the poset of open subsets of a smooth closed manifold M. If M has boundary, define $\mathcal{O}^{\partial}(M)$ category of open subsets of M containing ∂M.

Definition. A functor $F : \mathcal{O}(M)^{op} \to \mathbf{Top}$ is said to be *good* if

1. F takes isotopy equivalences to homotopy equivalences.
2. For a filtration $U_1 \subset \cdots \subset U_i \subset \cdots$, there is a homotopy equivalence

$$F(\cup_i U_i) \xrightarrow{\sim} \operatorname{holim}_i F(U_i)$$

   i.e. F behaves nicely with respect to certain homotopy limits.

Examples. Fix a smooth manifold N. Then the following functors are good functors on

$$\operatorname{Map}(-, N), \operatorname{Imm}(-, N), \operatorname{Emb}(-, N)$$

when seen as **Top** valued functors on $\mathcal{O}(M)^{op}$.

From now on we'll assume that all our functors are good functors of the form $\mathcal{O}(M)^{op} \to \mathbf{Top}$, unless otherwise specified.

1 Polynomial functors

A linear function f satisfies the identity

$$f(x+y) - f(x) - f(y) + f(0) = 0$$

The following should be thought of as a homotopy theoretic generalization of this.

Definition. A functor F is said to be *linear* if for all $V, W \subset M$, the total homotopy fiber[1] of

$$\begin{array}{ccc} F(V \cup W) & \longrightarrow & F(V) \\ \big\downarrow & & \big\downarrow \\ F(W) & \longrightarrow & F(V \cap W) \end{array} \tag{1}$$

is contractible.

It is not obvious how to generalize this definition. Instead, we use the following equivalent definition of a linear functor which naturally generalizes to higher degrees.

Definition. A functor F is said to be *polynomial of degree* ≤ 1 if for all $U \in \mathcal{O}(M)$ and disjoint, closed, non-empty subsets A_0, A_1 of U, the total homotopy fiber of

$$\begin{array}{ccc} F(U) & \longrightarrow & F(U \setminus A_0) \\ \big\downarrow & & \big\downarrow \\ F(U \setminus A_1) & \longrightarrow & F(U \setminus (A_0 \cup A_1)) \end{array}$$

is contractible.

This now has an obvious generalization.

Definition. A functor F is said to be *polynomial of degree* $\leq k$ if for all $U \in \mathcal{O}(M)$ and disjoint, closed, non-empty subsets $A_0, \dots, A_k$ of U, the homotopy fiber of the $(k+1)$-cube

$$\begin{aligned} \mathcal{P}(k+1) &\to \mathbf{Top} \\ \{0, \dots, k\} \supset S &\mapsto F\left(U \setminus \cup_{i \in S} U_i\right) \end{aligned}$$

is contractible.

Fact.
- $Map(-, N)$ is polynomial of degree ≤ 1.
- $\mathrm{Imm}(-, N)$ is polynomial of degree ≤ 1 if the dimension of N ¿ handle dimension of M ¿ 0, see Section 3.
- $Emb(-, N)$ is not polynomial of degree $\leq k$ for any k.

2 The Taylor tower

We'll now define a way to construct a polynomial approximation of an arbitrary good functor.

Definition. Let $\mathcal{O}_k(M)$ be the full subcategory of $\mathcal{O}(M)$ containing open subsets of M which are diffeomorphic to up to k open balls in M. If M has boundary, the full subcategory $\mathcal{O}_k^\partial(M)$ consists of open subsets $U = V_1 \sqcup V_2$ where V_1 is a collar nbhd of ∂M and V_2 is diffeomorphic up to k open balls.

[1] The total homotopy fiber of the square 1 is obtained by first taking the homotopy fibers of the vertical maps, then taking homotopy fiber of resulting horizontal map

$$\mathrm{hofib}(F(V \cup W) \to F(W)) \to \mathrm{hofib}(F(V) \to F(V \cap W)).$$

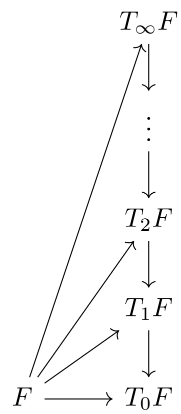
Definition. For a functor $F : \mathcal{O}(M)^{op} \to \mathbf{Top}$ the *k-th polynomial approximation* is a functor $T_kF : \mathcal{O}(M)^{op} \to \mathbf{Top}$ defined as

$$T_kF(V) = \operatorname{holim}_{U \in \mathcal{O}_k(V)} F(U).$$

where $V \in \mathcal{O}(M)$.

We have natural transformations $F \implies T_kF$ induced by the inclusion $\mathcal{O}_k(V) \hookrightarrow \mathcal{O}(V)$ and $T_kF \implies T_{k-1}F$ induced by the inclusion $\mathcal{O}_{k-1}(V) \hookrightarrow \mathcal{O}_k(V)$.

These assemble into a *Taylor tower*



Thus we can think of T_kF as the the k^{th} stage in the Taylor tower. The *k-th layer* is denoted $L_k = \operatorname{hofib}(T_kF \to T_{k-1}F)$.

Definition. We say T_kF *converges to* F if the naturally induced map

$$F(V) \xrightarrow{\sim} T_\infty F(V)$$

is an equivalence for all $V \in \mathcal{O}(M)$, where $T_\infty F$ is the inverse limit of the Taylor tower $T_\infty F := \operatorname{holim}_k T_k$.

