TALBOT 2018: THE MODEL INDEPENDENT THEORY OF ∞-CATEGORIES

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Goals

The aim of the 2018 Talbot workshop is to develop the theory of ∞-categories from first principles in a "model-independent" fashion, that is, using a common axiomatic framework that is satisfied by a variety of models. By the end of the week, we will also demonstrate that even "analytic" theorems about ∞-categories — which, in contrast to the "synthetic" proofs that may be interpreted simultaneously in many models, are proven using the combinatorics of a particular model — transfer across specified "change of model" functors to establish the same results for other equivalent models.

In more detail, the "synthetic" theory is developed in any ∞-*cosmos*, which axiomatizes the universe in which ∞-categories live as objects. Here the term "∞-category" is used very broadly to mean any structure to which category theory generalizes in a homotopy coherent manner. Several models of $(\infty, 1)$-categories are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories as well as sliced versions of all of the above. This usage is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional category, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.

Much of the development of the theory of ∞-categories takes place not in the full ∞-cosmos but in a quotient that we call the *homotopy 2-category*. The homotopy 2-category is a strict 2-category — like the 2-category of categories, functors, and natural transformations — and in this way the foundational proofs in the theory of ∞-categories closely resemble the classical foundations of ordinary category theory except that the universal properties that characterize, e.g. when a functor between ∞-categories defines a cartesian fibration, are slightly weaker than in the familiar case.

Over the course of the workshop, we will define and develop the notions of equivalence and adjunction between ∞-categories, limits and colimits in ∞-categories, homotopy coherent adjunctions and monads borne by ∞-categories as a mechanism for universal algebra, cartesian and cocartesian fibrations and their groupoidal variants, the calculus of modules between ∞-categories, Kan extensions, representable functors, the Yoneda lemma, and the Yoneda embedding.

Prerequisites

In advance of the workshop, we hope that each participant will take some time to familiarize themselves with the following topics.

- the combinatorics of simplicial sets;
- the calculus of pasting diagrams to define composite 2-cells in a 2-category, such as the 2-category of categories, functors, and natural transformations;

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- the construction and 2-categorical universal property of the comma category $f \downarrow g$ associated to a pair of functors $f : B \to A$ and $g : C \to A$; and
- the construction of the "pushout-product" or "pullback-hom" maps and the lifting properties satisfied in a cartesian monoidal model category by certain instances of this construction by virtue of the "SM7 axiom."[1]

A prerequisite problem set intending to guide one's reading related to these topics is available here:

www.math.jhu.edu/~eriehl/616/616preq.pdf

Talks

1. Preview of coming attractions. The mentors will give an overview of the week to come.

2. ∞-Cosmoi and their homotopy 2-categories. An ∞-*cosmos* axiomatizes the universe in which ∞-*categories* live as objects. We use the term "∞-category" very broadly to mean any structure to which category theory generalises in a homotopy coherent manner. Several models of $(\infty, 1)$-categories are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories as well as sliced versions of all of the above. This usage is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional category, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.

This talk starts by reviewing the basic homotopy theory of quasi-categories and then introduces ∞-cosmoi, deduces some elementary consequences of their axioms, and constructs a *homotopy 2-category* associated with each one. We relate some common homotopical structures in the ∞-cosmos, such as homotopy equivalences and isofibrations, to their 2-categorical counterparts in its homotopy 2-category.

References: [RV4, §2.1], [RVx, §1], [RV, chapter 1].

3. A menagerie of ∞-cosmological beasts. An ∞-cosmos is not intended to axiomatise all of the ∞-category notions to be found in the literature; this talk will, however, establish that it does encompass very many of them. In particular we shall see that quasi-categories, complete Segal spaces, Segal categories, and naturally marked quasi-categories all define ∞-cosmoi. The objects in these models all deserve to be regarded as $(\infty, 1)$-categories.

This talk starts by reviewing the basic homotopy theory of quasi-categories. We shall also see that complete Segal objects in any well behaved Quillen model category provide a further example, and by iterating that observation we extend our observations to various models of (∞, n)-categories. Other higher examples discussed here include Θ_n-spaces and (weak) complicial sets.

In search of various *fibred* ∞-categorical notions, we introduce a slice construction for ∞-cosmoi and prove that every such slice is again an ∞-cosmos. Finally we relate the animals in our ∞-cosmos zoo by introducing a theory of ∞-cosmological *functors* and *biequivalences*.

References: [RV4, §2], [JT], [Re1], [V2], [B].

[1] We refer to the "pushout-product" associated to a "tensor" bifunctor valued in a category with pushouts as the *Leibniz tensor*. The general categorical properties of this construction are reviewed in [RV0, §4-5].

