Talbot 2019 Talk 10

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Beyond $W_{g,1}$

Let W a $2n$-dimensional manifold, $n \geq 3$, compact, connected. When discussing general such manifolds, tangential structures become *essential.* Throughout the talk, let $\theta : B \to BO(2n)$ be a tangential structure and $\ell_W : W \to B$ a chosen θ-structure on W.

We define *the moduli space of W with θ-structures* to be

$$\mathcal{M}^\theta(W) := \mathrm{Emb}(W, \mathbb{R}^\infty) \times \mathrm{Bun}(TW, \theta^*\gamma_{2n}) / \operatorname{Diff}(W)$$

As $B\operatorname{Diff}(W)$ has a model given by the submanifolds of $\mathbb{R}^\infty$ diffeomorphic to W, this moduli space $\mathcal{M}^\theta(W)$ has a model consisting of all submanifolds of $\mathbb{R}^\infty$ diffeomorphic to W together with a choice of θ-structure.

If we fix a θ-structure ℓ_W, we denote by $\mathcal{M}^\theta(W, \ell_W)$ is the path component of $\mathcal{M}^\theta(W)$ containing ℓ_W.

When we worked with $W_{g,1}$'s, homological stability depended on *genus.* What do we use here?

$$\text{genus} = \max\{g \mid \exists \text{ embedding } e : \#_g S^n \times S^n \setminus D^{2n} \to W\}$$

when considering θ-structures, we require that such embeddings be *admissible*, i.e. that they respect the tangential structure.

1 Theorems

Theorem (A). *If ℓ_W is n-connected, B is simply-connected,*

$$\mathcal{M}^\theta(W, \ell_W) \to \Omega^\infty MT\theta$$

induces a homology isomorphism onto the path component of the image in degrees $\leq \frac{g-4}{3}$. If θ is spherical, in degrees $\leq \frac{g-3}{2}$.

Remark. θ is *spherical* if S^n possesses a θ-structure.

Theorem A depends on the very strong hypothesis that ℓ_W is n-connected, which means that for a fixed manifold W, it cannot be applied to any tangential structures. However, we can adapt this result to hold in a wider range of cases.

General tangential structures Given a tangential structure θ and ℓ_W a θ-structure on W which is not n-connected, we can use the Moore-Postnikov tower for ℓ_W:

$$\begin{array}{ccccc} & & B' & & \\ & {\scriptstyle n\text{-connected cofibration}}\nearrow & & \searrow{\scriptstyle n\text{-coconnected fibration}}\ u & \\ W & & \xrightarrow[\ell_W]{} & & B \xrightarrow{\theta} BO(2n) \end{array}$$

which gives a new tangential structure $\theta' : B' \to B \to BO(2n)$ and the map $W \to B'$ is an n-connected θ'-structure for W.

Definition. We call *homotopy automorphisms over* u the topological monoid given by

$$h\operatorname{Aut}(u) := \{\text{weak equivalences of } B' \text{ over } B\}$$

with multiplication given by composition.

There is an action of $h\operatorname{Aut}(u)$ on $\mathcal{M}^{\theta'}(W)$ by precomposing the θ-structures on W with such weak equivalences. This action gives us fiber sequences:

$$h\operatorname{Aut}(u) \to \mathcal{M}^{\theta'}(W) \to \mathcal{M}^{\theta}(W)$$
$$\mathcal{M}^{\theta'}(W) \to \mathcal{M}^{\theta}(W) \to Bh\operatorname{Aut}(u)$$
$$\mathcal{M}^{\theta'}(W, \ell_W) \to \mathcal{M}^{\theta}(W, \ell_W) \to BH$$

where $H \subset h\operatorname{Aut}(u)$ is a submonoid.

Theorem (B)**.** *If ℓ_W any θ-structure, W simply-connected, then*

$$\mathcal{M}^{\theta}(W, \ell_W) \to \Omega_0^{\infty} MT\theta' /\!\!/ h\operatorname{Aut}(u)$$

induces a H_*-isomorphism in degrees $\leq \frac{g-4}{3}$. If θ spherical, then in degrees $\leq \frac{g-3}{2}$.

