Talbot Notes

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Contents

1 Introduction (Dom and Emily)

The task we are engaged in this week is understanding ∞-categories, which is a topic that has been under development for the last 30 years. These should be roughly 'categories weakly enriched in spaces'. The spirit of this is to think of morphisms as things up to homotopy and therefore, composition and naturality become matters up to homotopy.

1.1 Why did (some) people become interested in ∞-categories?

Wave 1: Quillen Model Categories Model categories are a tool for studying homotopy categories, but these present some structure that was (originally) not well understood. This context however did not help us to understand homotopy limits and colimits, and therefore there should be some other, more precise, picture for that.

Wave 2: Derivators Derivators are functors that take 'shapes over which to take homotopy (co)limits' to 'homotopy categories of diagrams'. They allow us to define homotopy limits and colimits as adjunctions. However, there still seemed to be some elusive underlying structure that was not yet made explicit.

Wave 3: ∞-categories Infinity categories aim to be appropriately well-behaved structures that are presented by model categories and allow us to deal well with homotopy (co)limits.

1.2 Analytic and synthetic approaches

We could look this topic via an analytic approach, or a synthetic one. The distinction between those two can be examplified for instance in thinking as a naive exposition of 17th century geometry. The analytic approach is clear in Descartes' work with analytic geometry, centered in cartisian presented points and then building more structure using these building blocks. The synthetic side, clear in Euclid's approach, builds on preliminary (non-defined) concepts of lines and points and then use them to build other structures (one could say that this is the axiomatic approach).

This "duality" can also be seen in areas like Differential Geometry: the analytic approach would be the traditional one, where one builds manifolds from topologising a set and then adding differentiable structure. On the other hand, synthetic differential geometry already starts with differential manifolds and maps as formal objects satisfying certain axioms.

Another example is to look as homotopy type theory as a synthetic approach to homotopy theory of simplicial sets or topological spaces, which would be the analytical approach.

In summary, the synthetic approach can be thought of as top-down, axiomatic, and concerned with the relationships between formal objects, whereas the analytic approach is bottom-up, concerned with how things are built and what they are built from, looking inside the objects of study.

The principal thing to keep in mind in these lecture notes is that we are not interested in working with a specific model of ∞-categories (that would be an analytic approach). In contrast to 1-categories, the structure of ∞-categories is not so simple to describe explicitly. Although the various models for ∞-categories have many advantages, the category-theoretic concepts, like adjunctions, are not easily obtained within these homotopically-oriented models. What we now try to develop is the synthetic approach, where all these models are considered somehow as prototypical examples satisfying the axiomatic system we choose.

We can draw inspiration from the Australian category theorists, who have been interested in studying 2-categories since the 1970s. The study of 2-categories arises quite naturally from the study of the category of categories. This led to the study of enriched, internal and fibred categories. There are many structures which can be studied in the context of 2-categories: functors, natural transformations, adjunctions, Grothendick fibrations, profunctors, limits and colimits, monads and Eilenberg-Moore algebras, and Yoneda embeddings. We would like to do these same constructions in the context of ∞-categories.

The approach we use now is, like for 1-categories, to look at a some sort of well-behaved ∞-category of ∞-categories. More precisely, we consider a category of ∞-categories strictly enriched in quasi-categories, with some additional structure, which will be called an ∞-cosmos. This concept turns out to be very appropriate from the categorical viewpoint and also encompass many of the examples we want, such as internal and fibered categories. Moreover, it avoids an infinite regress. Additionally, we make our lives easier by considering a particular quotient of an ∞-cosmos, called the homotopy 2-category, which captures a surprising amount of structure, whilst allowing us to make use of all of the 2-category theory we already have.

1.3 Goals for the week

We might want to:

- do research **in** ∞-Categories;
- do research **using** ∞-categories;
- learn the theory of ∞-categories for its own sake-why not?

A goal for everyone: being able to prove some theorems from talks 2 and 4. This should be achievable through the exercises suggested.

A goal for the people in the second category is to prove things from talks 5 and 6.

Recaps through the weeks will focus on takeaway points for everyone.

2 ∞-cosmoi and their homotopy 2-categories (Maru)

2.1 Quasi-categories

Definition 2.1 (quasi-categories)**.** A *quasi-category* is a simplicial set X such that every inner horn has a filler. Explicitly, this means that for every $0 < k < n$ and horn $\Lambda^k[n] \to X$ there exists an extension along the inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

By the Yoneda lemma, the map $\Delta[n] \to X$ identifies an n-simplex in X whose faces agree with those specified by the horn.

One of the most important examples come from categories themselves:

Example 2.2. For any category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a quasicategory.

Note that a horn $\Lambda^1[2] \to N\mathcal{C}$ can be represented by

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd}$$

where f and g are morphisms in $\mathcal{C}$, and so asking for this horn to have a filler is the same as asking for the existence of a 1-cell in $N\mathcal{C}$ that acts as a composite of f and g; this cell will of course be gf. For an example of a higher dimension, a horn $\Lambda^1[3] \to N\mathcal{C}_3$ can be represented by

$$\begin{tikzcd} & x_1 \arrow[dr, "hg"] \arrow[dd, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "(hg)f"] \arrow[dr, "gf"'] & & x_3 \\ & x_2 \arrow[ur, "h"'] & \end{tikzcd}$$

and so asking for this horn to have a filler is the same as asking for $(hg)f = h(gf)$; similarly, all other fillers for horns of dimension $n \geq 3$ are given by the associativity in $\mathcal{C}$ of compositions of n maps.

Remark 2.3. We can see that, in this case, all fillers will be unique. In fact, the converse is also true: any quasi-category with unique fillers comes from the nerve of a category.

Remark 2.4. It's not hard to show that the nerve functor

$$\mathbf{Cat} \xrightarrow{N} \mathbf{sSet}$$

is full and faithful; this means we can study categories by looking at them as quasi-categories, and so quasi-categories are a generalization of categories via the nerve functor.

Another important example of quasi-categories comes from topological spaces:

Example 2.5. If X is a topological space, recall that its singular complex is

$$\mathrm{Sing}_n X = \mathbf{Top}(\Delta_n, X)$$

where Δ_n denotes the geometric n-simplex (i.e. the convex hull of the canonical basis in $\mathbb{R}^{n+1}$).

We know that the functor

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$$

has a left adjoint given by geometric realization

$$|-| : \mathbf{sSet} \to \mathbf{Top}$$

which we can use to easily show that $\mathrm{Sing}\, X$ is a quasi-category: a diagram

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathrm{Sing}\, X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

in **sSet** transposes to a diagram

$$\begin{tikzcd} |\Lambda^k[n]| \arrow[r] \arrow[d, hook] & X \\ |\Delta[n]| \arrow[ur, dashed] & \end{tikzcd}$$

in **Top**. Then, since a topological (n, k)-horn is a deformation retract of the geometric n-simplex $\Delta_n = |\Delta[n]|$, this last lift always exists.

Remark 2.6. In this case, we don't necessarily have unique fillers, but using the same argument we find fillers for *all* horns, not just inner ones. Such a simplicial set is called *Kan complex*, and they play an important role in studying the homotopy theory of **sSet**.

As we now see, quasi-categories already come with a natural notion of homotopy.

Definition 2.7 (homotopy relation on 1-simplices)**.** Given a parallel pair of 1-simplices f and g in a quasi-category X, we say that there is a *homotopy* from f to g if there exists a 2-simplex of either of the following forms:

$$\begin{tikzcd} & x_1 \arrow[dr, equal] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & x_1 \end{tikzcd} \qquad \begin{tikzcd} & x_0 \arrow[dl, equal] \arrow[dr, "f"] & \\ x_0 \arrow[rr, "g"'] & & x_1 \end{tikzcd}$$

It's not hard (but probably enlightening if one is not used to working with quasi-categories) to show that the relation witnessed by any of the types of 2-simplex on display in this definition is an equivalence relation, and these equivalence relations coincide. We use this to define the following:

Definition 2.8 (homotopy category)**.** If X is a quasi-category, its *homotopy category* hX has

- as objects, the set X_0,
- as morphisms, the set of homotopy classes of 1-simplices in X_1,
- a composition relation $h = g \circ f$ if and only if, for any choices of 1-simplices representing these maps, there exists a 2-simplex

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

Remark 2.9. A very careful person might have realized it's not obvious that this definition actually works; we're taking a quotient of X_1 by something that at plain sight may not be an equivalence relation. However, the nice properties of quasi-categories make it so, and allow us to define composition very in a very descriptive way, without the need of phrases such as "the equivalence generated by this relation" or such.

Definition 2.10 (isomorphisms in a quasi-category)**.** A 1-simplex in a quasi-category is an *isomorphism* if it represents an isomorphism in the homotopy category. Explicitly, this means that $f : a \to b$ is an isomorphism if and only if there exist a 1-simplex $f^{-1} : b \to a$ together with 2-simplices

$$\begin{tikzcd} & x_1 \arrow[dr, "f^{-1}"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, equal] & & x_1 \end{tikzcd} \qquad \begin{tikzcd} & x_0 \arrow[dr, "f"] & \\ x_0 \arrow[ur, "f^{-1}"] \arrow[rr, equal] & & x_1 \end{tikzcd}$$

Remark 2.11. Just like an arrow in a quasi-category A is represented by a simplicial map $\mathbb{2} \to A$ from the nerve of the free-living arrow, an isomorphism in A is represented by a simplicial map $\mathbb{I} \to A$ from the nerve of the free-living isomorphism.

We now define some important classes of maps.

Definition 2.12 (isofibrations)**.** A simplicial map $f : X \to Y$ is an *isofibration* if it lifts against the inner horn inclusions, and against the inclusion of either vertex into the free standing isomorphism $\mathbb{I}$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Notation: $X \twoheadrightarrow Y$.

Remark 2.13. Note that X is a quasi-category if and only if the map $X \to *$ is an isofibration. This gives a characterization of quasi-categories by a right lifting property, which may come in handy later.

Definition 2.14 (equivalences between quasi-categories)**.** A map $f : A \to B$ of quasi-categories is an *equivalence* if it extends to the data of a "homotopy equivalence" with the free-living isomorphism

$\mathbb{I}$ serving as the interval; that is, if there exist maps $g : B \to A$, α and β such that

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\ & B \end{tikzcd}$$

Notation: $A \simeq\!\!\Rightarrow B$.

Definition 2.15 (trivial fibrations)**.** A simplicial map $f : X \to Y$ is a *trivial fibration* if it lifts against all boundary inclusions

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Notation: $A \simeq\!\!\twoheadrightarrow B$

Remark 2.16. If this last nomenclature reminds you of model categories, that is exactly right: there exists a model structure on simplicial sets (the Joyal model structure) whose fibrant objects are the quasi-categories. The fibrations, weak equivalences, and trivial fibrations between fibrant objects are precisely the classes of isofibrations, equivalences, and trivial fibrations, respectively. If you don't know what any of this means, that's totally fine; you can simply remember the following.

Proposition 2.17. *As suggested by the notation,*

$$\textit{Trivial fibration} = \textit{isofibration} + \textit{equivalence}$$

2.2 ∞-cosmoi

Like Dom mentioned today, we are not going to define exactly what an ∞-category should be; rather, we will axiomatize the "universe" in which ∞-categories live, and give an idea of how they interact with each other via some special classes of maps, and as usual in category theory, these probings should give us some idea of what these things look like.

Definition 2.18 (∞-cosmoi)**.** An ∞-*cosmos* $\mathscr{K}$ is a category enriched over quasi-categories, meaning that it has

- objects A, B, that we call ∞-categories, and
- its morphisms define the vertices of functor-spaces $\mathrm{Fun}(A, B)$, which are quasi-categories,

that is also equipped with a specified class of maps that we call isofibrations and denote by "$\twoheadrightarrow$".

From these classes, we define a map $f : A \to B$ to be an equivalence if and only the induced map $f_* : \mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ on functor-spaces is an equivalence of quasi-categories for all $X \in \mathscr{K}$,

and we define f to be a trivial fibration just when f is both an isofibration and an equivalence; these classes are denoted by $\xrightarrow{\sim}$ and $\overset{\sim}{\twoheadrightarrow}$ respectively.

These classes must satisfy the following three axioms:

(i) (completeness) $\mathscr{K}$ has a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with all simplicial sets, each of these limit notions satisfying a universal property that is enriched over simplicial sets.

(ii) (isofibrations) The class of isofibrations contains all isomorphisms and any map whose codomain is the terminal object; is closed under composition, product, pullback, forming inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and has the property that if $f : A \to B$ is an isofibration and X is any object then

$$\mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$$

is an isofibration of quasi-categories.

(iii) (cofibrancy) Every trivial fibration admits a section

$$\begin{tikzcd} & E \arrow[d, two heads, "\sim"] \\ B \arrow[ur, dashed] \arrow[r, equal] & B \end{tikzcd}$$

Example 2.19. Show that axiom (iii) can be deduced from axioms (i) and (ii)

I like to pack some of this information as "isofibrations behave like fibrations, and everything is fibrant and cofibrant".

At this point you're probably wishing for some examples, but bear with me; Joj will take care of that in the next talk.

As a consequence of the axioms in definition 2.18, we see that the class of trivial fibrations enjoys the same stability properties as the class of fibrations.

Lemma 2.20. *If you replace "isofibrations" by "trivial fibrations" in axiom (ii), everything is still true.*

Another thing that works just like in quasi-categories is that we can characterize equivalences as "homotopy equivalences".

Lemma 2.21 (equivalences are homotopy equivalences)**.** *A map $f : A \to B$ in an ∞-cosmos $\mathscr{K}$ is an equivalence if and only if it extends to the data of a "homotopy equivalence", that is, if there exist maps $g : B \to A$, α and β such that*

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\ & B \end{tikzcd}$$

where $B^{\mathbb{I}}$ is the cotensor of B with $\mathbb{I}$ required by the axioms.

2.3 The homotopy 2-category

In future talks, a lot of the definitions and constructions will be given not in an ∞-cosmos, but in a more tractable 2-category that we now define.

Definition 2.22 (homotopy 2-category)**.** The *homotopy 2-category* of an ∞-cosmos $\mathscr{K}$ is the strict 2-category $h\mathscr{K}$ whose

- objects are the objects of $\mathscr{K}$, i.e. the ∞-categories
- 1-cells $f : A \to B$ are the 0-arrows in the simplicial set $\mathrm{Fun}(A, B)$, i.e. the ∞-functors
- 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ are homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$, which we call ∞-natural transformations.