4. Adjunctions, limits, and colimits in homotopy 2-categories. We have already seen that some 2-categorical notions may be imported into the world of ∞-cosmoi directly from their associated homotopy 2-categories. In this talk we continue on that journey, applying this insight to develop a theory of *adjunctions* between ∞-categories in an ∞-cosmos.

We start by reviewing the theory of (equationally defined) adjunctions and *mates* in 2-categories and discussing their generalisation to a theory of *absolute lifting diagrams*. We apply these notions in the homotopy 2-category of an ∞-cosmos, and derive some elementary consequences. These observations lead us to a discussion of the *internal* limits and colimits that can live within an ∞-category, and we give elementary proofs of familiar results such as the preservation of limits by right adjoints.

References: [RV1, §4-5], [KS], [RV, chapter 2].

5. Arrow and comma ∞-categories. In classical category theory, the equational account of adjunctions provides only one promontory from which to survey the world of universal constructions. For many purposes, notions of *representability* play an equally important role, and in abstract category theory these are often expressed in the language of *modules* (sometimes called *profunctors* or *correspondences*) between categories.

One well worn route to a theory of modules in traditional (internal) category theory is to study the abstract properties of *comma categories*. In 2-category theory these generalise to 2-dimensional limit structures called *comma objects*, and we review their theory with a view to re-interpreting such notions within the theory of ∞-cosmoi. We show that any ∞-cosmos admits the construction of the comma ∞-category associated to any cospan that possesses a homotopically well-behaved and simplicially enriched variant of the 2-universal property enjoyed by comma objects.

Our hope is to simplify some computations involving comma objects by executing them within the homotopy 2-category associated with our ∞-cosmos, and this leads us to investigating their universal properties in there. In doing so we discover that they only satisfy a certain *weak 2-universal* property, which we establish and apply.

References: [RV1, §3], [RV4, §3], [RV, chapter 3].

6. The universal properties of adjunctions, limits, and colimits. In this talk, we present a variety of results that describe the universal properties of adjunctions, limits, and colimits. A general theme is that such universal properties can be described by fibered equivalences between comma ∞-categories. For example, a functor $u : A \to B$ in an ∞-cosmos $\mathscr{K}$ has a left adjoint if and only if the associated comma $B \downarrow u$ is represented by some $f : B \to A$, in the sense that $B \downarrow u \simeq f \downarrow A$ in the sliced ∞-cosmos $\mathscr{K}_{/A\times B}$. This fibered equivalence pulls back to define an equivalences between the internal mapping spaces of A and B.

Comma ∞-categories can also be used to define the ∞-category of cones above or below a fixed or varying diagram. A limit of a diagram d is then an element that represents this comma ∞-category of cones over d and the limit defines a terminal element in this ∞-category of cones. For diagrams indexed by a simplicial set J, the limit cone can also be understood as the right Kan extension to a diagram indexed by the simplicial set $J^{\triangleleft}$ which has a cone point adjoined above the diagram. Specializing this result and its dual to the case of pullbacks and pushouts allows us to define the loops $\vdash$ suspension adjunction in any pointed ∞-category.

References: [RV1, §4-5], [RV, §3.4-5, chapter 4].

7. Homotopy coherent adjunctions and monads. In this talk, we discover how the 2-categorically defined adjunctions discussed in the last two talks extend, in a homotopically unique way, to give *homotopy coherent adjunctions*. These structures encapsulate all of the higher coherence data that one would hope for from a fully homotopical adjunction and they are realised as simplicial functors $\underline{\mathrm{Adj}} \to \mathscr{K}$ mapping a certain combinatorially defined simplicial category $\underline{\mathrm{Adj}}$ into our ∞-cosmos $\mathscr{K}$. It comes as somewhat of a surprise to discover that this simplicial category $\underline{\mathrm{Adj}}$ is actually no-more-nor-less than the 2-category long dubbed the *generic adjunction* by 2-category theorists. The proofs of these results will take us deep into the homotopy theoretic weeds of simplicial computads, local horn extension arguments, and the rather quaint theory of *squiggles* used to describe the simplices in the hom-spaces of $\underline{\mathrm{Adj}}$.

Any self-respecting adjunction of ∞-categories should give rise to a monad, those in turn should admit the construction of monadic adjunctions. While the 2-categorical notion of an adjunction encodes a defining universal property that allows us to extend them to fully homotopy coherent structures, it would be naïve of us to hope that monads of ∞-categories might be captured simply as 2-categorical monads in homotopy 2-categories. Since monads are purely equational beasts that possess no corresponding universal property from which to derive higher coherence data, all of this must be given explicitly.