These theorems are extremely important tools to understand the moduli space of manifolds, in particular because we know how to compute the cohomology of $\Omega^\infty MT\theta$ through different methods (for instance using the Thom isomorphism). Explicitly, we have:

Theorem. $H^*(\Omega^\infty MT\theta; \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c \text{ basis element of } H^{>d}(B; \mathbb{Q}^{w_1})]$

2 Example

Let $V_d \subset \mathbb{C}P^{n+1}$ be a smooth hypersurface determined by the roots of a nondegenerate homogeneous complex polynomial of degree d.

Fact. The diffeomorphism type of such a smooth V depends only on the degree d of the polynomial. The resulting $2n$-dimensional manifold is what we denote V_d.

Goal. To compute $H^*(\mathcal{M}^{or}(V_d); \mathbb{Q})$ for $V_d \subset \mathbb{C}P^4$ a 6-dimensional manifold.

STRATEGY:

1. Understand the algebraic topology of V_d;
2. Look at the Moore-Postnikov 3-stage for an orientation and apply Theorem A;
3. Use Theorem B to get a result about orientation.

2.1 Algebraic topology of V_d

The cohomology of V_d can be understood through some classical results in Algebraic Topology and Algebraic Geometry.

Lefschetz hypersurface theorem tells us that $V_d \overset{i}{\hookrightarrow} \mathbb{C}P^4$ is 3-connected. In particular, V_d is simply-connected and

$$H^0(V_d) = \mathbb{Z} \qquad H^1(V_d) = 0 \qquad H^2(V_d) = \mathbb{Z}\{t\}$$

By Poincaré duality,

$$H^6(V_d) = \mathbb{Z}\{u\} \qquad H^5(V_d) = 0 \qquad H^4(V_d) = \mathbb{Z}\{?\}$$

The Euler characteristic can be computed using methods from Algebraic Geometry, and we get

$$\chi(V_d) = d(10 - 10d + 5d^2 - d^3)$$

Combining all of the above we see that $H^3(V_d)$ is free of rank $4 - \chi(V_d) = d^4 - 5d^3 + 5d^2 - 10d + 4$.

In particular, this tells us what the genus of this manifold could be, since it is measured by the number of embedded $S^3 \times S^3$. A precise result on the genus can be obtained using the following:

Theorem (Wall's theorem)**.** *If W simply-connected, 6-dimensional manifold, then there exists M such that $H_3(M)$ is finite and $W = M\#_g S^3 \times S^3$.*

In particular, applying this result for the V_d's, we get that their genus is given by

$$g = \frac{1}{2}(\text{3rd Betti number}) = \frac{1}{2}(d^4 - 5d^3 + 5d^2 - 10d + 4)$$

2.2 Moore-Postnikov 3-stage

Given a choice of orientation $V_d \to BSO(6)$, we can take its Moore-Postnikov 3-stage

$$\begin{array}{ccc} & B^d & \\ {}^{\ell_{V_d}}\nearrow & & \searrow^{\theta_d} \\ V_d & \longrightarrow & BSO(6) \end{array}$$

Where ℓ_{V_d} is a 3-connected cofibration and θ_d is a 3-coconnected fibration.

Applying Theorem A to θ_d: we have a H_*-isomorhpism for $* \leq \frac{g-4}{3} = \frac{d^4+5d^3+10d^2-10d+4}{4}$

$$\mathcal{M}^{\theta_d}(V_d, \ell_{V_d}) \to \Omega^\infty MT\theta_d$$

which gives us an isomorphism in the same range between

$$H^*(\mathcal{M}^{\theta_d}(V_d, \ell_{V_d})) \xleftarrow{\cong} \mathbb{Q}[\kappa_c \mid c \text{ basis element of } H^{>d}(B^d;\mathbb{Q})] \tag{1}$$

What is $H^*(B^d;\mathbb{Q})$? By the (co)-connectivity hypotheses,

$$\begin{aligned} 0 = \pi_1(V_d) &\cong \pi_1(B^d) \\ \mathbb{Z} = \pi_2(V_d) &\xrightarrow{\cong} \pi_2(B^d) \xrightarrow{?} \pi_2(BSO(6)) = \mathbb{Z}/2 \\ \pi_3(V_d) &\twoheadrightarrow \pi_3(B^d) \hookrightarrow \pi_3 BSO(6) = 0 \\ &\pi_i(B^d) \xrightarrow{\cong} \pi_i BSO(6), \qquad i \geq 4 \end{aligned}$$