In other words, $h\mathscr{K}$ is the 2-category with the same objects as $\mathscr{K}$ and with hom-categories defined by

$$h\,\mathrm{Fun}(A, B) = h(\mathrm{Fun}(A, B))$$

Like any 2-category, $h\mathscr{K}$ comes equipped with a notion of equivalence.

Definition 2.23 (equivalence in a 2-category)**.** An *equivalence in a 2-category* is given by

- two objects A and B,
- two 1-cells $f : A \to B$ and $g : B \to A$,
- two invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left=60, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\simeq\Downarrow\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right=60, equal] \arrow[r, phantom, "\simeq\Downarrow\beta"] & B \end{tikzcd}$$

However, we also have a notion of equivalence in $h\mathscr{K}$ from the fact that $\mathscr{K}$ is an ∞-cosmos: that of 1-cells $f : A \xrightarrow{\sim} B$ inducing an equivalence of quasi-categories $f_* : \mathrm{Fun}(X, A) \xrightarrow{\sim} \mathrm{Fun}(X, B)$ for any $X \in \mathscr{K}$.

One of the reasons why the approach that we will be using (of working in $h\mathscr{K}$ instead of $\mathscr{K}$) actually works is that these two notions of equivalence coincide. All the constructions that we will introduce, and the universal properties that we will define, in the context of $h\mathscr{K}$, will of course be invariant under 2-categorical equivalence, and since these agree with the equivalences we have in the ∞-cosmos $\mathscr{K}$, they will be homotopically correct.

In simpler words, the things that our constructions in $h\mathscr{K}$ won't be able to tell apart are precisely the things that we do not wish to distinguish in $\mathscr{K}$ to begin with.

Theorem 2.24 (equivalences are equivalences)**.** *A functor $f : A \to B$ between ∞-categories defines an equivalence in the ∞-cosmos $\mathscr{K}$ if and only if it defines an equivalence in the 2-category $h\mathscr{K}$.*

Proof. Given an equivalence $f : A \xrightarrow{\sim} B$ in $\mathscr{K}$, lemma 2.21 stated that this is equivalent to the existence of an inverse equivalence $g : B \xrightarrow{\sim} A$ and homotopies $\alpha : A \to A^{\mathbb{I}}$ and $\beta : B \to B^{\mathbb{I}}$ in $\mathscr{K}$. But recall that cotensors are defined by the universal property

$$\mathrm{Fun}(-, A^{\mathbb{I}}) \simeq \mathrm{Fun}(-, A)^{\mathbb{I}}$$

so the 0-cells α and β give 0-cells $\hat{\alpha} : \mathbb{I} \times \Delta[0] \simeq \mathbb{I} \to \mathrm{Fun}(A, A)$ and $\hat{\beta} : \mathbb{I} \to \mathrm{Fun}(B, B)$ which specify an equivalence in $h\mathscr{K}$.

For the converse (Maru: I'm leaving this here because it's cool and someone might want to read it, but I didn't get to show this), we claim that if two parallel 1-cells $h, k : A \to B$ in the homotopy 2-category are connected by an invertible 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\simeq\Downarrow\alpha"] & B \end{tikzcd}$$

then h is an equivalence in the ∞-cosmos $\mathscr{K}$ if and only if k is. Using this, we see that the existence of invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\simeq\Downarrow\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right, equal] \arrow[r, phantom, "\simeq\Downarrow\beta"] & B \end{tikzcd}$$

implies that gf and fg are equivalences, and then the fact that equivalences satisfy the 2-out-of-6 property means that f and g must be equivalences too.

So, finally, why is the claim true? First, note that the evaluation maps $ev_0, ev_1 : B^{\mathbb{I}} \to B$ present in a homotopy equivalence are always trivial fibrations, which can be easily deduced by applying the Leibniz cotensor property of lemma 2.20 to the isofibration $B \twoheadrightarrow *$ and the simplicial inclusion $\mathbb{1} \hookrightarrow \mathbb{I}$.

Then, the invertible 2-cell from h to k can be represented by a map $\mathbb{I} \to \mathrm{Fun}(A, B)$, which in turn (by the universal property of the cotensor) corresponds to a map $A \to B^{\mathbb{I}}$ in $\mathscr{K}$ that fits in the following diagram

$$\begin{tikzcd} & B \\ A \arrow[ur, "h"] \arrow[r] \arrow[dr, "k"'] & B^{\mathbb{I}} \arrow[u, two heads, "ev_0"'] \arrow[d, two heads, "ev_1"] \\ & B \end{tikzcd}$$

Since equivalences satisfy the 2-out-of-3 property, we deduce our claim. $\square$

2.4 Appendix: unpacking the limit conditions

Definition 2.25 (cotensor)**.** Let $\mathcal{A}$ be a simplicial category. The *cotensor* of an object $A \in \mathcal{A}$ by a simplicial set U is characterized by an isomorphism of simplicial sets

$$\mathcal{A}(X, A^U) \simeq \mathcal{A}(X, A)^U$$

natural in $X \in \mathcal{A}$. Assuming such objects exist, the simplicial cotensor defines a bifunctor

$$\mathbf{sSet}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$$

$$(U, A) \mapsto A^U$$

in a unique way making the isomorphism natural in U and A as well.

Example 2.26. Cotensors of simplicial sets are exponentials (which supports the abuse of notation).

Definition 2.27 (enriched limits)**.** Enriched limits, when they exist, correspond to the usual limits in the underlying category, but the usual universal property is strengthened. Applying the covariant representable functor

$$\mathcal{A}(X, -) : \mathcal{A}_0 \to \mathbf{sSet}$$

to a limit cone $(\lim_{j \in J} A_j \to A_j)_{j \in J}$ in $\mathcal{A}_0$, there is natural comparison map

$$\mathcal{A}(X, \lim_{j \in J} A_j) \to \lim_{j \in J} \mathcal{A}(X, A_j)$$

and we say that $\lim_{j \in J} A_j$ defines a *simplicially enriched limit* when this is an isomorphism (of simplicial sets) for all $X \in \mathcal{A}$.

Definition 2.28 (towers)**.** A *tower* is a diagram of the shape of the poset of natural numbers

$$\cdots \to X_2 \to X_1 \to X_0$$

A limit over this type of diagram is sometimes called an inverse limit, or a directed limit, or sequential limit.

A *tower of isofibrations* is a special instance of a tower where all the maps involved are isofibrations.

Definition 2.29 (Leibniz cotensors)**.** Given an ∞-functor $f : A \twoheadrightarrow B$ and a simplicial map $i : X \to Y$, the *Leibniz cotensor* map is the induced map to the pullback

$$\begin{tikzcd}
A^Y \arrow[dr, dashed] \arrow[drr, bend left, "A^i"] \arrow[ddr, bend right, "f^Y"'] & & \\
& P \arrow[r] \arrow[d] & A^X \arrow[d, "f^X"] \\
& B^Y \arrow[r, "B^i"'] & B^X
\end{tikzcd}$$

In the case where $i : X \hookrightarrow Y$ is an inclusion of simplicial sets, this pullback exists; we show this by proving that the map $f^X : A^X \to B^X$ is an isofibration, and then appealing to the completeness axiom 2.18(i).

For this, consider the special case $i : \varnothing \to X$; the diagram reduces to

$$\begin{tikzcd} & A^{\varnothing} \arrow[d, "f^{\varnothing}"] \\ B^X \arrow[r, "B^i"'] & B^{\varnothing} \end{tikzcd} \quad = \quad \begin{tikzcd} & * \arrow[d, equal] \\ B^X \arrow[r] & * \end{tikzcd}$$

and the pullback of the latter always exists, since it is given by

$$\begin{tikzcd} B^X \arrow[r] \arrow[d, equal] & * \arrow[d, equal] \\ B^X \arrow[r] & * \end{tikzcd}$$

Now, since the pullback exists, this axiom ensures that the induced map $A^X \to B^X$ is an isofibration, which concludes the explanation.

3 A menagerie of ∞-categorical beasts (Joj)

The purpose of this section is to provide a catalogue of examples of ∞-cosmoi. We start with the most intuitive one given our previous remarks.

3.1 The ∞-cosmos of quasi-categories

We define the ∞-cosmos **QCat** whose

- objects are quasi-categories;
- functor complexes are the exponents in **sSet**s

$$Fun(X, Y) = Y^X = \{X \times \Delta[\cdot] \to Y\}$$

- isofibrations are the ones defined previously for quasi-categories.

We need to understand some model category theory to see how these form an ∞-cosmos: a model category is a category with three classes of morphisms

- fibrations, denoted $\twoheadrightarrow$
- cofibrations, denoted $\hookrightarrow$
- weak equivalences, denoted $\xrightarrow{\sim}$

These satisfy a series of axioms, in particular the lifting axiom: the diagram below has a lift whenever one of the vertical arrows is a weak equivalence.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & A \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Importantly, *the "converse" of the lifting axiom always holds*. Explicitly, a map is a cofibration if and only if it has the left lifting property (LLP) with respect to all trivial fibrations, and analogously, it is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations. This implies that cofibrations are determined by fibrations and weak-equivalences, and the respective analogous hold as well. As consequence, we get closure properties regarding composition, small products, pullbacks and limits of countable towers.

Exercise 3.1. Show that given a class $S \subset Ar(C) = Mor(C)$, the class of morphisms having the right lifting property with respect to S is closed under composition, small products, pullbacks and limits of countable towers.

This gives us closure properties for the isofibrations of quasi-categories since there exists a model structure on **sSet**, precisely the Joyal model structure, in which fibrations between quasi-categories are isofibrations and weak-equivalences between quasi-categories are the ones defined previously (in this setting, the cofibrations are monomorphisms).

To see that the above description of **QCat** indeed defines an ∞-cosmos, there are still characteristics we must verify:

- Enrichment in **qCat**: we need to show that Y^X, the hom-simplicial set, is in **qCat** if X and Y are.
- Cotensoring with simplicial sets is well-defined: we need to check, more generally that then Y^X is in **qCat** whenever Y is (and X is any simplicial set).
- Isofibrations are closed under cotensoring: even more generally, we need $E^X \to B^X$ to be an isofibration whenever $E \to B$ is.

The strategy to verify this property is to show that $E^X \to B^X$ has the right lifting property with respect to inner horn inclusions by transposing the question to the adjunct representative with products and show that the respective map $\Lambda^k[n] \times X \to \Delta[n] \times X$ is a trivial cofibration:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & E^X \arrow[d] \\ \Delta[n] \arrow[r] & B^X \end{tikzcd} \quad \to \quad \begin{tikzcd} \Lambda^k[n] \times X \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta[n] \times X \arrow[r] & B \end{tikzcd}$$

Exercise 3.2. Show that the above closure of isofibrations under cotensoring automatically gives "stability under Leibniz cotensors": given maps $A \rightarrowtail B$ and $X \twoheadrightarrow Y$, the induced dotted map is an isofibration.

$$\begin{tikzcd} X^B \arrow[dr, dashed, two heads] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X^A \times_{Y^A} Y^B \arrow[r] \arrow[d] & X^A \arrow[d] \\ & Y^B \arrow[r] & Y^A \end{tikzcd}$$

3.2 Other notions of ∞-categories

To define quasi-categories we characterized of the nerve $N(C)$ among simplicial sets and generalised this to a definition. We will now explore this strategy in other contexts.

In any **sSet** we have n maps $X_n \xrightarrow{e_i} X_1$, $0 \leq i \leq n-1$ corresponding to the map

$$\begin{aligned} [1] &\to [n] \\ 0 &\to i \\ 1 &\to i+1 \end{aligned}$$

in Δ, and these satisfy

$$\begin{tikzcd} & X_1 \arrow[dr, "\partial_0"] & \\ X_n \arrow[ur, "e_i"] \arrow[dr, "e_{i+1}"'] & = & X_0 \\ & X_1 \arrow[ur, "\partial_1"'] & \end{tikzcd} .$$

Hence, we get a map $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ for each n, this last set thought of as the set of sequences of n composable edges.

Note that for nerves of categories the map defined is an isomorphism, since any n composable maps span a unique n-simplex.

To generalize this, we recall that we think of an ∞-category as having a **space**, rather than a set X_1 of morphisms, and similarly a space of sequences of n composable maps for each n.

Definition 3.3. A bisimplicial set is a functor $\Delta^{op} \to$ **sSet** (or equivalently $\Delta^{op} \times \Delta^{op} \to Set$).

Definition 3.4. A Segal category is a bisimplicial set X such that

- X_0 is discrete;
- Each $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a weak-equivalence in the Kan-Quillen model structure in **sSet**.

The nerve of a simplicial space is an example of a complete Segal category.

We obtain a variant of this in the following way. The condition, in the definition of Segal category, that X_0 is discrete, reflects the idea that only the hom-sets of an ∞-category, and not the object set, are "spaces" (for example, given two spaces, there is a space of maps between them; however, there isn't an obvious "space of all spaces"). However, there turns out to be a fruitful way to think of the objects as also forming a "space". Here, for a 1-category, the "space of objects" is taken to be $NC_{\simeq}$, where $C_{\simeq}$ is the grupoid of isomorphisms of C. We then we get a simplicial space $X_n = N(C^{[n]})_{\simeq}$ and again we have that $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a isomorphism. To generalize this, we weaken "isomorphism" to "weak equivalence", and retain a version of the condition that "X_0 is the nerve of the underlying groupoid":

Definition 3.5. A complete Segal space (or Rezk category) is a bisimplicial set X such that

- X is Reedy fibrant
- $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a weak equivalence
- The map $s_0 : X_0 \to X_1$ maps isomorphically onto the subsimplicial set $X_{isos} \hookrightarrow X_1$ – the "space of invertible morphisms" (whose precise and somewhat technical definition we omit)

With these concepts, we can construct convenient ∞-cosmos as we have done with quasi-categories, because there are model structures on the category **ssSet** of bisimplicial sets, namely the Segal and Rezk model strucutres, such that the fibrant objects are the (Reedy fibrant) Segal categories and complete Segal spaces, respectively. That is, we take the objects of our ∞-cosmos to be the fibrant objects in these model categories, and the isofibrations to be the fibrations.