We round out this discussion by defining *homotopy coherent monads*, or simply just ∞-*monads*, to be simplicial functors $\underline{\mathrm{Mnd}} \to \mathscr{K}$, where $\underline{\mathrm{Mnd}}$ is the simplicial full subcategory of $\underline{\mathrm{Adj}}$ spanning one of its objects. Now we see that our adjunctions extend to homotopy coherent structures, parameterised by $\underline{\mathrm{Adj}}$, which themselves restrict to give homotopy coherent monads, parameterised by $\underline{\mathrm{Mnd}}$, just as we had hoped. Concretely the simplicial category $\underline{\mathrm{Mnd}}$ has the nerve of the algebraist's ordinal category $\mathbf{\Delta}_+$ as its unique endo-hom-space and ordinal sum as its composition.

References: [RV2], [SS], [RV, chapter 8].

8. Homotopy coherent monadicity and descent. Continuing the narrative of the last talk, we show how to derive an ∞-cosmological *Eilenberg-Moore object*, constructing the ∞-*category of algebras* associated to each ∞-monad. We prove a variant of the *Beck monadicity theorem* and examine a few applications to homotopy coherent algebra, ∞-category theory, and higher descent theory.

This talk commences with a review of the rubric of *weighted limits and colimits* in enriched category theory. Proceeding by analogy with classical 2-categorical accounts, we define the Eilenberg-Moore object of an ∞-monad to be a certain *flexibly weighted* simplicial limit. We briefly examine some of the properties of these flexible limits, including the key fact that the flexible limit of a diagram of ∞-categories that admit (and whose connecting maps preserve) a class of (co)limits again admits such (co)limits.

We derive an adjunction between the Eilenberg-Moore object of an ∞-monad and its underlying object, showing that the ∞-monad associated with this adjunction is simply just the ∞-monad we started with. This is the *monadic adjunction* characterized by Beck's theorem.

Now we consider an arbitrary adjunction and construct a comparison between its domain and the Eilenberg-Moore object of its associated ∞-monad. After a brief discussion of (split) simplicial objects in ∞-categories and their realisations, we examine some consequences of properties of that comparison and prove an ∞-categorical re-imagining of Beck's monadicity and descent theorems. Our primary goal here shall be to illustrate the way in which the proofs of these results proceed as direct analogues of their traditional 1-categorical counterparts.

References: [RV2, §5-7], [RV3], [Su], [Z], [RV, chapter 9].

9. Cartesian fibrations and the Yoneda lemma. The theory of *cartesian* or *Grothendieck fibrations* play a fundamental role in traditional internal category theory. For example, they arise naturally in the theory of modules to be developed in the subsequent talk, which may be described as certain spans whose legs are cartesian fibrations and in the theory of *fibred* (or *indexed*) *categories* over a base.

Joyal and Lurie both introduce a cartesian fibration notion for quasi-categories, described in terms of certain outer horn lifting properties. We can generalise that notion *representably* to an arbitrary ∞-cosmos $\mathscr{K}$; that is to say we might adopt the definition that a fibration $p: E \to B$ in $\mathscr{K}$ is a cartesian fibration if and only if for all objects $X \in \mathscr{K}$ the functor $\mathrm{Fun}(X, p): \mathrm{Fun}(X, E) \to \mathrm{Fun}(X, B)$ of hom-spaces is itself a cartesian fibration of quasi-categories and if any $Y \to X \in \mathscr{K}$ defines a cartesian functor between these cartesian fibrations.

As is our wont, however, we choose to take a more 2-categorical approach to this subject, by defining cartesian fibrations in terms of adjunctions and comma objects. We establish an equivalence between three characterisations of these structures defined internally to the homotopy 2-category. We also discuss *groupoidal* cartesian fibrations (called *right fibrations* by Joyal and Lurie), whose properties and applications mirror those of *discrete fibrations* in 1-category theory.

Now given a *point* $b: 1 \to B$ of an ∞-category, our categorical experience draws us to regarding the associated projection $p_B: B \downarrow b \to B$ as being the *representable* cartesian fibration defined by b. To validate that intuition, we discuss how the (external) Yoneda lemma may be formulated in this context and we prove that result.[2] This proof takes place entirely within the homotopy 2-category of our cosmos, as indeed do most of the proofs referenced in this talk.

References: [RV4, §4-6], [J], [L1, §2], [RV, chapter 5].

10. Two-sided fibrations and modules. Our goal in this talk, and its successor, is to provide a *modular* (or *profunctorial*) foundation for the category theory of ∞-categories in an ∞-cosmos $\mathscr{K}$. Specifically, we follow Street by developing a theory of *two-sided cartesian fibrations* over a pair of ∞-categories $A, B \in \mathscr{K}$. We might think of these as families of ∞-categories, indexed jointly by A and B, which possess compatible actions of A on the left (a covariant action) and of B on the right (a contravariant action). In Street's presentation of this notion, two-sided fibrations are represented as spans $q: A \twoheadleftarrow C \twoheadrightarrow B : p$ in which

- p is a cartesian fibration whose cartesian arrows map to isomorphisms under q,
- q is a cocartesian fibration whose cocartesian arrows map to isomorphisms under p, and
- (co-)cartesian lifts along p and q satisfy a *Beck-Chevalley condition.*

We take a slightly different approach, also originally suggested by Street, which realises two-sided fibrations as certain cartesian (cocartesian) fibrations in an ∞-cosmos of cocartesian (cartesian) fibrations.

