Talbot 2019 Talk 1

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-1.1 The Cobordism Category as a Topological Category

In this talk, we want to define the cobordism category as a topological category in the following sense:

Definition. A *topological category* $\mathcal{C}$ consists of the data of

- spaces of objects respectively morphisms $\mathcal{C}_0, \mathcal{C}_1 \in \mathbf{Top}$
- source and target maps $d_0, d_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$
- a degeneracy (identity) map $s^0 \colon \mathcal{C}_0 \to \mathcal{C}_0$
- a multiplication map $\mu : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$, where $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ is the pullback

$$\begin{array}{ccc} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \longrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow{\scriptstyle d^1} \\ \mathcal{C}_1 & \xrightarrow{d^0} & \mathcal{C}_0 \end{array}$$

satisfying the conditions:

- Associativity, i.e.

$$\begin{array}{ccc} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{id \times \mu} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \\ \downarrow{\scriptstyle \mu \times id} & = & \downarrow{\scriptstyle \mu} \\ \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{\mu} & \mathcal{C}_1 \end{array}$$

- Unitality, i.e. the composite of $\mathcal{C}_1 \xrightarrow{(d^1, id)} \mathcal{C}_0 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{(s^0, id)} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{\mu} \mathcal{C}_1$ is the identity.
- Compatibility of d_0, d_1 and s^0: $id = d^0 s^0 = d^1 s^0$.

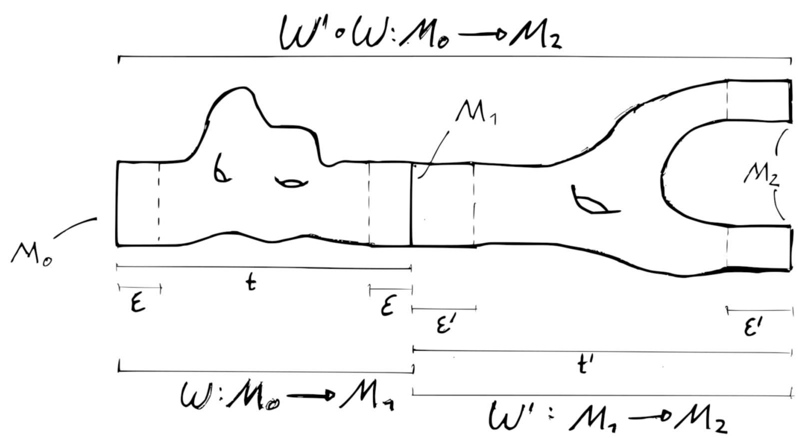
Example. Any (small) topologically enriched category, e.g. $\mathbf{Man}_X$ (manifolds with a map to X), becomes a topological category by equipping the set of objects with the discrete topology and declaring $\mathcal{C}_1$ to be the disjoint union of all mapping spaces.

Definition (see also [GMTW09], 2.1)**.**
$\mathbf{Cob}_d^n$ is the topological category of d-dimensional cobordisms in $\mathbb{R}^n$, defined by

- $(\mathbf{Cob}_d^n)_0 = \{X \subseteq \mathbb{R}^{n-1} \mid X \text{ closed compact smooth manifold of dimension d}\}$

- $(Cob_d^n)_1 = \{(W,t) \mid W \subseteq \mathbb{R}^{n-1} \times [0,t]$ compact submanifold $, \partial W = (\mathbb{R}^{n-1} \times \{0\}) \cap W \cup (\mathbb{R}^{n-1} \times \{t\}) \cap W = W_0 \cup W_1\}$

such that there exist $\varepsilon_0, \varepsilon_1 \in \mathbb{R}_{>0}$ such that $(\mathbb{R}^{n-1} \times [0, \varepsilon_0)) \cap W = W_0 \times [0, \varepsilon_0)$ and $\mathbb{R}^{n-1} \times (t - \varepsilon_1, t] \cap W = W_0 \times (t - \varepsilon_1, t]$ with maps $d^i : W \mapsto W_i$ and $s^0 : X \mapsto X \times [0,1]$. (This is usually called a collar condition.)
The composition of morphisms is given by gluing along the shared boundary, as depicted in the following picture, which depicts the composition of two cobordisms $W\colon M_0 to M_1$ and $W'\colon M_1 \to M_2$.



Remark. Strictly speaking, $\mathbf{Cob}_d^n$ is not a topological category in the previously defined sense, as it does not have (strict) identity morphisms. This issue will however not be of any consequence in our applications.

Important construction.

$$\begin{aligned} \Gamma : C^\infty([0,1], (\mathbf{Cob}_d^n)_0) &\to (\mathbf{Cob}_d^n)_1 \\ \gamma &\mapsto \bigcup_t \gamma(t) \times \{t\} \subset \mathbb{R}^{n-1} \times [0,1] \end{aligned}$$

Γ takes a smooth path γ from a manifold X_0 to X_1 and constructs a cobordism between X_0 and X_1.

-1.2 The Classifying Space Construction

We now construct the *classifying space* $B\mathcal{C}$ of a category $\mathcal{C} \in \mathbf{Cat}(\mathbf{Top})$, where $\mathbf{Cat}(\mathbf{Top})$ denotes the category of topological categories, by applying two functors which we now define.

Definition.

