Talbot 2019 Talk 0

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4/8/2019

0 Introduction

0.1 Classification problems

Inspiration for classification problems:

Let $\mathcal{G}$ be a groupoid. The classification problem is to determine $\pi_0\mathcal{G} = \mathrm{Obj}\,\mathcal{G}/\mathrm{iso}$.

Principle. Many interesting objects have a *parametrized* version, i.e. it makes sense to talk about families of them (over a parameter/base space X). Often, the parametrized version is at least as interesting as the non-parametrized version.

Examples.

1. $\mathcal{G}$ = groupoid of finite sets and bijections. There is an isomorphism $\pi_0\mathcal{G} \xrightarrow{\sim} \mathbb{N}$ given by cardinality.
   The parametrized version corresponds to finite covering spaces $\pi : E \to X$. Two such coverings are isomorphic if there exists a homeomorphism over the base space.
2. $\mathcal{G}$ = finite dimensional vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) and linear isomorphisms. There is an isomorphism $\pi_0\mathcal{G} \simeq \mathbb{N}$ given by taking the dimension. The parametrized version is vector bundles $\pi : V \to X$, and two such are isomorphic if there exists a continuous fiberwise linear isomorphism over X.
3. $\mathcal{G}$ = smooth manifolds and diffeomorphisms. The parametrized version is $\pi : E \to X$ a smooth fiber bundle, and two such are isomorphic if they are related by a diffeomorphism over the base space.
   Typically, one places conditions on the objects of $\mathcal{G}$ or asks for extra data:
   E.g. taking $\mathcal{G}$ with objects being manifolds that are compact, connected, 2-dimensional (conditions); and oriented (extra data), and looking only at diffeomorphisms which preserve orientation, we have $\pi_0\mathcal{G} \simeq \mathbb{N}$ given by genus[1].
   E.g. taking $\mathcal{G}$ with objects being manifolds which are compact (condition), oriented, and homotopy equivalent to S^7 (extra data), Kervaire and Milnor showed that $\pi_0\mathcal{G} \simeq \mathbb{Z}/28\mathbb{Z}$ has precisely 28 elements, and moreover that there is a group structure.

Today we'll talk about the parametrized version of 3., i.e. the classification of smooth fiber bundles.

[1]'maybe you've never seen the proof but you've heard it so many times you're sure it's true'

Definition. $\pi : E \to X$ is a *smooth fiber bundle* if one of the following (equivalent) conditions holds

i) $\forall x \in X$ $\exists$ neighborhood $x \in U \subseteq X$ and a F a smooth manifold such that there exists a diffeomorphism $\pi^{-1}(U) \xrightarrow{(\pi|_{\pi^{-1}(U)}, q)} U \times F$
ii) $\pi : E \to X$ is a smooth ($\mathbb{C}^\infty$) proper (i.e. the preimage $\pi^{-1}(K)$ of a compact set K is compact) submersion ($D_e\pi : T_eE \twoheadrightarrow T_{\pi(e)}X$ for all $e \in E$)

Two fiber bundles are *equivalent* if there is an equivalence (homeomorphism, diffeo, etc.) over X, i.e.

$$\begin{array}{ccc} E & \xrightarrow{\sim} & E' \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

Here we usually require that the fiber be compact and X a manifold (but X not necessarily compact), then (i) $\iff$ (ii) using Ehresmann's lemma.

0.2 Classifying spaces

A classifying/universal space B for $\mathcal{G}$ is a topological space giving a natural isomorphism of functors

$$\{\text{families over } X\}/\text{iso of families} \xrightarrow{\sim} [X, B] = \{\text{continuous maps } X \to B\}/\text{homotopy}$$

where $\leftarrow$ is given by pullback of the universal bundle.

Question. How do you prove such a B exists, and construct it?

0.2.1 Construction

Two approaches

Abstract $B\mathcal{G} = |N_\bullet\mathcal{G}|$. Given a groupoid, take its nerve $N_\bullet$ (either a simplicial set or a simplicial space) where $N_p\mathcal{C} = p$-tuples of composable morphisms and take its geometric realization.
Good for proving existence, but not helpful for understanding

Tailor-made (Better for understanding, less 'functorial') The prototypical example of this is the Grassmannian for vector bundles (see Milnor-Stasheff). This is an illustrative example.
$\mathrm{Gr}_d(\mathbb{R}^{n+d}) = \{V \subset \mathbb{R}^{n+d} \mid \text{ linear subspace, } \dim V = d\}$

$$\mathrm{Gr}_d(\mathbb{R}^\infty) = \operatorname*{colim}_{n\to\infty} \mathrm{Gr}_d(\mathbb{R}^{n+d})$$

Under some mild condition (e.g. X paracompact), we have

$$\left[X, \coprod_{d\geq 0} \mathrm{Gr}_d(\mathbb{R}^\infty)\right] \simeq \{\text{ vector bundles over } X\}/\text{iso}$$