We start by observing that if $\mathscr{K}$ is an ∞-cosmos then the simplicial category $\underline{\mathrm{Cart}}(\mathscr{K})_{/B}$ of cartesian fibrations and *cartesian functors* over a fixed base ∞-category $B \in \mathscr{K}$ is again an ∞-cosmos. Indeed, we can go a little further and describe a ∞-cosmos $\underline{\mathrm{Cart}}(\mathscr{K})$ of cartesian fibrations over arbitrary base ∞-categories and cartesian functors; this is also an ∞-cosmos and which admits a functor $\mathrm{cod}: \underline{\mathrm{Cart}}(\mathscr{K}) \to \mathscr{K}$ of ∞-cosmoi whose fibres are the $\underline{\mathrm{Cart}}(\mathscr{K})_{/B}$.

Now we are ready to define our two-sided fibrations between ∞-categories $A, B \in \mathscr{K}$ to be cocartesian fibrations over the projection $A \times B \twoheadrightarrow B$ in the ∞-cosmos $\underline{\mathrm{Cart}}(\mathscr{K})_{/B}$ of cartesian

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[2] An analogous analytic development of cartesian fibrations and the Yoneda lemma in the complete Segal space model can be found in [Ra1, Ra2].

fibrations over B. We show that this definition unwinds to give a more symmetric one, in the spirit of Street, and that in the dual this is no-more-nor-less than a cartesian fibration over the projection $A \times B \twoheadrightarrow A$ in the ∞-cosmos $\underline{\mathrm{coCart}}(\mathscr{K})_{/A}$ of cocartesian fibrations over A. Consequently, by iterating the ∞-cosmos of (co)cartesian fibrations construction, we obtain an ∞-cosmos of two-sided fibrations

$$ {}_{A\backslash}\underline{\mathrm{TwoSided}}(\mathscr{K})_{/B} \coloneqq \underline{\mathrm{coCart}}(\underline{\mathrm{Cart}}(\mathscr{K})_{/B})_{/(\pi_B\colon A\times B\to B)} \cong \underline{\mathrm{Cart}}(\underline{\mathrm{coCart}}(\mathscr{K})_{/A})_{/(\pi_A\colon A\times B\to A)} $$

over fixed ∞-categories A and B in $\mathscr{K}$. Furthermore, we define the simplicial category ${}_{A\backslash}\underline{\mathrm{Mod}}(\mathscr{K})_{/B}$ of *modules* from A to B to be the full simplicial sub-category of *groupoidal* two-sided fibrations in ${}_{A\backslash}\underline{\mathrm{TwoSided}}(\mathscr{K})_{/B}$.

The utility of framing our definitions in this way lies in the fact that we may now apply any result that lifts structures or properties from an ∞-cosmos $\mathscr{K}$ to the associated ∞-cosmoi of (co)cartesian fibrations to provide a corresponding lifted entity for the ∞-cosmos ${}_{A\backslash}\underline{\mathrm{TwoSided}}(\mathscr{K})_{/B}$.

References: [RV7, §5], [RV9], [RV5, §3], [RV, chapter 10].

11. The calculus of modules. Having introduced a module notion, our challenge now is to organise these together into a structure that abstracts their role in formal category theory. To that end, we shall exploit a formal analogy between the categorical calculii of bimodules between commutative rings and that of modules $M : A \nrightarrow B$ (between categories), an intuition often given concrete realisation by assembling the categories, functors, and modules of an abstract category theory together into a structure dubbed a *pro-arrow equipment* (or simply just an *equipment*) by Wood.

Naïvely we might hope to assemble the ∞-categories, functors, and modules in an ∞-cosmos $\mathscr{K}$ into an equipment, but this is not an utopia universally available to us. Specifically, Wood's framework assumes that a pair of modules $M : A \nrightarrow B$ and $N : B \nrightarrow C$ may be composed to give a *tensor product* module $M \otimes N : A \nrightarrow C$ but our ∞-cosmos axiomatization is too sparse to construct tensor products in general. Consequently, we are forced to frame the labours of this talk within the more general theory of *virtual equipments* as presented by Cruttwell and Shulman.