To define the enrichment in quasi-categories and cotensoring with simplicial sets, we make use of functors

$$\mathbf{sSet}_{Joyal} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{ssSet}_{Segal} \qquad\qquad \mathbf{sSet}_{Joyal} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{ssSet}_{Rezk}$$

(which are induced by certain functors between Δ and $\Delta \times \Delta$) which are "Quillen equivalences" – that is, they are adjunctions satisfying some properties, in particular inducing equivalences on homotopy categories.

For the enrichment, we can then define the "hom quasi-category" from A to B to be $R(\underline{\mathrm{ssSet}}(A, B))$ (here, we use that R, being a right-Quillen functor, preserves fibrant objects), and we can define the cotensor A^U, where U is a simplicial set, as $\underline{\mathrm{ssSet}}(L(U), A)$.

Further examples of ∞-cosmoi can are:

- slices of ∞-cosmoi: which can be thought of as "category of isofibrations over an object"
- 1-categories: using the categorical isofibrations and equivalences.

To finish, we discuss the concept of cosmological functors. These are simplicially enriched functors between ∞-cosmoi $\mathcal{K}_1 \to \mathcal{K}_2$ which preserve all the structure: isofibrations, relevant limits (products, pullbacks of isofibrations, limits of towers) and cotensors.

Example 3.6. Given a morphism $f : X \to Y$ between objects of an ∞-cosmos $\mathcal{K}$, we get a cosmological functor $f^* : \mathcal{K}/Y \to \mathcal{K}/X$.

Example 3.7. Given an ∞-cosmos $\mathcal{K}$ and an object A, then $\mathrm{Hom}(A, -) : \mathcal{K} \to \mathbf{QCat}$ is a cosmological functor.

Example 3.8. We have cosmological functors

$$\mathrm{CSS} \xrightarrow{\sim} \mathbf{QCat} \xleftarrow{\sim} \mathrm{Segal}$$

4 Adjunctions, limits and colimits in homotopy 2-categories (Emma)

4.1 Adjunctions

We know that every ∞-cosmos has a canonical 2-category assigned, its homotopy 2-category – formed from the ∞-categories, ∞-functors, and ∞-natural transformations. We will use the definition of adjunction in 2-categories for the particular case of the 2-homotpy categories of ∞-cosmoi to define adjunctions between ∞-categories.

As an example of adjunction in 2-categories, we will see that the usual 1-categorical notion of adjunction is an instance of its 2-categorical counterpart.

Definition 4.1. An adjunction in a 2-category $\mathcal{C}$ consists of:

- a pair of objects, A, B in $\mathcal{C}$,
- a pair of 1-cells, $A \xrightarrow{u} B$ and $B \xrightarrow{f} A$,
- a pair of 2-cells, $1_B \overset{\eta}{\Longrightarrow} uf$ and $fu \overset{\varepsilon}{\Longrightarrow} 1_A$, called the unit and counit respectively, such that the triangle equalities hold:

$$\begin{tikzcd} B \arrow[r, equal] \arrow[dr, "u"'] & B \arrow[d, phantom, "\Downarrow\varepsilon"] \\ A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, shift left=3, "\Downarrow\eta"] \arrow[ur, phantom, shift right=3, "f"] & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend left, "u"] \\ A \arrow[u, bend left, "u"] \arrow[loop right, "="] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"'] & B \arrow[d, phantom, "\Downarrow\varepsilon"] \\ A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, shift left=3, "\Downarrow\eta"] \arrow[ur, phantom, shift right=3, "f"] & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \\ A \arrow[u, bend right, "f"'] \arrow[loop right, "="] \end{tikzcd} \tag{1}$$

In this scenario, f is called the left adjoint and u is called the right adjoint. We denote this relationship by writing $f \dashv u$ or via the diagram

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f"] \end{tikzcd}$$

Let's recall the definition in 1-category theory.

Definition 4.2. Given two functors $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$, we say that (F, G) is an adjoint pair– or that F is a left adjoint of G, or that G is a right adjoint of F – if there exist natural transformations $\varepsilon : FG \Rightarrow \mathrm{id}_D$ and $\eta : \mathrm{id}_C \Rightarrow GF$ such that the following diagrams commute

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, equal, "\mathrm{id}"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal, "\mathrm{id}"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & F \end{tikzcd} \tag{2}$$

Remark 4.3. An adjunction in the 1-categorical setting is precisely an adjunction in the 2-category **Cat** – where the objects are the 1-categories, the 1-cells are the functors, and the 2-cells are the natural transformations.

To see this, let's rewrite the triangle equalities in 2 with 2-categorical symbolism. First, we can rephrase the left diagram in 2 as

$$\varepsilon F \cdot F\eta = \mathrm{id}_F \tag{3}$$

Now, we have

$$F\eta = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "\mathrm{id}_{\mathcal{C}}"{name=U}] \arrow[r, bend right=50, "GF"'{name=D}] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\Downarrow\eta"] \end{tikzcd} \qquad \varepsilon F = \begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "GF"{name=U}] \arrow[r, bend right=50, "\mathrm{id}_{\mathcal{D}}"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\Downarrow\varepsilon"] \end{tikzcd}$$

Pasting these diagrams we see that the triangle equality 3 is diagrammatically represented by

$$\begin{tikzcd} \mathcal{C} \arrow[rr, equal] \arrow[dr, "F"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[dl, "F"] \\ & \mathcal{D} \arrow[ur, "G"] \arrow[r, equal] \arrow[ur, phantom, "\downarrow\varepsilon"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \arrow[d, bend left, "F"] \arrow[d, phantom, "(=)"] \\ \mathcal{D} \end{tikzcd}$$

which is exactly the right diagram in 1 in the definition of adjunctions for 2-categories. We can proceed analogously for the second triangle equality.

Since the definition of adjunction in 2-categories (4.1) is stated in terms of objects, 1-cells, 2-cells, and composites of them, we have the following lemma.

Lemma 4.4. *Adjunctions in a* 2*-category are preserved by* 2*-functors.*

Definition 4.5. We define an adjunction between ∞-categories in an ∞-cosmos $\mathcal{K}$ to be an adjunction between them in the homotopy 2-category $\flat\mathcal{K}$.

This definition might seem suspiciously simple. However, the following evidence makes a case for its adequacy. Firstly, in the ∞-cosmos of quasi-categories, this definition of adjunction recovers Lurie's. Secondly, addressing its seemingly low dimensional nature, we will see in Martina's talk that any adjunction in the homotopy 2-category $\flat\mathcal{K}$ can be promoted to an homotopy coherent adjunction in $\mathcal{K}$, and these extensions are homotopically unique.

Proposition 4.6. *Let* $\begin{tikzcd} A \arrow[r, bend right=50, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right=50, "f"'] \end{tikzcd}$ *and* $\begin{tikzcd} A' \arrow[r, bend right=50, "u'"'] \arrow[r, phantom, "\perp"] & B' \arrow[l, bend right=50, "f'"'] \end{tikzcd}$ *be two adjunctions, and* $a : A \to A'$, $b : B \to B'$ 1*-cells. Then there exists a bijection between* 2*-cells* $\lambda : bu \Rightarrow u'a$ *and* $\mu : f'b \Rightarrow af$*, where* μ *is the composite*

$$\begin{tikzcd} & A \arrow[r, "a"] \arrow[dr, "u"] \arrow[d, phantom, "\Uparrow\eta"] & A' \arrow[r, equal] \arrow[dr, "u'"] \arrow[d, phantom, "\Uparrow\lambda"] & A' \\ B \arrow[ur, "f"] \arrow[r, equal] & B \arrow[r, "b"'] & B' \arrow[ur, "f'"'] \arrow[u, phantom, "\Uparrow\varepsilon'"] & \end{tikzcd}$$

We call λ *and* μ *the mates of the adjunctions* $f \dashv u$ *and* $f' \dashv u'$.

Proposition 4.7. *Let* $A \underset{u}{\overset{f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} B$ *be an adjunction between* ∞*-categories. Then:*

1. *for any* ∞*-category* X*,*

$$\mathrm{Fun}(X, A) \underset{u_*}{\overset{f_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Fun}(X, B)$$

   defines an adjunction between quasi-categories;

2. *for any* ∞*-category* X

$$\mathrm{hFun}(X, A) \underset{u_*}{\overset{f_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{hFun}(X, B)$$

   defines an adjunction between categories;

3. *for any simplicial set* $\mathcal{U}$*,*

$$A^{\mathcal{U}} \underset{u^{\mathcal{U}}}{\overset{f^{\mathcal{U}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} B^{\mathcal{U}}$$

   defines an adjunction between ∞*-categories.*

Proof. It's enough to observe that these three diagrams are obtained from the original adjunction after an application of certain 2-functors, and to recall from Lemma 4.4 that 2-functors preserve 2-categorical adjunctions. □

Proposition 4.8. *Adjunctions compose. This is, given adjoint functors as below-left, their composites (shown below-right) also form an adjunction.*

$$C \underset{u'}{\overset{f'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} B \underset{u}{\overset{f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} A \qquad\qquad C \underset{u'u}{\overset{ff'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} A$$

Proof. Write $\eta : \mathrm{id}_B \Rightarrow uf$, $\varepsilon : fu \Rightarrow \mathrm{id}_A$, $\eta' : \mathrm{id}_C \Rightarrow u'f'$ and $\varepsilon' : f'u' \Rightarrow \mathrm{id}_B$ for the units and counits respectively.

We define the unit and counit of the adjunction $ff' \dashv uu'$ by the pasting diagrams below

$$\begin{tikzcd}
C \arrow[rr, equal] \arrow[dr, "f'"'] \arrow[drr, phantom, "\Downarrow \eta'"] & & C \\
& B \arrow[r, equal] \arrow[dr, "f"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \eta"] & B \arrow[u, "u'"'] \\
& & A \arrow[u, "u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& C \arrow[dr, "f'"] \arrow[dr, phantom, shift right=3ex, "\Downarrow \varepsilon'"] & \\
& B \arrow[u, "u'"] \arrow[r, equal] \arrow[dr, "f"] \arrow[r, phantom, shift right=3ex, "\Downarrow \varepsilon"] & B \\
A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd}$$

We now show now that they satisfy the triangle equalities for $ff' \dashv uu'$.

From the adjunctions $f \dashv u$ and $f' \dashv u'$, we know

$$\begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"] & B \\ A \arrow[u, "u"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \arrow[u, "u"'] \arrow[ul, phantom, "\Downarrow\eta"] \end{tikzcd} = \begin{tikzcd} B \arrow[loop below, "="] \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, equal] \arrow[dr, "f'"] & C \\ B \arrow[u, "u'"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\varepsilon'"] & B \arrow[u, "u'"'] \arrow[ul, phantom, "\Downarrow\eta'"] \end{tikzcd} = \begin{tikzcd} C \arrow[loop below, "="] \\ B \arrow[u, bend left, "u'"] \arrow[u, bend right, "u'"'] \end{tikzcd}$$

Thus, we have

$$\begin{tikzcd} C \arrow[rr, equal] \arrow[dr, "f'"] & & C \\ B \arrow[u, "u'"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\varepsilon'"] & B \arrow[r, equal] \arrow[dr, "f"] \arrow[ur, phantom, "\Downarrow\eta'"] & B \arrow[u, "u'"'] \\ A \arrow[u, "u"] \arrow[rr, equal] \arrow[ur, phantom, "\Downarrow\varepsilon"] & & A \arrow[u, "u"'] \arrow[ul, phantom, "\Downarrow\eta"] \end{tikzcd} = \begin{tikzcd} C \arrow[loop above, "="] \\ B \arrow[u, bend left, "u'"] \arrow[u, bend right, "u'"'] \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[loop below, "="] \end{tikzcd} = \begin{tikzcd} C \arrow[loop above, "="] \\ A \arrow[u, bend left, "u'u"] \arrow[u, bend right, "u'u"'] \arrow[loop below, "="] \end{tikzcd}$$

which is one of the triangle equalities. The other one is obtained analogously. □

Proposition 4.9 (uniqueness of adjoints)**.** *If $f \dashv u$ and $f' \dashv u$, then $f \cong f'$. Conversely, if $f \dashv u$ and $f \cong f'$, then $f' \dashv u$.*

Proof. For the first statement, we wish to find an invertible 2-cell $f \Rightarrow f'$. Write $\eta : \mathrm{id}_B \Rightarrow uf$, $\varepsilon : fu \Rightarrow \mathrm{id}_A$, $\eta' : \mathrm{id}_B \Rightarrow uf'$ and $\varepsilon' : f'u \Rightarrow \mathrm{id}_A$ for the respective units and counits, and consider the pasting diagrams

$$\begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f'"'] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\eta'"] & \\ & A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"'] & B \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\eta"] & \\ & A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon'"] & A \end{tikzcd}$$

These diagrams define 2-cells $f \Rightarrow f'$ and $f' \Rightarrow f$; we want to show that the composites $f \Rightarrow f' \Rightarrow f$ and $f' \Rightarrow f \Rightarrow f'$ are identities.

We will show that the composite $f \Rightarrow f' \Rightarrow f$ – computed by pasting horizontally the diagrams above – is the identity, since the other argument is the same.

This is easily seen: applying one of the triangle equalities for the adjunctions $f' \dashv u'$ and $f \dashv u$ we get

$$\begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"'] & B \arrow[r, equal] \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\eta'"] & \\ & A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon'"] & A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd} = \begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"'] & B \arrow[r, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\eta"] & B \\ & A \arrow[r, equal] \arrow[ur, "u"] & A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd} = \begin{tikzcd} B \arrow[loop above, "="] \\ A \arrow[u, bend left, "f"] \arrow[u, bend right, "f"'] \arrow[loop below, "="] \end{tikzcd}$$

Thus, we conclude that $f \simeq f'$.