The (semi-simplicial, topological) *nerve* is the functor

$$\begin{aligned} N\colon \mathbf{Cat}(\mathbf{Top}) &\to \mathrm{Fun}(\Delta_{inj}^{op}, \mathbf{Top}) \\ \mathcal{C} &\mapsto \begin{pmatrix} 0 \mapsto \mathcal{C}_0 \\ 1 \mapsto \mathcal{C}_1 \\ 1 < n \mapsto \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1 \end{pmatrix} \end{aligned}$$

The *(fat) geometric realization* functor is

$$\begin{aligned} \|\cdot\| : \mathrm{Fun}(\Delta_{inj}^{op}, \mathbf{Top}) &\to \mathbf{Top} \\ X &\mapsto \coprod_{n\geq 0} X_n \times \Delta^n / \sim \end{aligned}$$

where $(x_n, r) \in X_n \times \Delta^{n-1} \sim (x_{n-1}, s) \in X_{n-1} \times \Delta_n \iff d^i x_n = x_{n-1}$ and $d_i s = r$.
The *classifying space* $B\mathcal{C}$ of a category $\mathcal{C} \in \mathbf{Cat}(\mathbf{Top})$ is the space $B\mathcal{C} = \|N\mathcal{C}\|$.

Example. Given a topological monoid M, we can regard it as an object of $\mathbf{Cat}(\mathbf{Top})$. In talk 4, we will encounter the group completion map $M \to \Omega BM$. Specialising to the topological monoid of endomorphisms of an object $X \in \mathbf{Cob}_d^n$ and composing with the inclusion $\Omega B \operatorname{Map}_{\mathbf{Cob}_d^n}(X,X) \hookrightarrow \Omega B\mathbf{Cob}_d^n$, we obtain $\operatorname{Map}_{\mathbf{Cob}_d^n}(X,X) \to \Omega B\mathbf{Cob}_d^n$. We can further precompose with Γ, which we can apply to loops, to obtain the composite

$$\Omega_X((\mathbf{Cob}_d^n)_0) \to \operatorname{Map}_{\mathbf{Cob}_d^n}(X,X) \to \Omega B\mathbf{Cob}_d^n$$

which will become important later.

-1.3 Madsen-Tillmann Spectra

We have a normal bundle over the Grassmannian

$$\begin{array}{ccc} \nu_d^n \oplus \varepsilon & \longrightarrow & \nu_d^{n+1} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Gr}_d(\mathbb{R}^n) & \hookrightarrow & \mathrm{Gr}_d(\mathbb{R}^{n+1}) \end{array}$$

where $\nu_d^n = \{(V,v) \mid V \in \mathrm{Gr}_d(\mathbb{R}^n), v \in V^\perp\}$. By applying Thom spaces and the adjunction $(\Sigma \dashv \Omega)$, this gives a map $\mathrm{Th}(\nu_d^n) \to \Omega\,\mathrm{Th}(\nu_d^{n+1})$. This defines a (pre-)spectrum called the *Madsen-Tillmann spectrum* MT_d with n-th space $\mathrm{Th}(\nu_d^n)$.
There is a diagram

$$\begin{array}{ccccc} B\mathbf{Cob}_d^n & \hookrightarrow & B\mathbf{Cob}_d^{n+1} & \xrightarrow{\text{colim}} & B\mathbf{Cob}_d^\infty \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ \Omega^{n-1}\,\mathrm{Th}(\nu_d^n) & \longrightarrow & \Omega^n\,\mathrm{Th}(\nu_d^{n+1}) & \xrightarrow{\text{colim}} & \Omega^{\infty-1} MT_d \end{array}$$

The vertical maps are homotopy equivalences by the theorem of Galatius-Madsen-Tillmann-Weiss (main theorem of [GMTW09]).

-1.4 Tangential Structures

Definition.
A *tangential structure* on BO_d is a map θ which is a Serre fibration

$$\begin{array}{ccc} \theta^*\gamma & \longrightarrow & \gamma \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\theta} \!\!\!\!\rightarrow & BO_d \end{array}$$

A θ*-structure* on M^d is a bundle map

$$\begin{array}{ccc} TM & \longrightarrow & \theta^*\gamma \\ \downarrow & \lrcorner & \downarrow \\ M & \longrightarrow & X \end{array}$$

such that $M \to X \twoheadrightarrow BO_d$ classifies the tangent bundle of M.

Examples.

θ:	A θ-structure is a(n):
$BSO_d \to BO_d$	orientation of TM.
$BU_{\frac{d}{2}} \to BO_d$	almost complex structure on TM.
$BSpin_d \to BO_d$	spin structure on TM.
$\{*\} \to BO_d$	framing of TM.

Given a θ, we can define a version of $\mathbf{Cob}_d^\infty$ that takes compatible θ-structures into account. In this enhanced setting, there also is a GMTW-theorem ([GMTW09], Main Theorem 2) which asserts the existence of the upper right homotopy equivalence in the following diagram.

$$\begin{array}{ccccc} \mathrm{Th}(\theta^*\nu_d^n \oplus \varepsilon) & \longrightarrow & \mathrm{Th}(\theta^*\nu_d^{n+1}) & \xrightarrow{\text{colim}} & \Omega^{\infty-1}MT\theta \simeq B\mathbf{Cob}_\theta^\infty \\ \downarrow & & \downarrow & & \\ X_n & \longrightarrow & X_{n+1} & \xrightarrow{\text{colim}} & X \\ \downarrow{\scriptstyle\theta_n} & & \downarrow{\scriptstyle\theta_n} & & \downarrow \\ \mathrm{Gr}_d(\mathbb{R}^n) & \longrightarrow & \mathrm{Gr}_d(\mathbb{R}^{n+1}) & \xrightarrow{\text{colim}} & BO_d \\ \uparrow & & & & \uparrow \\ \nu_d & & & & \gamma \end{array}$$

References

[GMTW09] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss. The homotopy type of the cobordism category. *Acta Math.*, 202(2):195–239, 2009.