Virtual equipments encapsulate a calculus whose primary protagonists are objects supporting two flavours of arrows, called *functors* and *modules*, and structures relating them called *cells* (which generalise natural transformations of modules). Cells may be depicted as rectangular tiles, whose vertical edges are functors and whose horizontal edges are (sequences of) modules, and they admit an associative vertical composition which acts to combine cells that abut along their horizontal edges. We shall display these composites as tiled regions called *pasting diagrams*, and very many of our arguments will be couched largely in diagrammatic terms.[3]

To motivate the claim that virtual equipments encapsulate a theory of modules suited to the expression of abstract category theory and to build some familiarity with this calculus, we re-derive some useful categorical results entirely within that formalism. We shall find that the functors $f : A \to B$ of any virtual equipment give rise to pairs of modules $B \downarrow f : A \nrightarrow B$ and $f \downarrow B : B \nrightarrow A$, that these are formally adjoint (in a suitable sense), that they admit certain (universally defined) tensor products with other modules, and that they satisfy various formulations of the Yoneda lemma. We also recover the module characterization of adjoint functors observed in talk 6.

We conclude this talk with a proof that the totality of ∞-categories, functors and modules in any ∞-cosmos $\mathscr{K}$ may indeed be collected together into a virtual equipment $\underline{\mathrm{Mod}}(\mathscr{K})$ whose cells are given as certain transformations of modules.

References: [CS], [RV5, §4], [W], [RV, chapter 11]

[3] See [M] for a graphical calculus describing such diagrams.

12. Pointwise Kan extensions. The adjoint formulation of (co)limits in ∞-categories, as presented in talk 4, is adequate for many purposes but is found wanting when we come to consider the theory of *Kan extensions*. Any 2-category supports a notion of Kan extension, couched in terms of a universal property of 2-cell bearing triangles, and this may be imported via the homotopy 2-category into the theory of ∞-categories in an ∞-cosmos. It is known, however, that even in the 2-category of categories this does not characterise the class of Kan extensions of primary interest, that is those that may be regarded as being constructed *pointwise* from the (co)limits that may exist in the target category. What is more, our existing theory of (co)limits does not easily provide us with an analogue of Kan's formula for constructing these extensions. This talk rectifies these deficiencies.

We start by defining what it means for an ∞-category to admit a family of (co)limits weighted by a module, and we derive some basic consequences. This then leads to a theory of *pointwise Kan extensions* which may be applied in any virtual equipment. At this level of generality, we also introduce an *exact square* notion and we examine the sense in which this provides a common language in which to discuss the theory of functors that are fully faithful, final, or initial.

It is also possible to characterise pointwise Kan extensions in terms expressible entirely within the homotopy 2-category associated with an ∞-cosmos. We discuss this alternative characterisation and demonstrate that it is equivalent to the equipment based notion. Finally we specialise these notions to ∞-cosmoi that are cartesian closed, we prove familiar properties of initial and final functors and a Beck-Chevalley result for pointwise Kan extensions.

Specialising all of this to the ∞-cosmos of quasi-categories, we prove the expected Kan extension existence result for functors that land in suitable (co)complete quasi-categories. Ultimately this leads us to a proof that any suitably complete and cocomplete quasi-category gives rise to a *derivator* in the sense of Heller [He] and Grothendieck [G].

References: [RV5, §5], [RV9, §6], [RV, chapter 12].

13. Proof of model independence. The biequivalences of ∞-cosmoi introduced in talk 3 might be referred to as "change-of-model" functors, converting complete Segal spaces to Segal categories or quasi-categories for instance. In this talk we prove that a biequivalence of ∞-cosmoi induces a biequivalence between the corresponding calculii of modules as expressed by the virtual equipment of ∞-categories, functors, modules, and module maps. The corollary is that categorical results proven with any of the biequivalent models of $(\infty, 1)$-categories apply to them all.

In more detail, a biequivalence of ∞-cosmoi $\mathscr{K} \xrightarrow{\sim} \mathscr{L}$ induces a biequivalence between their homotopy 2-categories — this being a 2-functor that is essentially surjective on objects up to equivalence and defines a local equivalence of functor spaces — that in addition preserves and indeed reflects and creates the comma ∞-category of any cospan. As a corollary, the biequivalence induces a bijection between equivalence classes of objects, a local bijection between isomorphism classes of parallel functors, a local bijection between natural transformations, a local bijection between equivalence classes of modules, and so on. It follows that a functor in $\mathscr{K}$ admits a right adjoint if and only if its image does so in $\mathscr{L}$ or a diagram valued in an ∞-category in $\mathscr{K}$ admits a limit if and only if any diagram in an ∞-category in $\mathscr{L}$ that is equivalent to its image admits a limit there. Since "formal category theory" enables us to rephrase categorical statements in terms of equivalences between modules, we conclude more generally that the formal category theory of ∞-categories in $\mathscr{K}$ is equivalent to the formal category theory of ∞-categories in $\mathscr{L}$.