Conversely, since $f \simeq f'$ there exist mutually inverse 2-cells $\beta : f \Rightarrow f'$ and $\beta' : f' \Rightarrow f$. The pasting diagrams

$$\begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f'"'] & B \arrow[r, equal] \arrow[d, phantom, "\Downarrow\beta"] \arrow[dr, "f"] & B \arrow[dl, "u"] \\ & A \arrow[r, equal] \arrow[d, phantom, "\Downarrow\eta"] & A \\ & & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"] & B \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\beta'"] & \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \arrow[r, equal] & A \end{tikzcd}$$

respectively define the unit and counit of $f' \dashv u$ so that the triangle equalities

$$\begin{tikzcd} B \arrow[r, equal] & B \arrow[r, equal] \arrow[dr, "f'"] & B \arrow[r, equal] \arrow[dr, "f"] & B \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\beta'"] & A \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\beta"] & A \arrow[ur, "u"] \arrow[u, phantom, "\Downarrow\eta"] \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f'"'] & B \arrow[r, equal] \arrow[d, phantom, "\Downarrow\beta"] \arrow[dr, "f"] & B \arrow[r, equal] \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\beta'"] & B \\ & A \arrow[r, equal] \arrow[d, phantom, "\Downarrow\eta"] & A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \\ & & & \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "f'"] \arrow[u, bend right, "f'"'] \arrow[u, phantom, "="] \end{tikzcd}$$

hold.

□

Proposition 4.10 (adjoint equivalences)**.** *Any equivalence can be promoted to an adjoint equivalence by modifying one of the invertible* 2*-cells.*

4.2 Absolute liftings

The main reason absolute lifting diagrams will be of use to us is that they allow us to characterize adjunctions.

Definition 4.11 (absolute liftings)**.** Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in a 2-category, an absolute left lifting of g through f is given by a pair (ℓ, λ) consisting of a 1-cell and a 2-cell as below

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, "\Uparrow\lambda"] & A \end{tikzcd}$$

such that any 2-cell as displayed below-left factors uniquely through (ℓ, λ) as displayed below-right

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] \arrow[ur, "\exists!\ \hat{\ell}"] \arrow[ur, phantom, "\Uparrow\lambda"] & A \end{tikzcd}$$

Dually, an absolute right lifting of g through f is given by a pair (r, ρ) of a 1-cell and a 2-cell as below

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\rho"] & A \end{tikzcd}$$

such that any 2-cell as displayed below-left factors uniquely through (r, ρ) as displayed below-right

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\varphi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\ \Downarrow\varphi"] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\rho"] & A \end{tikzcd}$$

The adjective "absolute" comes from the following property.

Lemma 4.12. *Absolute liftings are stable under the restriction of their domain objects. That is, if (ℓ, λ) is an absolute left lifting of g through f, then for any $c : X \to C$, the restricted diagram $(c\ell, c\lambda)$ defines an absolute left lifting diagram of gc through f.*

$$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\lambda"] & A \end{tikzcd}$$

Remark 4.13. If instead of pre-composing as before, we post-compose from g with a function $a : A \to X$, stability trivially holds.

Lemma 4.14 (composition and cancelation of absolute lifting diagrams)**.** *Suppose (r, ρ) defines an absolute right lifting diagram of h through f and that we have the diagram below*

$$\begin{tikzcd} & C \arrow[d, "g"] \\ & B \arrow[d, "f"] \arrow[dl, phantom, shift right=3ex, "\Downarrow\sigma"] \\ D \arrow[ur, "r"] \arrow[r, "h"'] \arrow[uur, bend left, "s"] \arrow[ur, phantom, shift right=3ex, "\Downarrow\rho"] & A \end{tikzcd}$$

Then (s, σ) defines an absolute right lifting of r through g if and only if $(s, \rho \cdot f\sigma)$ defines an absolute right lifting diagram of h through fg.

Proof. Let's first assume that (s, σ) is an absolute right lifting of r through g. We want to show that any 2-cell as below left uniquely factors as below right.

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\varphi"] & C \arrow[d, "fg"] \\ D \arrow[r, "h"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & C \arrow[d, "fg"] \\ D \arrow[ur, "s"] \arrow[r, "h"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\rho\cdot f\sigma"] & A \end{tikzcd}$$

Using consecutively that (r, ρ) defines an absolute right lifting of h through f to factor φ, and that (s, σ) defines one of r through g to factor ψ_1, we get

$$
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\varphi"] & C \arrow[d, "fg"] \\
D \arrow[r, "h"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[dd, "d"'] \arrow[ddr, phantom, "\exists\Downarrow\psi_1"] & C \arrow[d, "g"] \\
 & B \arrow[d, "f"] \\
D \arrow[r, "h"'] \arrow[ur, "r"] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[dd, "d"'] \arrow[ddr, phantom, "\exists\Downarrow\psi_2"] & C \arrow[d, "g"] \\
 & B \arrow[d, "f"] \arrow[dl, "s"'] \arrow[dl, phantom, shift left=3, "\Downarrow\sigma"] \\
D \arrow[r, "h"'] \arrow[ur, "r"] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd}
$$

Then ψ_2 is a factorization of φ through $\rho \cdot f\sigma$, as we were looking for. For uniqueness, any other such factorization, say, ψ_2', must verify $\sigma \cdot g\psi_2' = \psi_1$ by uniqueness of the factorization coming from (r, ρ) being an absolute right lifting. But then, by uniqueness of the factorization guaranteed by (s, σ) being an absolute right lifting, it must be that $\psi_2' = \psi_2$.

Conversely, we want to show that any 2-cell φ as below left uniquely factors as below right

$$
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\varphi"] & C \arrow[d, "g"] \\
D \arrow[r, "r"'] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\text{'''}"] & C \arrow[d, "g"] \\
D \arrow[r, "r"'] \arrow[ur, "s"] \arrow[ur, phantom, shift right=3, "\Downarrow\sigma"] & B
\end{tikzcd}
$$

assuming that $(s, \rho \cdot f\sigma)$ defines an absolute right lifting of h through fg.

By that assumption, the pasting diagram displayed below left factors as showed in the middle.

$$
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\varphi"] & C \arrow[d, "g"] \\
D \arrow[r, "r"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, "f"] \\
D \arrow[r, "h"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\text{'''}"] & C \arrow[d, "g"] \\
D \arrow[r, "r"] \arrow[ur, "s"] \arrow[ur, phantom, shift right=3, "\Downarrow\sigma"] \arrow[dr, "h"'] \arrow[dr, phantom, shift left=3, "\Downarrow\rho\cdot f\sigma"] & B \arrow[d, "f"] \\
 & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\text{'''}"] & C \arrow[d, "g"] \\
D \arrow[r, "r"] \arrow[ur, "s"] \arrow[ur, phantom, shift right=3, "\Downarrow\sigma"] \arrow[dr, "h"'] \arrow[dr, phantom, shift left=3, "\Downarrow\rho"] & B \arrow[d, "f"] \\
 & A
\end{tikzcd}
$$

Since (r, ρ) defines an absolute right lifting, factorizations are unique and therefore the upper part of the left and right diagrams above must coincide. Thus, we have obtained a factorization of the 2-cell φ through σ, as we were seeking. Finally, to show the uniqueness of such factorization, we reason as we did before. □

Lemma 4.15. *A 2-cell $\eta : \mathrm{id}_B \Rightarrow uf$ defines the unit of an adjunction $f \dashv u$ if and only if (f, η) defines an absolute left lifting diagram*

$$
\begin{tikzcd}
 & A \arrow[d, "u"] \\
B \arrow[ur, "f"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & B
\end{tikzcd}
$$

Dually, a 2-cell $\varepsilon : fu \Rightarrow \mathrm{id}_A$ defines the counit of an adjunction $f \dashv u$ if and only if (u, ε) defines an absolute right lifting diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow\varepsilon"] \\ A \arrow[ur, "u"] \arrow[r, equal] & A \end{tikzcd}$$

Proof. Suppose we have an absolute right lifting diagram as in the picture; let's prove it defines the counit of an adjunction $f \dashv u$. To find the unit of the adjunction, we consider the unique factorization

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} = \begin{tikzcd} B \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow r"] & B \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow\varepsilon"] \\ A \arrow[r, equal] \arrow[ur, "u"] & A \end{tikzcd}$$

Note that one of the triangle identity diagrams is precisely the equality we used to define η. Now, to prove

$$\begin{tikzcd} & B \arrow[r, equal] \arrow[dl, phantom, "\Downarrow\varepsilon"] \arrow[dr, phantom, "\Downarrow\eta"] & B \\ A \arrow[ur, "u"] \arrow[r, equal] & A \arrow[ur, "u"'] & \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & B \end{tikzcd}$$

we show that we can append both cells to ε to factor the same cell:

$$\begin{tikzcd} & B \arrow[r, equal] \arrow[dl, phantom, "\Downarrow\varepsilon"] \arrow[dr, phantom, "\Downarrow\eta"] & B \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow\varepsilon"] \\ A \arrow[ur, "u"] \arrow[r, equal] & A \arrow[r, equal] & A \end{tikzcd} = \begin{tikzcd} B \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & B \arrow[d, "f"] \\ A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

Thus, since factorizations are unique, we conclude that the triangle equality holds.

Conversely, start from an adjunction whose counit is ε; we want to show that (u, ε) defines an absolute right lifting. For this, consider a diagram

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

that is, a 2-cell $\alpha : fb \Rightarrow a$. From Proposition 4.7, we know $f \dashv u$ induces an adjunction

$$\begin{tikzcd} \mathrm{hFun}(X, A) \arrow[rr, bend left, "u_*"] \arrow[rr, phantom, "\perp"] & & \mathrm{hFun}(X, B) \arrow[ll, bend right, "f_*"'] \end{tikzcd}$$

and we transpose $\alpha : fb \Rightarrow a$ across this adjunction to get a 2-cell $\beta : b \Rightarrow ua$, which will satisfy[1]

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\beta"] & B \arrow[d, "f"] \\ A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd}$$

yielding the desired factorization. Finally, uniqueness is given by the fact that transposition is a bijective correspondence, since any cell γ that could replace β in the factorization would have α as its transpose, and then $\gamma = \beta$. □

4.3 Limits and colimits

We will use the previous lemma to define limits and colimits in our ∞-categories; but first, let's recall how we can view limits and colimits in any 1-category through an adjunction.

Let $\mathcal{C}$ be a 1-category, and $F : J \to \mathcal{C}$ be a diagram of shape J. Let $\Delta : \mathcal{C} \to \mathcal{C}^J$ denote the constant diagram functor, taking an object X in $\mathcal{C}$ to the diagram $\Delta_X : J \to \mathcal{C}$ constant at X. Then, the limit of the diagram F is an object $\lim F$ in $\mathcal{C}$ such that

$$\mathcal{C}(-, \lim F) \cong \mathcal{C}^J(\Delta_{(-)}, F);$$

in other words, $\lim F$ is the representative of the functor of cones over F (note that, thanks to Yoneda, this definition also yields the legs of the limit cone).

Now, if we allow F to vary in $\mathcal{C}^J$ (which amounts to considering all possible diagrams of shape J in $\mathcal{C}$), we can say that $\mathcal{C}$ will have *all limits* if for any X in $\mathcal{C}$ and any diagram F in $\mathcal{C}^J$, we have

$$\mathcal{C}(X, \lim F) \cong \mathcal{C}^J(\Delta_X, F);$$

i.e. if there exists a functor $\lim : \mathcal{C}^J \to \mathcal{C}$ that is right adjoint to $\Delta : \mathcal{C} \to \mathcal{C}^J$. Dually, $\mathcal{C}$ will have *all colimits* if there exists a functor $\mathrm{colim} : \mathcal{C}^J \to \mathcal{C}$ that is left adjoint to Δ.

Thus, following the cue from 1-categories, we define what it means for an ∞-category to admit all (co)limits of a given shape J.

Definition 4.16 (diagram ∞-categories)**.** Given a simplicial set J and an ∞-category A, we refer to the cotensor A^J as the *∞-category of J-shaped diagrams in A*.

Note that the unique map to the terminal object $J \to 1$ induces a functor

$$\Delta : A \cong A^1 \to A^J$$

which we call the *constant diagram functor*. This name is justified since, if we study the case of quasicategories, we see that Δ takes an n-simplex of A, in the form of a simplicial map $\Delta[n] \xrightarrow{x_n} A$, to the n-simplex in $(A^J)_n = \{J \times \Delta[n] \to A\}$ that picks x_n for every value in J, i.e. the one given by $J \times \Delta[n] \xrightarrow{!\times \mathrm{id}} 1 \times \Delta[n] \cong \Delta[n] \xrightarrow{x_n} A$.

[1] Recall that since β is the transpose of α, the transpose of β should be α, and that we transpose a map $\beta : b \Rightarrow ua$ by composing $fb \xrightarrow{f\beta} fua \xrightarrow{\varepsilon a} a$.

Definition 4.17 (all (co)limits)**.** An ∞-category A *admits all colimits* of shape J if the constant diagram functor $\Delta : A \to A^J$ admits a left adjoint, while A *admits all limits* of shape J if the constant diagram functor admits a right adjoint.

$$\begin{tikzcd} A^J \arrow[r, bend left, "\text{colim}"] \arrow[r, bend right, "\text{lim}"'] \arrow[r, phantom, "\perp"] \arrow[r, phantom, "\perp"] & A \arrow[l, "\Delta"'] \end{tikzcd}$$

Thanks to Lemma 4.15, we can rephrase this definition in the following manner: an ∞-category A admits all colimits (limits) of shape J if there exists an absolute left (right) lifting of $\mathrm{id} : A^J \to A^J$ through $\Delta : A \to A^J$ as in the diagram below left (right)

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^J \arrow[ur, "\text{colim}"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow \eta"] & A^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^J \arrow[ur, "\text{lim}"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow \varepsilon"] & A^J \end{tikzcd}$$

It is of course possible for an ∞-category to have *some*, but not *all*, (co)limits of a given shape. Fortunately, the definition via absolute liftings admits a natural generalization.

Definition 4.18 (some (co)limits)**.** A *colimit* of a family of diagrams $d : D \to A^J$ of shape J is given by an absolute left lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\text{colim}"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Uparrow \eta"] & A^J \end{tikzcd}$$

comprised of a colimit functor $\mathrm{colim} : D \to A$ and a colimit cone $\eta : d \Rightarrow \Delta\,\mathrm{colim}$.

Dually, a *limit* of a family of diagrams $d : D \to A^J$ of shape J is given by an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\text{lim}"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow \varepsilon"] & A^J \end{tikzcd}$$

comprised of a limit functor $\mathrm{lim} : D \to A$ and a limit cone $\varepsilon : \Delta\mathrm{lim} \Rightarrow d$.

In particular, letting $D = 1$ we define the (co)limit of a single diagram $d : 1 \to A^J$.