Theorem ([Wei99])**.** *If $F : \mathcal{O}(M)^{op} \to \mathbf{Top}$ is good then,*

1. *T_kF is polynomial of degree $\leq k$*
2. *If F is polynomial of degree $\leq k$ then $F \to T_kF$ is a homotopy equivalence.*

2.1 Homogeneous functors

Definition. A functor $E : \mathcal{O}(M)^{op} \to \mathbf{Top}$ is *homogeneous of degree* $\leq k$ if E is polynomial of degree $\leq k$, and $T_{k-1}E(V) \simeq *$ for all V.

Example. For any functor $F : \mathcal{O}(M)^{op} \to \mathbf{Top}$, the k^{th} layer in the Taylor tower L_kF is homogeneous of degree k

It is possible to a complete classification of homogeneous functors. Let $\binom{M}{k}$ be the space of unordered configurations of k points in M. Let $\rho : Z \to \binom{M}{k}$ be a fibration with section. Denote the space of sections by $\Gamma\left(\binom{M}{k}, Z; \rho\right)$.

Definition. Define

$$\Gamma\left(\partial\binom{M}{k}, Z;\rho\right) = \operatorname{hocolim}_{Q\in\mathcal{N}}\Gamma\left(\binom{M}{k}\cap Q, Z;\rho\right)$$

where Q is a neighborhood of the fat diagonal in M^k/Σ_k. When M has boundary, we take $\mathcal{N}$ also including elements of M^k/Σ_k where one coordinate is in ∂M.

Let

$$\Gamma^c\left(\binom{M}{k}, Z;\rho\right) = \operatorname{hofib}\left(\Gamma\left(\binom{M}{k}, Z;\rho\right)\to\Gamma\left(\partial\binom{M}{k}, Z;\rho\right)\right)$$

be the space of compactly supported sections.

The space of compactly supported sections provides a canonical example for homogeneous functors of degree k.

Example. The functor $\Gamma^c\left(\binom{-}{k}, Z;\rho\right)$ is homogeneous of degree k.

Theorem. *Let E homogeneous of degree k. Then there exists a fibration $p : Z \to \binom{-}{k}$ such that E is naturally homotopy equivalent to the space of compactly supported sections of Z,*

$$E(-)\xrightarrow{\simeq}\Gamma^c\left(\binom{-}{k}, Z;\rho\right).$$

If $E = L_kF$ for some functor F, then the fibers of the classifying fibration are the called the derivatives *and are denoted $F^k(\varnothing)$.*

Definition. Let $B_1,\ldots,B_k$ pairwise disjoint open balls in M. Then $F^{(k)}(\varnothing) = t\operatorname{hofib}\left(S\mapsto F\left(\bigcup_{i\notin S}B_i\right)\right)$

Example. $F = \operatorname{Emb}(-,\mathbb{R}^n), k = 2$

$$\begin{array}{ccc} \operatorname{Emb}(B_1\sqcup B_2,\mathbb{R}^n) & \longrightarrow & \operatorname{Emb}(B_1,\mathbb{R}^n) \\ \downarrow & & \downarrow \\ \operatorname{Emb}(B_2,\mathbb{R}^n) & \longrightarrow & \operatorname{Emb}(\varnothing,\mathbb{R}^n) \end{array} \quad\to\quad \begin{array}{ccccc} S^{n-1} & \longrightarrow & S^{n-1} & \longrightarrow & * \\ & & \downarrow & & \downarrow \\ & & \operatorname{Conf}(2,\mathbb{R}^n) & \longrightarrow & \operatorname{Conf}(1,\mathbb{R}^n) \\ & & \downarrow & & \downarrow \\ & & \operatorname{Conf}(1,\mathbb{R}^n) & \longrightarrow & \operatorname{Conf}(0,\mathbb{R}^n)\simeq * \end{array}$$

3 Convergence

Recall: We say that the Taylor tower for F converges to F if $F\to T_\infty F$ is an equivalence.

Proposition. *If $F^{(k)}(\varnothing)$ is c_k-connected, then $L_k(F(M))$ is (c_k-km)-connected, where $m = \dim M$.*

Definition. The *handle dimension* is the least positive integer j such that M admits a handlebody decomposition with handles of index $\leq j$.

Theorem. *If M is a smooth manifold of handle dimension m and N is a smooth manifold of dimension n, then the map*

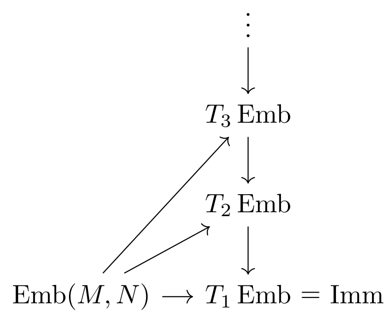
$$\mathrm{Emb}(M, N) \to T_k \mathrm{Emb}(M, N)$$

is $(k(n-m-2)+1-m)$-connected. In particular, if $n-m-2>0$, then

$$\mathrm{Emb}(M, N) \to T_\infty \mathrm{Emb}(M, N)$$

is an equivalence.

Fact. $T_1 \mathrm{Emb} = \mathrm{Imm}$



Thus the tower for the embeddings functor can be thought of as a way to remove self-intersections iteratively.

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