After establishing model independence, we embark upon a guided tour through applications of the speaker's choosing, illustrating how a change-of-model functor can be used to transfer an "analytically-proven" result about one model of $(\infty, 1)$-categories to another model. Sample applications of this kind can be found in [RV10] but we encourage the speaker to search for their own.

References: [RVx], [RV10], [RV, part IV].

14. Comprehension and the Yoneda embedding. Given a cocartesian fibration $p : E \twoheadrightarrow B$ between ∞-categories and an ∞-category A, the *comprehension construction* defines a homotopy coherent diagram that we call the *comprehension functor* indexed by the quasi-category $\mathrm{Fun}(A, B)$ of functors from A to B and valued in the $(\infty, 1)$-categorical core of the ∞-cosmos $\underline{\mathrm{coCart}}(\mathscr{K})_{/B}$ of cocartesian fibrations over B. In the case $A = 1$, the comprehension functor defines a "straightening" of the cocartesian fibration. In the case where the cocartesian fibration is the universal one over the quasi-category of small ∞-categories, the comprehension functor converts a homotopy coherent diagram of shape A into its "unstraightening," a cocartesian fibration over A.

To explain the name, there is an analogy first observed by Street between the comprehension construction in set theory and Grothendieck's construction of the category of elements of a functor $F : C \to \underline{\mathrm{Set}}$ as the category formed by pulling back the cocartesian fibration $\underline{\mathrm{Set}}_* \to \underline{\mathrm{Set}}$. In the ∞-categorical context, the Grothendieck construction is christened "unstraightening" by Lurie. In this context, its inverse, the "straightening" of a cocartesian fibration into a homotopy coherent diagram is particularly important, because such functors are intrinsically tricky to specify, in practice requiring an infinite hierarchy of homotopy coherent data.

The fact that the comprehension construction can be applied in any ∞-cosmos has an immediate benefit. The codomain projection functor $\mathrm{cod} : A^2 \to A$ defines a cocartesian fibration in the slice ∞-cosmos $\mathscr{K}_{/A}$, in which case the comprehension functor specializes to define the Yoneda embedding, a map from the *underlying quasi-category* $\mathrm{Fun}(1, A)$ of A into the quasi-category $\wp(A)$ of groupoidal cartesian fibrations over A. This homotopy coherent diagram carries an element $a : 1 \to A$ to $\mathrm{dom} : A \downarrow a \to A$, a module from 1 to A. A direct analysis of this construction proves that the Yoneda embedding is fully faithful.

References: [RV6], [L1], [KV].

15. On the construction of limits and colimits. To this point we've talked in great generality about the meta-theory of limits and colimits in ∞-categories, but we have not as yet demonstrated the completeness or cocompleteness of any specific ∞-category. In this talk we rectify this oversight by discussing how limits and colimits arise in the homotopy coherent nerves of Kan complex enriched categories. While most of the (co-)completeness results we discuss in this talk apply specifically to the ∞-cosmos of quasi-categories, they may be transported along the change of model functors discussed in talk 13 to provide analogous results in the biequivalent models of $(\infty, 1)$-categories.

We start by considering the theory of categories enriched in Kan complexes and defining what it means for these to admit a flexibly weighted *homotopy (co-)limit*. We observe, in particular, that the full sub-category of fibrant and cofibrant objects in any simplicially enriched model category, in the sense of Quillen, admits all (small) flexibly weighted homotopy (co-)limits, thus providing us with a substantial stock of examples.

Given a simplicial set X we consider its *homotopy coherent realisation* $\mathfrak{C}(X)$ and diagrams $D : \mathfrak{C}(X) \to \mathscr{A}$ of that shape in a Kan complex enriched category $\mathscr{A}$. We define an associated flexible weight W_p on $\mathfrak{C}(X)$ and show that homotopy limits in $\mathscr{A}$ of diagrams $D : \mathfrak{C}(X) \to \mathscr{A}$ and weighted by W_p actually provide a limit for the dual diagram $\hat{D} : X \to \mathfrak{N}(\mathscr{A})$ in the homotopy coherent nerve $\mathfrak{N}(\mathscr{A})$, a (typically large) quasi-category.

Applying these results to the Kan complex enriched category $\underline{\mathrm{Cart}}^g(\underline{\mathrm{QCat}})_{/B}$, of groupoidal fibrations over a fixed quasi-category B, we show that its homotopy coherent nerve $\wp(B)$ is small complete and cocomplete. Now an entirely formal argument, expressed in the virtual equipment of modules in $\underline{\mathrm{QCat}}$, establishes that the Yoneda embedding $\mathscr{Y} : B \to \wp(B)$, as introduced in the last talk, preserves any limits that happen to exist in B. Now the observation that products, pullbacks of isofibrations, and limits of countable chains of isofibrations are all homotopy limits in $\underline{\mathrm{Cart}}^g(\underline{\mathrm{QCat}})_{/B}$ allows us to demonstrate that a quasi-category admits all small limits if it possesses all pullbacks, products, and limits of countable chains.