Using the abstract approach, we have $B\mathcal{G} = |N_\bullet\mathcal{G}|$ a topological space where $N_0\mathcal{G}$ = objects and $N_1\mathcal{G}$= morphisms. Therefore

$$\pi_0 B\mathcal{G} = \mathrm{coeq}(N_1\mathcal{G} \rightrightarrows N_0\mathcal{G}) = \pi_0\mathcal{G}$$

So the study of the classification problem becomes the study of the homotopy type of $B\mathcal{G}$. This can be done, in particular by understanding its cohomology $H^*(B\mathcal{G})$ = characteristic classes. e.g. $H^*(\mathrm{Gr}_d;\mathbb{Q})$ is a polynomial ring on the Pontryagin classes.

0.3 Manifold bundles

Manifold bundles also admit classifying spaces. Let W be a fixed smooth compact manifold, then we can construct the classifying space of smooth fibre bundles with fibre W: following the abstract approach, we consider $\mathrm{Diff}(W)$, the diffeomorphism group, give it the C^∞-topology, i.e. a sequence of diffeomorphisms converges if and only if all derivatives converge. Then $B\,\mathrm{Diff}(W) = |N_\bullet \mathrm{Diff}(W)|$ = classifying space. We can also do:

0.3.1 A Grassmannian-type construction

Let

$$B := \operatorname*{colim}_{n\to\infty}\{Q \subset \mathbb{R}^{n+d} \mid \text{ smooth compact submanifold of dimension } d\}$$

given a suitable topology. If X a smooth manifold,

$$[X, B] \simeq \{ \text{ smooth fiber bundles } \pi : E \to X \mid \pi^{-1}(x) \text{ compact, } d\text{- dimensional } \} \tag{1}$$

We have $B = \coprod_W B\,\mathrm{Diff}(W)$ where W ranges over all diffeomorphism classes.

A corresponding model for $B\,\mathrm{Diff}(W)$ is

$$\operatorname*{colim}_{n}\{Q \subset \mathbb{R}^{n+d} \mid Q \text{ diffeo to } W\} = \mathrm{Emb}(W, \mathbb{R}^{n+d})/\,\mathrm{Diff}(W)$$

. *Sketch of proof of (1).* Given $\pi : E \to X$ a smooth fiber bundle. Choose an embedding

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & X \times \mathbb{R}^{n+d} \\ \downarrow \pi & & \downarrow \\ X & = & X \end{array}$$

Then we get a map

$$\begin{aligned} X &\to B \\ x &\mapsto \pi^{-1}(x) \subset \{x\} \times \mathbb{R}^{n+d} = \mathbb{R}^{n+d} \end{aligned}$$

0.4 Understanding $B\,\mathrm{Diff}(W)$

The goal is now to study $H^*(B\,\mathrm{Diff}(W))$. Think of this as a higher homotopy version of classification of manifolds, but also analogue of the Grassmannian story.

What we know so far:

Pattern. The $B\,\mathrm{Diff}(W)$ have homological stability which is, vaguely speaking, the phenomenon of different W having somewhat similar $H^*(B\,\mathrm{Diff}(W))$.

History. (symmetric groups) we have a map $S_{n-1} \to S_n$ which induces a map of classifying spaces $BS_{n-1} \to BS_n$ where BS_n classifies n-fold covering maps.

Nakaoka (1961) showed that $H^k(BS_{n-1}) \leftarrow H^k(BS_n)$ is an isomorphism for $k \leq \frac{n}{2}$

Harer (1985) showed that (Σ_g = surface of genus g). Fix an embedding $D^2 \subset \Sigma_g$. Then we get a map $B\,\mathrm{Diff}(\Sigma_g; D^2) \to B\,\mathrm{Diff}(\Sigma_{g+1}, D^2)$ induced by a map of Diff-groups given by connect sum with a torus. This maps induces an iso on H^* when $* << g$. When we have a pattern like this, talk about the *stable range*.

Description of H^* in the stable range as the cohomology of a space of maps. In the symmetric group case, we have the Barratt-Priddy-Quillen-Segal theorem which says we have a map $BS_n \to \mathrm{colim}_{N\to\infty}\{S^N \to S^N \mid \text{ pointed, of degree } n\}$ which is a H_* isomorphism in the stable range. This is surprising because LHS is a $K(\pi, 1)$ and RHS is far from being a $K(\pi, 1)$–it's an infinite loop space.

Credit goes to Ulrike Tillmann, Madsen-Weiss.

Question. Is the last map n-connected for some n? (if a map of *simply-connected* spaces is a H_* iso in a range of degrees, then it's a π_*-iso in same range of degrees).

No–this example shows why the assumption of simple-connectivity in this version of the Hurewicz theorem is essential. This map 'throws away the homotopy and keeps the homology.'