Remark 4.19. Clearly, if A has (co)limits of shape J for all possible families of diagrams $d : D \to A^J$, then A has all (co)limits of shape J, by considering the case $D = A^J$ and $d = \mathrm{id}$. Conversely, if A

has all (co)limits of shape J, then we obtain the (co)limit for any family of diagrams $d : D \to A^J$ by restricting

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ D \arrow[r, "d"] & A^J \arrow[ur, "\mathrm{colim}"] \arrow[r, equal] \arrow[ur, phantom, "\Uparrow \eta"] & A^J \end{tikzcd} \qquad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ D \arrow[r, "d"] & A^J \arrow[ur, "\lim"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow \varepsilon"] & A^J \end{tikzcd}$$

and using Lemma 4.12.

In certain ∞-cosmoi, it's also true that if A has (co)limits of all diagrams $d : 1 \to A^J$, then A has all (co)limits, but unfortunately, this isn't always the case.

We finish this section by proving an important and very useful result: preservation of (co)limits by adjoints. The fact that we're already able to prove such a thing in detail is evidence of the fact that this approach to ∞-categories is much more tractable than others found in the literature, for which this theorem would require a great deal more work.

Theorem 4.20 (RAPL/LAPC)**.** *Right adjoints preserve limits and left adjoints preserve colimits.*

Proof. We will show that right adjoints preserve limits, since the other argument is dual.

Let $u : A \to B$ be a functor that admits a left adjoint $f : B \to A$, with unit $\eta : \mathrm{id}_B \Rightarrow uf$ and counit $\varepsilon : fu \Rightarrow \mathrm{id}_A$. Suppose that a family of diagrams $d : D \to A^J$ has a limit in A, i.e. there exists an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[r, "d"'] \arrow[ur, "\lim"] \arrow[ur, phantom, "\Downarrow \rho"] & A^J \end{tikzcd}$$

Our goal is to show that u preserves this limit; that is, the following diagram is also an absolute right lifting

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ D \arrow[r, "d"'] \arrow[ur, "\lim"] \arrow[ur, phantom, "\Downarrow \rho"] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd} \tag{4}$$

According to Proposition 4.7, the cotensor functor $(-)^J : \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ carries the adjunction $f \dashv u$ to an adjunction $f^J \dashv u^J$ with unit η^J and counit ε^J; this implies that (u^J, ε^J) is an absolute right lifting of id_{A^J} through f^J, which is then preserved by restriction along the diagram functor d.

$$\begin{tikzcd} & & B^J \arrow[d, "f^J"] \\ D \arrow[r, "d"] & A^J \arrow[ur, "u^J"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow \varepsilon^J"] & A^J \end{tikzcd} \tag{5}$$

Therefore, Lemma 4.14 ensures that (1) will be an absolute lifting diagram if and only if the composition of (1) and (2), displayed below left, is one too.

$$
\begin{tikzcd}
 & A \arrow[r, "u"] \arrow[d, "\Delta"] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] & A^J \arrow[r, "u^J"] \arrow[dr, equal] \arrow[dr, phantom, "\Downarrow\varepsilon^J"] & B^J \arrow[d, "f^J"] \\
 & & A^J
\end{tikzcd}
=
\begin{tikzcd}
 & & B \arrow[d, "f"] \\
 & A \arrow[ur, "u"] \arrow[r, equal] \arrow[d, "\Delta"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] & A^J \arrow[r, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A^J \\
 & &
\end{tikzcd}
=
\begin{tikzcd}
 & & B \arrow[d, "f"] \\
 & A \arrow[ur, "u\lim"] \arrow[r, equal] \arrow[d, "\Delta"] \arrow[dr, phantom, "\Downarrow\varepsilon\lim"] & A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] & A^J \arrow[r, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A^J \\
 & &
\end{tikzcd}
$$

From the 2-functoriality of the simplicial cotensor in its exponent variable, $f^J\Delta = \Delta f$ and $\varepsilon^J\Delta = \Delta\varepsilon$. Then, the diagram above left is the same as the one displayed in the middle, and since naturality of whiskering implies that

$$(\varepsilon^J d)(f^J u^J \rho) = \rho(\varepsilon^J \Delta\lim)$$

which is in turn equal to

$$\rho(\Delta\varepsilon\lim)$$

we see that the diagrams are equal to the one above right. But this is a composition of two absolute liftings: the one given by the limit diagram we started with, and the whiskering of the lifting diagram given by (u, ε) with the map $\lim : D \to A$. This shows that the diagrams above are absolute liftings, which concludes our proof. □

Together with Proposition 4.10, we deduce the following.

Corollary 4.21. *Equivalences preserve both limits and colimits.*

Remark 4.22. One might observe that this looks nothing like the proof of RAPL/LAPC in the usual 1-categorical setting. There, we proceeded as follows: consider a limit cone over a diagram $d : J \to A$[2]

$$
\begin{tikzcd}
 & \lim d \arrow[dl] \arrow[d] \arrow[dr] & \\
d_1 \arrow[r] & d_2 \arrow[r] & d_3
\end{tikzcd}
$$

Apply the functor u to obtain a cone over a J-shaped diagram in B (more precisely, in the image of u)

$$
\begin{tikzcd}
 & u\lim d \arrow[dl] \arrow[d] \arrow[dr] & \\
ud_1 \arrow[r] & ud_2 \arrow[r] & ud_3
\end{tikzcd}
$$

To prove that this is a limit cone, consider another cone over it

$$
\begin{tikzcd}
 & c \arrow[dl] \arrow[dd] \arrow[dr] & \\
 & u\lim d \arrow[dl] \arrow[d] \arrow[dr] & \\
ud_1 \arrow[r] & ud_2 \arrow[r] & ud_3
\end{tikzcd}
$$

[2]For simplicity, we assume in the pictures that $J = \bullet \to \bullet \to \bullet$.

and transpose this to get a diagram in A

$$\begin{tikzcd}
 & fc \arrow[dl] \arrow[d, "\lim d"'] \arrow[dr] & \\
d_1 \arrow[r] & d_2 \arrow[r] & d_3
\end{tikzcd}$$

Then, the induced factorization $fc \dashrightarrow \lim d$ transposes to a factorization $c \dashrightarrow u\lim d$. For uniqueness, any other such factorization in B through $u\lim d$ would transpose to a different factorization in A through $\lim d$, which contradicts the assumption that $\lim d$ is a limit in A.

In the case of ∞-categories, Lemma 4.14 allows us to skip this back and forth process, but one could choose not to resort to this result and proceed in a similar way to 1-categories, by choosing a cone over the limit cone candidate and finding a unique factorization.

Remark 4.23. If instead of the statement of RAPL, where we show that right adjoints commute with limits of any families of diagrams our category has, we assume both categories involved have *all* limits, the proof becomes much easier.

Indeed, in this case, the limit functor is right adjoint to the constant diagram functor

$$\begin{tikzcd}
A^J \arrow[r, bend left, "\lim"] \arrow[r, phantom, "\top"] & A \arrow[l, bend left, "\Delta"]
\end{tikzcd}$$

and we show that $u\lim \cong \lim u^J$ by proving they both have the same left adjoint, and appealing to the uniqueness of adjoints proven in Lemma 4.9.

Since adjunctions compose, left adjoints for the aforementioned functors are given by the compositions

$$\begin{tikzcd}
A^J \arrow[r, bend left, "\lim"] \arrow[r, phantom, "\top"] & A \arrow[l, bend left, "\Delta"] \arrow[r, bend left, "u"] \arrow[r, phantom, "\top"] & B \arrow[l, bend left, "f"]
\end{tikzcd} \qquad \begin{tikzcd}
A^J \arrow[r, bend left, "u^J"] \arrow[r, phantom, "\top"] & B^J \arrow[l, bend left, "f^J"] \arrow[r, bend left, "\lim"] \arrow[r, phantom, "\top"] & B \arrow[l, bend left, "\Delta"]
\end{tikzcd}$$

and noting that the diagram

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\
A \arrow[r, "f^J"'] & A^J
\end{tikzcd}$$

commutes up to isomorphism (in fact, this is true when considering any functor in place of f) we conclude that the two left adjoints $f^J\Delta \cong \Delta f$ are isomorphic, and therefore their right adjoints must be too.

5 Comma ∞-categories (Laura)

Comma ∞-categories are one of the tools we need to talk about universal properties of adjunctions, limits, and colimits in an ∞-cosmos.

5.1 Motivation and background

Before giving the definition of a comma category in an ∞-cosmos, let's first recall the definition from classical category theory. Given a cospan $C \xrightarrow{G} A \xleftarrow{F} B$ in the 2-category $\mathscr{C}at$, the **comma category** $F \downarrow G$ is the category where

- objects are triples $(b, c, \psi : Fb \to Gc)$ with $b \in B$, $c \in C$, and $\psi \in A$
- a morphism $(b, c, \psi) \to (b', c', \psi')$ is a pair $(g : b \to b', f : c \to c')$ such that the following diagram commutes

$$\begin{tikzcd} Fb \arrow[r, "\psi"] \arrow[d, "Fg"'] & Gc \arrow[d, "Gh"] \\ Fb' \arrow[r, "\psi'"'] & Gc' \end{tikzcd}$$

More generally, in any 2-category $\mathcal{C}$ we have the notion of a comma object. Let $C \xleftarrow{g} A \xrightarrow{f} B$ be a cospan in $\mathcal{C}$. A (weak) **comma object** consists of an object $f \downarrow g$ together with morphisms $p_1 : f \downarrow g \to C$ and $p_0 : f \downarrow g \to B$ along with a 2-cell

$$\begin{tikzcd} f \downarrow g \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\varphi \swarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

such that for all objects X in $\mathcal{C}$, the map

$$\hom(X, f \downarrow g) \to \hom(X, f) \downarrow \hom(X, g)$$

is a **smothering functor**, i.e. it is surjective on objects, full, and conservative. Here, $\hom(X, f)$ denotes the functor $\hom(X, B) \to \hom(X, A)$ defined by post-composition with f, and $\hom(X, g)$ is analogous.

Notice here the weak universal property. As we'll see later, this is what will characterize the comma constructions in homotopy 2-category $\mathfrak{h}\mathscr{K}$ of an ∞-cosmos $\mathscr{K}$. (One says that $f \downarrow g$ is a *strong* comma object if the map is an isomorphism.)

Remark 5.1. The comma category $F \downarrow G$ is a comma object in $\mathcal{C} = \mathscr{C}at$

Here's an example of a smothering functor.

Lemma 5.2. *Let $\mathbb{2}$ denote $\Delta[1]$. For any quasi-category A, the functor of 1-categories*

$$\mathsf{h}(A^{\mathbb{2}}) \to (\mathsf{h}A)^{\mathbb{2}}$$

is smothering.

Proof. (Sketch) An arrow in $\mathsf{h}A$ is represented by a 1-simplex in A, so this map is surjective on objects. It is full since we can choose representatives witnessing composition. It is conservative almost by definition. □

The following lemmas entail some facts about smothering functors.

Lemma 5.3. *(Exercise 3.1.i (i) in ICWM) The class of smothering functors is closed under composition, retract, products and pullbacks.*

Lemma 5.4. *(Lemma 3.1.3 in ICWM) Each fiber of a smothering functor is a non-empty connected groupoid.*

Lemma 5.5. *(Lemma 3.1.5 in ICWM) For any pullback of quasi-categories of the form*

$$\begin{tikzcd} E \times_B A \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where p is an isofibration, the induced functor

$$\mathsf{h}(E \times_B A) \to \mathsf{h}E \times_{\mathsf{h}B} \mathsf{h}A$$

is smothering.

Lemma 5.6. *For any cospan of quasi-categories $C \xrightarrow{g} A \xleftarrow{f} B$, the canonical functor*

$$\mathsf{h}\mathsf{Hom}_A(f, g) \to \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f, \mathsf{h}g)$$

defined by the pullback

$$\begin{tikzcd} \mathsf{Hom}_A(f, g) \arrow[r] \arrow[d, "{(p_1, p_0)}"'] & A^{\mathbb{2}} \arrow[d, "{(cod, dom)}"] \\ C \times B \arrow[r, "g \times f"'] & A \times A \end{tikzcd}$$

is smothering.

5.2 Comma ∞-categories

Now we are equipped to present the notion of comma categories in an ∞-cosmos.

Definition 5.7. Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a cospan in an ∞-cosmos $\mathscr{K}$. The **comma ∞-category** $f \downarrow g$ is the pullback

$$\begin{tikzcd} f \downarrow g \arrow[r] \arrow[d, two heads, "(p_1, p_0)"'] & A^{\mathbb{2}} \arrow[d, two heads, "(p_1, p_0)"] \\ C \times B \arrow[r, "g \times f"'] & A \times A \end{tikzcd}$$

together with a specified isofibration $(p_1, p_0) : f \downarrow g \twoheadrightarrow C \times B$, which are projections onto the domain and codomain, and a canonical 2-cell

$$\begin{tikzcd} f \downarrow g \arrow[r] \arrow[d, two heads, "(p_1, p_0)"'] \arrow[dr, phantom, "\varphi \Swarrow"] & A^{\mathbb{2}} \arrow[d, two heads, "(p_1, p_0)"] \\ C \times B \arrow[r, "g \times f"'] & A \times A \end{tikzcd}$$

called the **comma cone.**

An example of a comma ∞-category is an arrow ∞-category.

Example 5.8. An **arrow ∞-category** is a comma category over a cospan of identities:

$$\begin{tikzcd} A^{\mathbb{2}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\varphi \Swarrow"] & A \arrow[d, equal] \\ A \arrow[r, equal] & A \end{tikzcd}$$

and the 2-cell is then called a "generic arrow."

5.3 Properties

Some properties of these constructions are as follows:

Proposition 5.9. *A commutative diagram*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"'] & B \arrow[l, "f"'] \arrow[d, "q"] \\ C' \arrow[r, "g'"] & A' & B' \arrow[l, "f'"'] \end{tikzcd}$$

induces a map $f \downarrow g \to f' \downarrow g'$ making the following diagram commute

$$\begin{tikzcd} f \downarrow g \arrow[r, dashed, "r \downarrow q"] \arrow[d, two heads] & f' \downarrow g' \arrow[d] \\ C \times B \arrow[r, "r \times q"] & C' \times B' \end{tikzcd}$$

Moreover, if r, p, q are all equivalences/isofibrations/trivial fibrations then so is $r \downarrow q$.