If time permits we shall extend these ideas to a study of homotopy colimits in an ∞-cosmos $\mathscr{K}$, and discuss conditions under which these may be lifted to the ∞-cosmos $\underline{\mathrm{Cart}}(\mathscr{K})_{/B}$. We also show that pullback along any functor $f : A \to B$ between ∞-categories in $\mathscr{K}$ gives rise to an ∞-cosmos functor $f^* : \underline{\mathrm{Cart}}(\mathscr{K})_{/B} \to \underline{\mathrm{Cart}}(\mathscr{K})_{/A}$ which preserves any homotopy colimits that lift to those ∞-cosmoi in this manner. This result extends, under suitable conditions, to the ∞-cosmos ${}_{A\backslash}\underline{\mathrm{TwoSided}}(\mathscr{K})_{/B}$ and we show that in the case $\mathscr{K} = \underline{\mathrm{QCat}}$ this leads to a construction which provides us with a reflection from two-sided fibrations to modules.

References: [RV7], [RV9].

16. Other approaches to model-independent ∞-category theory. The synthetic theory of ∞-categories developed internally to an ∞-cosmos and its homotopy 2-category provides one approach to developing the "model-independent" theory of ∞-categories. Furthermore, the model independence theorem discussed in talk 12 proves that even "analytic" theorems, proven in a particular model of ∞-categories, can be transferred across any biequivalence of ∞-cosmoi to demonstrate that result in other equivalent models. In this way, we conclude that the theories of $(\infty, 1)$-categories presented in the quasi-category, complete Segal space, Segal category, or naturally marked quasi-category models are all the same.

But this is not the only approach to model independent ∞-category theory. Even in the case of $(\infty, 1)$-categories, the ∞-cosmos axioms were deliberately chosen to exclude certain models[4] out of a desire to simply the proofs in the development of ∞-category theory within an ∞-cosmos. In this concluding talk, we invite the speaker to sketch other approaches to model independent ∞-category theory chosen at his or her discretion and the group as a whole to comment on their relative advantages and disadvantages.

References: [L0], [To], [BS-P], [AFR], [C], …

17 future vistas. The mentors will outline work in progress and survey open problems.

References

[AFR] Ayala, D.; Francis, J.; Rozenblyum, N. Factorization homology from higher categories, arXiv:1504.04007

[BS-P] Barwick, C. and Schommer-Pries, C. On the Unicity of the Homotopy Theory of Higher Categories, (2013), arXiv:1112.0040

[B] Bergner, J.E. *The homotopy theory of* $(\infty, 1)$*-categories*, London Mathematical Society Student Texts **90**, 2018.

[4] Simplicially enriched categories or relative categories, both strictly-defined categorical objects that nonetheless define the objects in a model category that is Quillen equivalent to the other models, do not have well-behaved function complexes and hence do not fit into the ∞-cosmological framework.

[C] Cisinski, D-C. *Higher categories and homotopical algebra*, to appear in Cambridge Studies in Advanced Mathematics by Cambridge University Press www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf

[CS] Cruttwell, G.S.H. and Shulman, M. A unified framework for generalized multicategories, *Theory Appl. Categ.* **24** (2010), no. 10, 580–655.

[G] Grothendieck, A. Les Dérivateurs, `http://people.math.jussieu.fr/~maltsin/groth/Derivateurs.html` (1991).

[J2] Joyal, A. The theory of quasi-categories and its applications. Quadern 45, Vol. II, Centre de Recerca Matemàtica Barcelona (2008).

[JT] Joyal, A. and Tierney, M. Quasi-categories vs Segal spaces. *Categories in algebra, geometry and mathematical physics* Contemp. Math. **431** (2007), 277–326.

[KV] Kazhdan, D and Varshavsky, Y., Yoneda Lemma for Complete Segal Spaces, (2014), arXiv:1401.5656

[KS] Kelly, G.M. and Street, R. Review of the elements of 2-categories, Category Seminar (Proc. Sem., Sydney, 1972/1973), 75–103. Lecture Notes in Math., Vol. 420, Springer, Berlin, 1974.

[He] Heller, A. Homotopy theories, Memoirs of the American Mathematical Society 71 (383).

[J] Joyal, A. Quasi-categories and Kan complexes, Journal of Pure and Applied Algebra 175 (2002) 207–222.

[Lo] Low, Z.L. The homotopy bicategory of $(\infty, 1)$-categories, (2013), arXiv:1310.0381

[L0] Lurie, J. On ∞-topoi (2003) arXiv:0306109

[L1] Lurie, J. *Higher Topos Theory.* Annals of mathematics studies, no. 170, Princeton University Press, 2009.