These maps tell us basically everything about the homotopy groups of B^d, except in degree 2. So it remains to understand the map on π_2, which is detected by w_2, the 2nd Stiefel-Whitney class:

$$w_2 \equiv 5 - d \mod 2$$

Then the map will be different depending on d being even or odd, however in either case,

$$B^d \xrightarrow{h} BSO(6) \times K(\mathbb{Z}, 2)$$

is a rational homotopy equivalence *over* $BSO(6)$. Therefore

$$H * (B^d;\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, e, t] \tag{2}$$

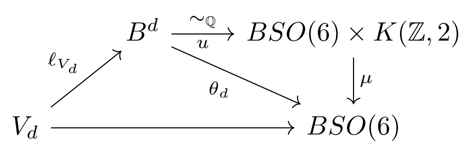
Combining 1 and 2, we get that

$$H^*(\mathcal{M}^{\theta_d}(V_d, \ell_{V_d}), \mathbb{Q}) \cong \mathbb{Q}[\kappa_{t^i c} | c \text{ is a monomial in } p_1, p_2, e, \text{ with } |c| + 2i > 6]$$

in degrees $* \leq \frac{d^4+5d^3+10d^2-10d+4}{4}$.

2.3 Change of tangential structure

We now want to use what we know about $\mathcal{M}^{\theta_d}(V_d, \ell_{V_d})$ to understand $\mathcal{M}^{or}(V_d)$, by using Theorem B. However, $hAut(-)$ is generally *hard* to understand, so to compare these two spaces, we will make use of the fact that B^d is rationally homotopy equivalent to $BSO(6) \times K(\mathbb{Z}, 2)$ over $BSO(6)$:



It is much easier to understand the homotopy automorphisms of $BSO(6) \times K(\mathbb{Z}, 2)$ over μ, so we will try to understand $\mathcal{M}^{or}(V_d)$ by comparing it to $\mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d})$. We will do this in two steps:

(a) *Comparing* $\mathcal{M}^{\theta_d}(V_d, \ell_{V_d})$ *with* $\mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d})$. We have a fiber sequence

$$\mathcal{M}^{\theta_d}(V_d, \ell_{V_d}) \to \mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d}) \to BH \tag{3}$$

where $H \subseteq h\,\mathrm{Aut}(h)$ is the subgroup that stabilizes that path component. Since h is a rational homotopy equivalence, $\pi_i(h\,\mathrm{Aut}(h)) \otimes \mathbb{Q} = 0, i > 0$ and $\pi_0(h\,\mathrm{Aut}(h)) = 0$ (by obstruction theory). Hence the first map in 3 is a rational homotopy equivalence and

$$H^*(\mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d}); \mathbb{Q}) \cong H^*(\mathcal{M}^{\theta_d}(V_d, \ell_{V_d}); \mathbb{Q})$$

(b) *Comparing* $\mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d})$ *with* $\mathcal{M}^{or}(V_d)$. Now we want to understand $hAut(\mu)$. We have maps

$$\begin{array}{ccc} & & BSO(6) \times K(\mathbb{Z}, 2) \\ & \nearrow & \downarrow \mu \\ V_d & \longrightarrow & BSO(6) \end{array}$$

and we want a self map of $BSO(6) \times K(\mathbb{Z}, 2)$ which respects the projection onto the first factor. We have no freedom in the first coordinate, only freedom in the second coordinate. Hence

$$h\,\mathrm{Aut}(\mu) \cong \mathrm{Map}(BSO(6), h\,\mathrm{Aut}(K(\mathbb{Z}, 2)))$$

We can understand $hAut(K(\mathbb{Z}, 2)) \subset \mathrm{Map}(K(\mathbb{Z}, 2), K(\mathbb{Z}, 2))$ looking at the following fibre sequence

$$\begin{array}{ccc} M_* := \mathrm{Map}_*(K(\mathbb{Z}, 2), K(\mathbb{Z}, 2)) & \longrightarrow & \mathrm{Map}(K(\mathbb{Z}, 2), K(\mathbb{Z}, 2)) \\ & & \downarrow \\ & & K(\mathbb{Z}, 2) \end{array}$$