A comma ∞-category $f \downarrow g$ satisfies a strict universal property in $\mathscr{K}$, but it satisfies a weak universal property in $\mathfrak{h}\mathscr{K}$ given by the three operations in the next proposition.

Proposition 5.10.

$$\begin{tikzcd} f \downarrow g \arrow[r, two heads, "p_0"] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\varphi \Swarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

is a comma object in $\mathfrak{h}\mathscr{K}$, i.e. for all $X \in \mathscr{K}$, $\hom(X, f \downarrow g) \to \hom(X, f) \downarrow \hom(X, g)$ is smothering. Explicitly, the universal property is given by 3 operations:

1. *1-cell induction (surjective on objects). Given a 2-cell α displayed below left,*

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\alpha \Swarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, dashed, "a"'] \arrow[rr, bend left, "b"] \arrow[dr, bend right, "c"'] & f \downarrow g \arrow[r, two heads, "p_0"] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\varphi \Swarrow"] & B \arrow[d, "f"] \\ & C \arrow[r, "g"'] & A \end{tikzcd}$$

   there exists a 1-cell $a : X \to f \downarrow g$ above right so that $b = p_0 a$, $c = p_1 a$, and $\alpha = \varphi a$.

2. *2-cell induction (fullness). Given $a, a' : X \to f \downarrow g$ and τ_0, τ_1 so that*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "a'"] \arrow[dr, phantom, "\tau_1 \Swarrow"] & f \downarrow g \arrow[r, two heads, "p_0"] \arrow[d, two heads, "p_1"] \arrow[dr, phantom, "\varphi \Swarrow"] & B \arrow[d, "f"] \\ f \downarrow g \arrow[r, two heads, "p_1"'] & C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[d, "g"'] \arrow[dr, phantom, "\Nwarrow \varphi"] & f \downarrow g \arrow[l, two heads, "p_1"'] \arrow[d, two heads, "p_0"'] \arrow[dr, phantom, "\Nwarrow \tau_0"] & X \arrow[l, "a'"'] \arrow[d, "a"] \\ A & B \arrow[l, "f"] & f \downarrow g \arrow[l, two heads, "p_0"] \end{tikzcd}$$

   then there exists a 2-cell $\tau : a \Rightarrow a'$ so that

$$\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow \tau"] & f \downarrow g \arrow[r, two heads, "p_1"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "a'"'] \arrow[dr, phantom, "\tau_0 \Swarrow"] & f \downarrow g \arrow[d, two heads, "p_1"] \\ f \downarrow g \arrow[r, two heads, "p_1"'] & C \end{tikzcd}$$

   and so that

$$\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow \tau"] & f \downarrow g \arrow[r, two heads, "p_0"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "a'"'] \arrow[dr, phantom, "\tau_1 \Swarrow"] & f \downarrow g \arrow[d, two heads, "p_0"] \\ f \downarrow g \arrow[r, two heads, "p_0"'] & B \end{tikzcd}$$

3. *2-cell conservativity (convervative). Any 2-cell*

$$\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow \tau"] & f \downarrow g \end{tikzcd}$$

   with $p_0\tau$ and $p_1\tau$ isomorphisms is itself an isomorphism.

Proof. Consider the cosmological functor $\mathsf{Fun}(X,-)\colon \mathscr{K} \to q\,\mathbf{Cat}$ that carries the diagram below left

$$\begin{tikzcd} f\downarrow g \arrow[r, "\varphi"] \arrow[d, two heads] & A^{\mathbb{2}} \arrow[d, two heads] \\ C\times B \arrow[r, "g\times f"'] & A\times A \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{Fun}(X, f\downarrow g) \arrow[r] \arrow[d, two heads] & \mathsf{Fun}(X,A)^{\mathbb{2}} \arrow[d, two heads] \\ \mathsf{Fun}(X,C)\times\mathsf{Fun}(X,B) \arrow[r] & \mathsf{Fun}(X,A)\times\mathsf{Fun}(X,\times A) \end{tikzcd}$$

to the pullback diagram of quasi-categories above right. By Lemma 5.6,

$$\mathsf{hFun}(X, f\downarrow g) \to \mathsf{hFun}(X,f)\downarrow \mathsf{hFun}(X,g)$$

is smothering, and hence satisfies the three conditions. □

Proposition 5.11. *Whiskering with the comma cone φ in Definition 5.7 induces a bijection between 2-cells*

$$\left\{ \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\varphi\swarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \right\} \rightsquigarrow \left\{ \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, "a"] & B \\ C & f\downarrow g \arrow[l, two heads, "p_1"] \arrow[u, two heads, "p_0"'] \end{tikzcd} \right\}$$

where the right consists of isomorphism classes of maps of spans

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d, "c"'] \arrow[drr, bend left, "a"] \arrow[drr, bend right, "a'"'] & & B \\ C & & f\downarrow g \arrow[ll, two heads, "p_1"] \arrow[u, two heads, "p_0"'] \arrow[ull, phantom, "\gamma\cong"] \end{tikzcd}$$

such that $p_0\gamma = id_B$ and $p_1\gamma = id_C$.

Definition 5.12. A **fibered equivalence** over an ∞-category A in $\mathscr{K}$ is an equivalence

$$\begin{tikzcd} E \arrow[rr, "\simeq"] \arrow[dr] & & F \arrow[dl] \\ & A & \end{tikzcd}$$

in the slice ∞-cosmos $\mathscr{K}_{/A}$.

Proposition 5.13. *(uniqueness of comma ∞-categories) For any isofibration $E \overset{(e_1,e_0)}{\twoheadrightarrow} C\times B$ that is fibered equivalent to $f\downarrow g \overset{(p_1,p_0)}{\twoheadrightarrow} C\times B$, the 2-cell*

$$\begin{tikzcd} E \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\varepsilon\swarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

encoded by the equivalence $E \xrightarrow{\simeq} f \downarrow g$ *satisfies the weak universal property of comma* ∞*-categories. Conversely, if given two* ∞*-categories with isofibrations*

$$E \xrightarrow{(e_1, e_0)} C \times B \quad \text{and} \quad D \xrightarrow{(d_1, d_0)} C \times B$$

equipped with 2-cells ∂ *and* ε *satisfying the weak universal property of comma* ∞*-categories, then* D *and* E *are fibered equivalent over* $C \times B$*.*

These comma cones have "pullback-like" properties. For example, see Lemmas 3.4.10 and 3.4.12 of *Fibrations and Yoneda's Lemma in an* ∞*-cosmos* by Riehl and Verity as well as Lemma 3.4.8 of *Infinity Categories for the Working Mathematician.*

6 Universal properties of adjunctions and co/limits (Lynne)

6.1 Plan and motivation

In the 2-category **Cat**, the data of an adjunction $u : A \leftrightarrows B : f$ can be described in terms of a bijection between hom sets $B(b, ua) \cong A(fb, a)$, natural in a and b. Equipped now with the notion of comma categories, we can translate this data as an equivalence $B \downarrow u \cong f \downarrow A$ over $A \times B$.

Analogously, saying that a diagram $d : J \to A$ has a limit $l \in A$ is equivalent to establishing a bijection $A(a, l) \cong A^J(\Delta a, d)$ which is natural in a. With the notion of comma categories, this becomes an equivalence $A \downarrow l \cong \Delta \downarrow d$ over A.

Our first goal is to describe the universal property of adjunctions and limits of ∞-categories in terms of fibered equivalences (i.e. equivalences in a slice ∞-cosmos) of ∞-comma categories.

Furthermore, in **Cat**, if $d : J \to A$, we can describe $\lim(d)$ as a right Kan extension

$$\begin{tikzcd} J \arrow[r, "d"] \arrow[d, hook] \arrow[dr, phantom, "\Swarrow"] & A \\ J^{\triangleleft} \arrow[ur, "\mathrm{Ran}(d)"'] & \end{tikzcd}$$

where $J^{\triangleleft}$ is the category obtained from J by freely adding an initial object $\top$, and $\lim(d) = \mathrm{Ran}(d)(\top)$.

If A has all J-shaped limits, we have

$$\begin{tikzcd} A^{J^{\triangleleft}} \arrow[rr, bend left, "\mathrm{Res}"] \arrow[dr, bend right, "\mathrm{ev}_{\top}"'] & \arrow[r, phantom, "\perp"] & A^J \arrow[ll, bend left, "\mathrm{Ran}"] \arrow[dl, bend left, "\lim"] \\ & A & \end{tikzcd}$$

So our second goal is, for any $J \in sSet$, to define a cone $J^{\triangleleft} \in sSet$ and prove that there is a similar adjunction as above.

Finally, we will apply this to pullbacks and pushouts to construct a loop-suspension adjunction.

Definition 6.1. An element $t : 1 \to A$ in an ∞-category is terminal if it is right adjoint to $! : A \to 1$. Dually, we define an initial object in an ∞-category.

6.2 Universal properties

In order to discuss the universal properties mentioned, recall that in an ∞-cosmos $\mathscr{K}$, we have an adjunction $f \dashv u$ iff

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=3ex, "\Swarrow \varepsilon"] & A \end{tikzcd}$$

is an absolute right lifting diagram.

Analogously, a diagram $d : D \to A^J$ admits a limit if there exists an absolute right lifting

$$\begin{tikzcd} & A \arrow[d, "l"] \\ D \arrow[ur, "\Delta"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow \lambda"] & A^J \end{tikzcd}$$

Theorem 6.2 (3.4.6)**.** *Given functors $l : C \to B$, $f : B \to A$ and $g : C \to A$, we have a bijection between absolute right lifting diagrams*

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[ur, "g"] \arrow[r, "h"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow \lambda"] & C \end{tikzcd}$$

and isomorphism classes of diagrams

$$\begin{tikzcd} & B \downarrow l \arrow[dl] \arrow[dr] \arrow[dd, "y"] & \\ C & & B \\ & f \downarrow g \arrow[ul] \arrow[ur] & \end{tikzcd}$$

given by the map that sends λ to the unique $y : B \downarrow l \to f \downarrow g$ such that

$$
\begin{tikzcd}
 & B \downarrow l \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \psi"] & \\
C \arrow[rr, "l"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=4ex, "\Leftarrow \lambda"] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & B \downarrow l \arrow[d, "y"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & f \downarrow g \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \varphi"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

Moreover, $\lambda : fl \Rightarrow g$ is an absolute right lifting if and only if the induced $y : B \downarrow l \to f \downarrow g$ is a fibered equivalence over $C \times B$.

Before proving the theorem, we consider some consequences of it.

Since we can rephrase the adjunction property in terms of an absolute right lifting diagram, we get the proposition below as an immediate corollary.

Proposition 6.3. *We have an adjuction $u : A \leftrightarrows B : f$ if and only if we have an equivalence $B \downarrow u \simeq_{A \times B} f \downarrow A$.*

Definition 6.4. Given $a, a' : 1 \to A$, the comma ∞-category $a \downarrow a'$ is called the internal hom of A from a to a', and denoted $\mathrm{Hom}_A(a, a')$.

Corollary 6.5. *An adjuction $f \dashv u$ induces an equivalence $b \downarrow ua \simeq fb \downarrow a$ for any element $a : 1 \to A$ and $b : 1 \to B$.*

Proof. We take the pullback of the diagram

$$
\begin{tikzcd}
B \downarrow u \arrow[rr, "\simeq"] \arrow[dr] & & f \downarrow A \arrow[dl] \\
 & A \times B &
\end{tikzcd}
$$

along the map $(a, b) : 1 \to A \times B$ to get

$$
\begin{tikzcd}
b \downarrow ua \arrow[rr, "\simeq"] \arrow[dr] & & fb \downarrow a \arrow[dl] \\
 & 1 &
\end{tikzcd}
$$

□

Corollary 6.6. *$t : 1 \to A$ is terminal if and only if $p_0 : A \downarrow t \to A$ is a trivial fibration.*

Proof. By definition, t is terminal if and only if $! \dashv t$ which happens if and only if there is an equivalence

$$\begin{tikzcd} A \downarrow t \arrow[rr, "\simeq"] \arrow[dr, "p_0"'] & & ! \downarrow 1 \cong A \arrow[dl, equal] \\ & A & \end{tikzcd}$$

which happens iff p_0 is an equivalence. □

Proposition 6.7. *A diagram $d : D \to A^J$ admits a limit if and only if the ∞-category of cones over d, $\Delta \downarrow d$ is right representable, i.e. $A \downarrow l \simeq \Delta \downarrow d$, and in this case, $l : D \to A$ is the limit.*

Remark 6.8. The limit l induces a terminal element in $\Delta \downarrow d$ (see 3.4.8, 4.3.2 in ICWM).

Proof of the second part of Theorem 6.2. Suppose we have $\lambda : fl \Rightarrow g$ an absolute right lifting, then any diagram

$$\begin{tikzcd} & f \downarrow g \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

by 1-cell induction is equal to

$$\begin{tikzcd} & f \downarrow g \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\exists!}{\Leftarrow}"] & \\ C \arrow[rr, "\rho"'] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\lambda}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

which, by the correspondence is

$$\begin{tikzcd} & f \downarrow g \arrow[d, "z"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & B \downarrow \rho \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\psi}{\Leftarrow}"] & \\ C \arrow[rr, "\rho"'] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\lambda}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

which can in turn be re-written as

$$\begin{tikzcd}
 & f \downarrow g \arrow[d, "z"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & B \downarrow \rho \arrow[d, "y"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & f \downarrow g \arrow[dl] \arrow[dr] & \\
C \arrow[dr, "g"'] & \overset{\varphi}{\Leftarrow} & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

By a previous proposition, we have that $yz \simeq_{C \times B} \mathrm{id}_{f \downarrow g}$.

The other composite follows from similar arguments; this is left as an exercise.

Conversely, if we supposed that the induced map $y : B \downarrow l \to f \downarrow g$ is a fibered equivalence over $C \times B$ then

$$\begin{tikzcd}
 & B \downarrow \rho \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\psi}{\Leftarrow}"] & \\
C \arrow[rr, "\rho"'] \arrow[dr, "g"'] & \overset{\lambda}{\Leftarrow} & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

is a comma cone, and therefore we have the following two bijections. By pasting with the comma cone diagram above, we have a bijection between diagrams of the form

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
C \arrow[dr, "g"'] & \overset{\alpha}{\Leftarrow} & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

and equivalence classes of diagrams of the form

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] \arrow[dd] & \\
C & & B \\
 & B \downarrow \rho \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

By pasting wth ψ we have a bijection between the equivalence classes of diagrams of the form shown above and diagrams of the form

$$\begin{tikzcd} & X \arrow[d] \\ C \arrow[ur] \arrow[r, "\rho"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow"] & B \end{tikzcd}$$

Then, the composite bijection gives that λ is an absolute lifting diagram.