[L2] Lurie, J. *Higher Algebra*, (2014), www.math.harvard.edu/∼lurie/papers/HigherAlgebra.pdf

[L3] Lurie, J. *Spectral Algebraic Geometry*, (2016), www.math.harvard.edu/∼lurie/papers/SAG-rootfile.pdf

[M] Myers, D.J. String Diagrams For Double Categories and (Virtual) Equipments, (2017), 1–51, arXiv:1612.02762

[Ra1] Rasekh, N. Yoneda Lemma for Simplicial Spaces, (2017), 1–75, arXiv:1711.03160

[Ra2] Rasekh, N. Cartesian Fibrations and Representability, (2017), 1–74, arXiv:1711.03670

[Re1] Rezk, C. A model for the homotopy theory of homotopy theory, *Trans. Amer. Math. Soc.* , **353** (2001) 973–1007

[R4] Riehl, E. *Categorical Homotopy Theory*, New Mathematical Monographs, 24. Cambridge University Press, 2014, xviii+352 pp. www.math.jhu.edu/∼eriehl/cathtpy.pdf

[R6] Riehl, E. *Category Theory in Context*, to appear in the Dover: Modern Math Originals series, 2016, www.math.jhu.edu/∼eriehl/context.pdf

[RV0] Riehl, E. and Verity, D. The theory and practice of Reedy categories, *Theory Appl. Categ.* **29** (2014), no. 9 (2014), 256–301. arXiv:1304.6871

[RV1] Riehl, E. and Verity, D. The 2-category theory of quasi-categories, *Adv. Math.* **280** (2015), 549–642. arXiv:1306.5144

[RV2] Riehl, E. and Verity, D. Homotopy coherent adjunctions and the formal theory of monads, *Adv. Math* **286** (2016), 802–888. arXiv:1310.8279

[RV3] Riehl, E. and Verity, D. Completeness results for quasi-categories of algebras, homotopy limits, and related general constructions, *Homol. Homotopy Appl.* **17** (2015), no. 1, 1–33. arXiv:1401.6247

[RV4] Riehl, E. and Verity, D. Fibrations and Yoneda's lemma in an ∞-cosmos, (2015), 1–75, to appear in *J. Pure Appl. Algebra*, arXiv:1506.05500

[RV5] Riehl, E. and Verity, D. Kan extensions and the calculus of modules for ∞-categories, with D. Verity, (2015), 1-66, to appear in *Algebr. Geom. Topol.*, arXiv:1507.01460

[RVx] Riehl, E. and Verity, D. ∞-category theory from scratch, (2015), 1–53, www.math.jhu.edu/∼eriehl/scratch.pdf

[RV6] Riehl, E. and Verity, D. The comprehension construction, (2017), 1–76, arXiv:1706.10023

[RV7] Riehl, E. and Verity, D. On the construction of limits and colimits in ∞-categories, in preparation.

[RV8] Riehl, E. and Verity, D. Cartesian exponentiation and monadicity, in preparation.

[RV9] Riehl, E. and Verity, D. The calculus of two-sided fibrations and modules, in preparation.

[RV10] Riehl, E. and Verity, D. A proof of the model independence of ∞-category theory, in preparation.

[RV] Riehl, E. and Verity, D. *Elements of ∞-Category Theory*, preliminary draft version available from www.math.jhu.edu/∼eriehl/elements.pdf

[SS] Schanuel, S. and Street, R. The free adjunction. Cahiers de Topologie et Géométrie Différentielle Catégoriques (1986). Volume: 27, Issue: 1, page 81–83.

[Si2] Simpson, C. *Homotopy Theory of Higher Categories: From Segal Categories to n-Categories and Beyond*, New Mathematical Monographs, 19. Cambridge University Press, 2011.

[Su] Sulyma, Y. J. F. ∞,-categorical monadicity and descent, New York J. Math. 23 (2017) 749–777. nyjm.albany.edu/j/2017/23-35.html

[To] Toën, B. Vers une axiomatisation de la théorie des catégories supérieures, *K-Theory* **34** (2005), no. 3, 233–263.

[V2] Verity, D. Weak complicial sets I, basic homotopy theory, Adv. Math. 219 (2008) 1081–1149.

[W2] Weber, M. Yoneda structures from 2-toposes, *Appl. Categ. Struct.* **15** (2007) no. 3, 259–323.

[W] Wood, R.J., *Abstract proarrows I.*, Cahiers de Topologie et Géom. Diff. **XXIII-3** (1982), 279–290.

[Z] Zaganidis, D. Towards an $(\infty, 2)$-category of homotopy coherent monads in an ∞-cosmos, EPFL PhD thesis, 2017 infoscience.epfl.ch/record/231037/files/EPFL_TH7748.pdf

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