Then

$$\begin{aligned} &\pi_0(M_*) \simeq H^2(K(\mathbb{Z}, 2)) = \mathbb{Z} \\ &\pi_1(M_*) \simeq [S^1 \wedge K(\mathbb{Z}, 2), K(\mathbb{Z}, 2)] \cong [K(\mathbb{Z}, 2), K(\mathbb{Z}, 1)] = H^1(K(\mathbb{Z}, 2)) = 0 \\ &\pi_i(M_*) = 0 \qquad i > 1 \end{aligned}$$

This gives us $h\,\mathrm{Aut}(K(\mathbb{Z}, 2)) \simeq \mathbb{Z}^\times \ltimes K(\mathbb{Z}, 2)$ and therefore

$$h\,\mathrm{Aut}(\mu) \simeq \mathbb{Z}^\times \rtimes \mathrm{Map}(BSO(6), K(\mathbb{Z}, 2)) \simeq \mathbb{Z}^\times \rtimes K(\mathbb{Z}, 2)$$

If we just want to consider elements which stabilize our path component, do not include -1 (which reverses orientation). We have a fiber sequence

$$\mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d}) \to \mathcal{M}^{or}(V_d) \to \Sigma K(\mathbb{Z}, 2) = K(\mathbb{Z}, 3) \tag{4}$$

where we know the rational cohomology of the base and the fiber: by item (a) and section 2.2,

$$H^*(\mathcal{M}^{\mu}(V_d, h \circ \ell_{V_d}); \mathbb{Q}) = \mathbb{Q}[\kappa_{t^i c} | c \text{ is a monomial in } p_1, p_2, e, \text{ with } |c| + 2i > 6]$$

and

$$H^*(K(\mathbb{Z},3);\mathbb{Q}) = \bigwedge[i_3]$$

This implies that the Serre spectral sequence associated to the fibration 4 has at most two non-zero columns: when $p = 0$ and $p = 3$. In particular, this implies that the only possible nontrivial differential is d_3.
Result: $d_3(\kappa_{t^n e}) = i_3 \otimes n \cdot \kappa_{t^{n-1}e} \implies d_3$ is surjective!
Therefore we have

$$H^*(\mathcal{M}^{or}(V_d);\mathbb{Q}) = \ker\left(d_3 \circlearrowleft \mathbb{Q}[\kappa_{t^i c} | c \text{ is a monomial in } p_1, p_2, e, \text{ with } |c| + 2i > 6]\right)$$

Remark. At first glance, it may seem from our computations that the cohomology of $\mathcal{M}^{or}(V_d)$ does not depend on d, however this is not the case. Actually, we can see that d_3 is immediately related to d, with $d_3(\kappa_{te}) = \chi(V_d)$, which is a function of d.

Interlude by Oscar

What are these κ classes? How can they be defined intrinsically?

Look at $\mathcal{M}^\theta(W)$ classifies fiber bundles $W \to E \xrightarrow{\pi} X$ and a θ-structure on the vertical tangent bundle $T_\pi E = \ker D\pi$

$$\begin{array}{ccccc} T_\pi E & \longrightarrow & \theta^*\gamma & & \\ \downarrow & & \downarrow & & \\ E & \xrightarrow{\ \ell\ } & B & \xrightarrow{\ \theta\ } & BO(2n) \\ \downarrow & & & & \\ X & & & & \end{array}$$

Let $c \in H^*(B;\mathbb{Q})$. Then $c(T_\pi E) = \ell^* c \in H^{|c|}(E;\mathbb{Q})$ and $\kappa_c(\pi) = \int_\pi c(T_\pi E) = \pi_! c(T_\pi E) \in H^{|c|-2n}(X;\mathbb{Q})$ where $\pi_!$ is referred to as (fiber integration, Gysin map, pushforward) give the *generalized Miller-Morita-Mumford classes*. This is the 'down-to-earth' way of describing them.