□

6.3 The ∞-category of cones and limits/colimits

For $J \in sSet$, let $J^{\triangleright} = \mathbb{1} \star J$ and $J^{\triangleleft} = J \star \mathbb{1}$. Think of these constructions as 'adding an initial element', and 'adding a terminal element',respectively.

Proposition 6.9.

$$\begin{tikzcd} & A^{J^{\triangleright}} \arrow[dl, "\mathrm{Res}"'] \arrow[dr, "\mathrm{ev}_{\top}"] & \\ A^J \arrow[dr, equal] & \Leftarrow & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A^{J^{\triangleleft}} \arrow[dl, "\mathrm{ev}_{\top}"'] \arrow[dr, "\mathrm{Res}"] & \\ A \arrow[dr] & \Leftarrow & A^J \arrow[dl, equal] \\ & A^J & \end{tikzcd}$$

are comma cones.

Proposition 6.10. *An ∞-category A admits a limits $d : D \to A^J$ if and only if there exists an absolute right lifting diagram*

$$\begin{tikzcd} & A^{J^{\triangleright}} \arrow[d, "\mathrm{Ran}"] \\ D \arrow[ur, "\mathrm{Res}"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\varepsilon \Swarrow"] & A^J \end{tikzcd}$$

Sketch of proof. By 3.5.13 in ICWM we have a bijective correspondence between absolute right lifting diagrams

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\rho"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow"] & A \end{tikzcd}$$

and absolute right lifting diagrams

$$\begin{tikzcd} & f \downarrow A \arrow[d, "p_1"] \\ C \arrow[ur, "i"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow"] & A \end{tikzcd}$$

Now, A has all limits of shape J iff we have, for each $d : D \to A^J$, an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow"] & A^J \end{tikzcd}$$

which, by the above correspondence, is the case iff we have an absolute right lifting diagram

$$\begin{tikzcd} & \Delta \downarrow A^J \arrow[d, "p_1"] \\ D \arrow[ur, "\mathrm{Ran}"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow"] & A^J \end{tikzcd}$$

but, $\Delta \downarrow A^J \cong A^{J^{\triangleright}}$, and using this replacement makes the vertical arrow Res, so we are done. $\square$

Corollary 6.11. *A admits all limits of shape J if and only if there is an adjunction*

$$\begin{tikzcd} A^{J^{\triangleright}} \arrow[r, bend left, "\mathrm{Res}"] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend left, "\mathrm{Ran}"] \end{tikzcd}$$

Proof. We know that A has all limits of shape J if and only if we have an absolute right lifting

$$\begin{tikzcd} & A^{J^{\triangleright}} \arrow[d, "\mathrm{Ran}"] \\ D \arrow[ur, "\mathrm{Res}"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\varepsilon \Swarrow"] & A^J \end{tikzcd}$$

But this gives the counit of the adjunction via the process described earlier. $\square$

We can now define pushouts and pullbacks:

Definition 6.12. A pushout is a colimit of a diagram of shape $\ulcorner = \Lambda^0[2]$.

Dually, a pullback is a limit of a diagram of shape $\lrcorner = \Lambda^2[2]$.

Now we can see that, since $\ulcorner^{\triangleleft} \cong \square \cong \lrcorner^{\triangleright}$ we have adjunctions

$$\begin{tikzcd} A^{\lrcorner} \arrow[r, bend left, "\mathrm{Res}"] \arrow[r, phantom, "\perp"] & A^{\square} \arrow[l, bend left, "\mathrm{Ran}"] \arrow[r, bend left, "\mathrm{Lan}"] \arrow[r, phantom, "\perp"] & A^{\ulcorner} \arrow[l, bend left, "\mathrm{Res}"] \end{tikzcd}$$

iff A has all pushouts and pullbacks.

6.4 The loop-suspension adjunction

This section makes sense for pointed ∞-categories: a pointed ∞-category A is one that admits a zero object $*\colon 1 \to A$, i.e. $*$ is both terminal and initial. We can see that, in that case, since $*$ is initial, we have an adjunction $* \dashv !$, whose counit $\rho : * ! \Rightarrow \mathrm{id}_A$ can be seen as a map $\overline{\rho} : A \to A^2$ sending a to the arrow $* \to a$. Also, since $*$ is terminal, we have an adjunction $! \dashv *$, whose unit $g : id_A \Rightarrow * !$ corresponds to a map $\overline{g} : A \to A^2$ sending a to the arrow $a \to *$. Then we can form the map $\widetilde{\rho}$ as follows:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\widetilde{\rho}"] \arrow[drr, bend left, "\overline{\rho}"] \arrow[ddr, bend right, "\overline{\rho}"'] & & \\
& A^{\lrcorner} \arrow[r] \arrow[d] & A^2 \arrow[d, two heads, "p_1"] \\
& A^2 \arrow[r, two heads, "p_1"'] & A
\end{tikzcd}$$

and similarly for $\widetilde{g}$.

Then, the loop functor Ω is defined as the limit

$$\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
A \arrow[ur, "\Omega"] \arrow[r, "\widetilde{\rho}"'] \arrow[ur, phantom, shift right=3ex, "\swarrow"] & A^{\lrcorner}
\end{tikzcd}$$

and analogously, we define the suspension functor Σ as the colimit

$$\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
A \arrow[ur, "\Sigma"] \arrow[r, "\widetilde{g}"'] \arrow[ur, phantom, shift right=3ex, "\nearrow"] & A^{\ulcorner}
\end{tikzcd}$$

Proposition 6.13. *If A has all pullbacks and pushouts, then $\Omega : A \leftrightarrows A : \Sigma$.*

Sketch of proof. The adjunctions

$$\begin{tikzcd}
A^{\lrcorner} \arrow[r, bend left, "\mathrm{Res}"] \arrow[r, phantom, "\perp"] & A^{\square} \arrow[l, bend left, "\mathrm{Ran}"] \arrow[r, bend left, "\mathrm{Lan}"] \arrow[r, phantom, "\perp"] & A^{\ulcorner} \arrow[l, bend left, "\mathrm{Res}"]
\end{tikzcd}$$

live over $A \times A$. Composing the two adjunctions and pulling back along $(*, *) : 1 \to A \times A$ gives the result. $\square$

7 Homotopy coherent adjunctions and monads (Martina)

7.1 Ideas about homotopy coherence

One can make sense of	in a 2-category ($h\mathcal{K}$)	in a simplicial category ($\mathcal{K}$)
adjunctions	adjunctions in $h\mathcal{K}$	homotopy coherent adjunctions in $\mathcal{K}$
monads	monads in $h\mathcal{K}$	homotopy coherent monads in $\mathcal{K}$
	(easier)	(meaningful)

Interestingly, the notions in the first row coincide, whereas the notions in the second row don't.

7.2 Adjunctions and monads in a 2-category

Definition 7.1. Let **Adj** be the 2-category with objects $\{+,-\}$ such that:

$$\begin{aligned}\mathbf{Adj}(+,+) &= \Delta_+\\ \mathbf{Adj}(-,-) &= \Delta_-\\ \mathbf{Adj}(+,-) &= \Delta_\perp\\ \mathbf{Adj}(-,+) &= \Delta_\top.\end{aligned}$$

Here Δ_+ represents the category of finite ordinals and order-preserving maps, $\Delta_\perp$ represents the category of finite non-empty ordinals with bottom- and order-preserving maps. The categories Δ_+ and $\Delta_\perp$ have dual descriptions. Composition is given by ordinal sum.

Definition 7.2. We define **Mnd** as the full 2-sub-category of **Adj** generated by $+$.

Remark 7.3. The 2-category **Adj** contains the adjunction:

$$[0] : + \rightleftarrows - : [0].$$

Proposition 7.4. *The* 2*-category* **Adj** *represents adjunctions. Formally, for every* 2*-category* C*, we have a natural equivalence:*

$$\mathbf{Adj}(C) \cong 2\text{-}\,\mathbf{Cat}(\mathbf{Adj}, C).$$

Proof. On the one hand, given a functor $\mathbf{Adj} \to C$ we have the adjunction given by its image.

On the other hand, given a 2-categorical ajunction $u : A \rightleftarrows B : f$ in C, with unit and counit η and ε, define:

$$\begin{aligned}F : \mathbf{Adj} &\to C\\ + &\mapsto B\\ - &\mapsto A\end{aligned}$$

$$
\begin{aligned}
F : \mathbf{Adj}(+,+) &\to C(B,B) \\
[-1] &\mapsto \mathrm{id}_B \\
[0] &\mapsto uf \\
[1] &\mapsto ufu \\
&\vdots
\end{aligned}
$$

with the various maps given by units and counits. And similarly for the other hom-categories. □

Definition 7.5. A **monad** in C consits of an arrow $t : B \to B$ together with 2-cells $\mu : t^2 \Rightarrow \mathrm{id}_B$ and $\eta : \mathrm{id}_B \Rightarrow t$ satisfying:

$$
\begin{tikzcd}
B \arrow[r] & B \arrow[dr] \arrow[d, phantom, "\Downarrow\mu"] & \\
B \arrow[u] \arrow[ur] \arrow[rr] \arrow[ur, phantom, "\Downarrow\mu"] & & B
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r] \arrow[dr] & B \arrow[d] \\
B \arrow[ur] \arrow[r] \arrow[ur, phantom, "\Downarrow\mu"] & B,
\end{tikzcd}
$$

$$
\begin{tikzcd}
& B \arrow[dr] \arrow[d, phantom, "\Downarrow\mu"] & \\
B \arrow[ur, bend left] \arrow[ur, bend right, "\Downarrow\eta"] \arrow[rr] & & B
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[loop left, "t"] \arrow[loop right, "t"] \arrow[d, phantom, "\Leftarrow\mathrm{id}"] \\
B,
\end{tikzcd}
$$

and the dual right unitary condition.

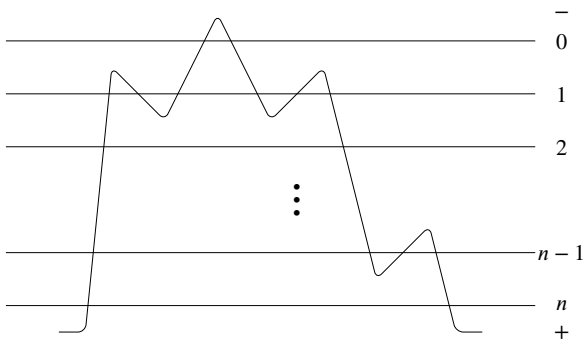
Remark 7.6. Analogously to **Adj**, the 2-category **Mnd** represents monads.

7.3 Homotopy coherent adjunctions and monads

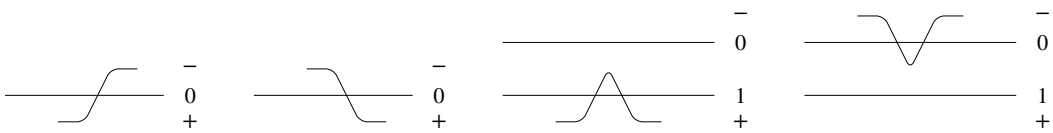
Let $\mathcal{K}$ denote a simplicial category. Given a 2-category C, there is a simplicial category N_*C with the same objects as C mapping spaces given by $N_*C(c,d) = N(C(c,d))$.

Definition 7.7. A **homotopy coherent adjunction** (resp. **monad**) in $\mathcal{K}$ is a simplicial functor $N_*\mathbf{Adj} \to \mathcal{K}$ (resp. $N_*\mathbf{Mnd} \to \mathcal{K}$).

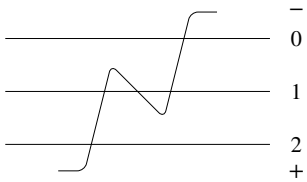
We want to give a better description of $N_*\mathbf{Adj}$ and $N_*\mathbf{Mnd}$. For this we define a simplicial category $\tilde{\mathbf{Adj}}$ with objects $\{+,-\}$, and $\tilde{\mathbf{Adj}}(+,+)_n$ given by "strictly ondulating squiggles over $(n+1)$ lines that start and end at the bottom". The face operations are given by removing lines. Degeneracies are given by duplicating lines. Composition is given by sticking squiggles one next to the other. As an example, consider the following squiggle, represented by the string $(+, n-1, n, 1, 2, -, 2, 1, +)$:



The other hom-spaces have analogous descriptions, but starting and ending at different places. Let us now give a name to some important squiggles in $\tilde{\mathbf{Adj}}$. The squiggles



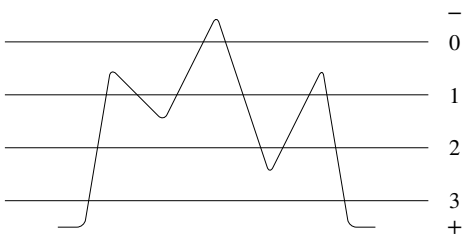
correspond to the arrows $\underline{u}$, $\underline{f}$, and 2-arrows $\underline{\eta}$, and $\underline{\varepsilon}$, respectively. As another example, the squiggle



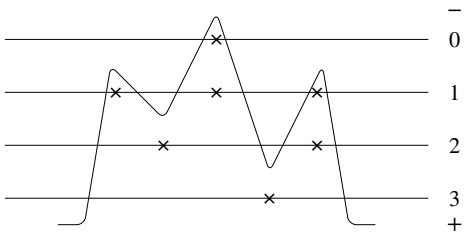
corresponds to one of the triangular equalities.

Proposition 7.8. *We have* $\tilde{\mathbf{Adj}} \simeq N_*\mathbf{Adj}$. □

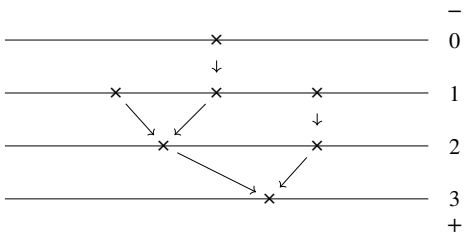
Let us give an idea of how this equivalence works. We show how to map a specific squiggle in $\tilde{\mathbf{Adj}}(+,+)_3$ to an element in $N_*\mathbf{Adj}(+,+)_3 = (N\Delta_+)_3$. Given the squiggle



we can mark the following points



to obtain a diagram



which we can interpret as a chain $[0] \to [2] \to [1] \to [0]$ of morphisms in Δ_+.

Similarly, there is a description of $N_*\mathbf{Mnd}$ in terms of a simplicial category $\tilde{\mathbf{Mnd}}$.

Observe that given a homotopy coherent adjunction $\mathbb{A} : N_*\mathbf{Adj} \to \mathcal{K}$ we have in particular $A = \mathbb{A}(-)$, $B = \mathbb{A}(+)$, $u = \mathbb{A}(\underline{u})$, and $f = \mathbb{A}(\underline{f})$, and similarly for η, ε, and the triangle equalities. So we have the following observation.

Remark 7.9. Any homotopy coherent adjunction in $\mathcal{K}$ induces an adjunction in the homotopy category $h\mathcal{K}$.

In this case, more is true.

Theorem 7.10. *Any adjunction in $h\mathcal{K}$ can be lifted to a homotopy coherent adjunction in $\mathcal{K}$.* □

This is not true for monads: although every homotopy coherent monad in $\mathcal{K}$ induces a monad in $h\mathcal{K}$, not every monad in $h\mathcal{K}$ lifts to a homotopy coherent monad in $\mathcal{K}$.

8 Homotopy coherent monads and descent (Kyle)

The aim of this talk is to give analogues of some constructions and results in classical category theory involving monads in the context of ∞-cosmoi.

Let **Adj** denote the 2-category which classifies adjunctions and **Mnd** the 2-category which classifies monads. We will consider these also as simplicial categories by taking the nerves of Hom-categories.

8.1 The classical picture

Let $T : \mathbf{Mnd} \to \mathbf{Cat}$ be a monad. Let X denote its "underlying category", $t : X \to X$ the corresponding endofunctor and $\mu : tt \Rightarrow t$ respectively $\eta : \mathrm{id}_X \Rightarrow t$ the multiplication respectively the unit.

Definition 8.1. A *T-algebra* is an object $x \in X$ equipped with a morphism $\rho : tx \to x$ such that the diagrams

$$\begin{tikzcd} ttx \arrow[r, "\mu"] \arrow[d, "t\rho"'] & tx \arrow[d, "\rho"] \\ tx \arrow[r, "\rho"'] & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & tx \arrow[dr, "\rho"] & \\ x \arrow[ur, "\eta"] \arrow[rr, equal] & & x \end{tikzcd}$$

commute.

A *morphism of T-algebras* is a morphism $\varphi : x \to y$ in X such that the diagram

$$\begin{tikzcd} tx \arrow[r, "t\varphi"] \arrow[d, "\rho_x"'] & ty \arrow[d, "\rho_y"] \\ x \arrow[r, "\varphi"'] & y \end{tikzcd}$$

commutes.

The *category of T-algebras* or the *Elienberg–Moore object of T* has T-algebras as objects and morphisms of T-algebras as morphisms. We will denote it by X^t.

The Eilenberg–Moore object has the following nice properties.

Proposition 8.2. *There is an adjunction $f^t : X \leftrightarrows X^t : u^t$ where on objects, the left adjoint is given by $f^t(x) = (ttx \xrightarrow{\mu_x} tx)$ and the right adjoint by $u^t(ty \xrightarrow{\rho} y) = y$.*

Proposition 8.3. *Suppose that $f : X \leftrightarrows A : u$ is an adjunction such that $uf = t$, η is the unit of the adjunction and $\mu = u\varepsilon f$ where ε is the counit of the adjunction. Then there is a unique "morphism of adjunctions" from (f, u) to (f^t, u^t) whose underlying functor $K : A \to X^t$ is given on objects by $K(a) = (tua = ufua \xrightarrow{(u\varepsilon)_a} ua)$.*

We would like to construct an analogue of the Eilenberg–Moore object for a homotopy coherent monad $T : \mathbf{Mnd} \to \mathcal{E}$ in an ∞-cosmos $\mathcal{E}$. For this, we will construct a right Kan extension

$$\begin{tikzcd} \mathbf{Mnd} \arrow[r, "T"] \arrow[d, hook] & \mathcal{E} \\ \mathbf{Adj} \arrow[ur, dashed] & \end{tikzcd}$$

via weighted limits which will be the analogue of the adjunction $X \leftrightarrows X^t$.

8.2 Weighted limits

Let $\mathcal{V}$ be a (co)complete closed symmetric monoidal category. Let D, $\mathcal{E}$ be $\mathcal{V}$-enriched categories where D is small. Let $F : D \to \mathcal{E}$ and $W : D \to \mathcal{V}$ be $\mathcal{V}$-enriched functors. From now on, everything (functors, limits etc.) will be $\mathcal{V}$-enriched even if it is not explicitly stated.

Definition 8.4. A *limit of F with weight W* is an object $\{W, F\}_D \in \mathcal{E}$ such that there is an isomorphism

$$\mathcal{V}^D(W, \mathcal{E}(e, F(\square))) \cong \mathcal{E}(e, \{W, F\}_D)$$

that is natural in $e \in \mathcal{E}$.

Weighted limits generalize some familiar constructions in the following sense.

Remark 8.5. If W is the constant functor on an object $v \in V$ of $\mathcal{V}$, the previous definition specializes to a cotensor of a limit:

$$\begin{aligned} \mathcal{E}(e, \{\mathrm{c}_v, F\}_D) &\cong \mathcal{V}^D(\mathrm{c}_v, \mathcal{E}(e, F(\square))) \\ &\cong \mathcal{V}(v, \lim \mathcal{E}(e, F(\square))) \cong \mathcal{V}(v, \mathcal{E}(e, \lim F(\square))) \cong \mathcal{E}(e, \lim F(\square)^v). \end{aligned}$$

Thus limits are special cases of weighted limits where the weight is the constant functor on the monoidal unit and cotensors are special cases of weighted limits where both the functor F and the weight W are constant.

Remark 8.6. If the weight W is a representable functor of the form $D(d, \square)$, then $\{W, F\}_D \cong F(d)$ by the Yoneda lemma:

$$\mathcal{E}(e, \{D(d, \square), F\}_D) \cong \mathcal{V}^D(D(d, \square), \mathcal{E}(e, F(\square))) \cong \mathcal{E}(e, F(d)).$$

The following fact will allow us to define the aforementioned Kan extension.

Remark 8.7. Given functors $i : C \to C'$ and $H : C \to \mathcal{D}$, the right Kan extension of H along i can be computed by the formula

$$\mathrm{Ran}_i\, H(c') = \{C'(c', i(\square)), H\}_C.$$

Thus, given a homotopy coherent monad $T : \mathbf{Mnd} \to \mathcal{E}$ in an ∞-cosmos, we would like to define its "Eilenberg–Moore object", i.e. its right Kan extension along the inclusion $j : \mathbf{Mnd} \to \mathbf{Adj}$ as the composite

$$\mathrm{Ran}_j T : \left(\mathbf{Adj}^{\mathrm{op}}\right)^{\mathrm{op}} \xrightarrow{\text{Yoneda embedding}} \left(\mathbf{sSet}^{\mathbf{Adj}}\right)^{\mathrm{op}} \xrightarrow{j^* : W \mapsto W \circ j} \left(\mathbf{sSet}^{\mathbf{Mnd}}\right)^{\mathrm{op}} \xrightarrow{\{\square, T\}_{\mathbf{Mnd}}} \mathcal{E}.$$

However, a priori it is not clear whether the required weighted limits exist in $\mathcal{E}$.

Recall that $\mathrm{ob}\,\mathbf{Adj} = \{+, -\}$, so we want to show that weighted limits with weight $\mathbf{Adj}(+, j(\square))$ and $\mathbf{Adj}(-, j(\square))$ exist. Note that $j : \mathbf{Mnd} \hookrightarrow \mathbf{Adj}$ is fully faithful with $j(+) = +$, which implies that $\mathbf{Adj}(+, j(\square)) \cong \mathbf{Mnd}(+, +)$ is a representable weight. Hence weighted limits with this weight exist by 8.6 and in particular we have

$$\mathrm{Ran}_j T(+) = \{\mathbf{Adj}(+, j(\square)), T\}_{\mathbf{Mnd}} \cong \{\mathbf{Mnd}(+, \square), T\}_{\mathbf{Mnd}} \cong T(+).$$

The case of $\mathbf{Adj}(-, j(\square))$ is trickier and we will appeal to a general result about the existence of weighted limits with certain kinds of weights in ∞-cosmoi to show that weighted limits with this weight exist.

Definition 8.8. A weight $W : D \to \mathbf{sSet}$ is called *flexible* if it is projectively cofibrant, i.e. if for every natural transformation $W \to Q$ in $\mathbf{sSet}^D$ and every natural transformation $P \overset{\sim}{\twoheadrightarrow} Q$ in $\mathbf{sSet}^D$ that is an objectwise acyclic fibration (in $\mathbf{sSet}_{\mathrm{Quillen}}$ or $\mathbf{sSet}_{\mathrm{Joyal}}$), the lifting problem

$$\begin{tikzcd} & P \arrow[d, two heads, "\sim"] \\ W \arrow[r] \arrow[ur, dashed] & Q \end{tikzcd}$$

has a solution.

Theorem 8.9. *∞-cosmoi admit all flexibly weighted limits.*

Proof sketch. We first observe that projectively cofibrant weights are cell complexes with cells of the form

$$\partial\Delta[n] \times D(d, \square) \hookrightarrow \Delta[n] \times D(d, \square).$$

Weighted limits with weights of the form $A \times D(d, \square)$ for some $A \in \mathbf{sSet}$ and $d \in D$ exist in every ∞-cosmos since, like the weighted limits in 8.5 and 8.6, they have the universal property of a cotensor:

$$\{A \times D(d, \square), F\}_D \cong F(d)^A.$$

Now one can do induction on the cell structure of W because the colimits in the cell structure (coproducts, pushouts and transfinite compositions) translate to certain kinds of limits (products, pullbacks and inverse limits) which exist in an ∞-cosmos. $\square$

There is an inclusion $\mathbf{Adj}(+, j(\square)) \hookrightarrow \mathbf{Adj}(-, j(\square))$ which is induced by "precomposing with the right adjoint of the free-living adjunction $+ \leftrightarrows -$ in $\mathbf{Adj}$" such that $\mathbf{Adj}(-, j(\square))$ can be obtained from $\mathbf{Adj}(+, j(\square))$ by attaching cells of the form $\partial\Delta[n] \times \mathbf{Mnd}(+, \square) \hookrightarrow \Delta[n] \times \mathbf{Mnd}(+, \square)$. Thus, by the description of projectively cofibrant weights in terms of cell complexes, $\mathbf{Adj}(-, j(\square))$ is projectively cofibrant (i.e. flexible) as $\mathbf{Adj}(+, j(\square))$ is. Hence, by the previous theorem, weighted limits with weight $\mathbf{Adj}(-, j(\square))$ exist in every ∞-cosmos.

All in all, the right Kan extension $\mathrm{Ran}_j T : \mathbf{Adj} \to \mathcal{E}$ of a homotopy coherent monad $T : \mathbf{Mnd} \to \mathcal{E}$ along the inclusion $j : \mathbf{Mnd} \hookrightarrow \mathbf{Adj}$ does exist, and we can define the *∞-category of T-algebras* as $\mathrm{Ran}_j T(-) \in \mathcal{E}$ which comes with a preferred homotopy coherent adjunction

$$T(+) \cong \mathrm{Ran}_j T(+) \leftrightarrows \mathrm{Ran}_j T(-).$$

8.3 Comparing $\mathrm{Ran}_j T$ with other adjunctions

Let $T : \mathbf{Mnd} \to \mathcal{E}$ be a homotopy coherent monad. We will denote by $X := T(+)$ the "underlying ∞-category of T" and by $X^t := \mathrm{Ran}_j T(-)$ the ∞-category of T-algebras. Further let $f : X \leftrightarrows A : u$ be given by a homotopy coherent adjunction $\mathbf{Adj} \to \mathcal{E}$ whose associated monad (i.e. restriction to $\mathbf{Mnd}$) is T. Then, similar to the classical case, one can construct a "comparison functor" $K : A \to X^t$. We will now investigate properties of this functor.

We start with a criterion for determining when K admits a left adjoint.

Definition 8.10. Given a functor $w : C \to Z$ between ∞-categories, the *∞-category of w-split simplicial objects in C* is defined as the pullback

$$\begin{tikzcd} S(w) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Z^{\Delta_\top} \arrow[d, two heads, "Z^{[0]\oplus\square}"] \\ C^{\Delta^{\mathrm{op}}} \arrow[r, "w^{\Delta^{\mathrm{op}}}"'] & Z^{\Delta^{\mathrm{op}}} \end{tikzcd}.$$

Theorem 8.11. *$K : A \to X^t$ admits a left adjoint if A has colimits of u-split simplicial objects, i.e. if there is an absolute left lifting diagram*

$$\begin{tikzcd} & A \arrow[d, "c_\square"] \\ S(u) \arrow[ur] \arrow[r, two heads] \arrow[ur, phantom, shift right=3ex, "\Nearrow"] & A^{\Delta^{\mathrm{op}}} \end{tikzcd}.$$

Next, we will describe fully faithfulness of K in terms of cocompleteness with respect to certain resolutions. Note that similar to the classical case, the homotopy coherent comonad $g := fu : A \to A$ gives rise to *comonadic resolutions* in A which are, morally speaking, augmented simplicial objects of the form

$$a \longleftarrow fua \leftleftarrows fufua \Lleftarrow \cdots$$

where the degeneracy maps are induced by the "comultiplication" $fu \to fufu$ and the face maps are induced by the "counit" $fu \to \mathrm{id}_A$. This construction yields a functor $g_\star : A \to A^{\Delta_+^{\mathrm{op}}}$. We will denote by $g_\bullet : A \to A^{\Delta^{\mathrm{op}}}$ the composite of $g_\star$ with the restriction along the opposite of the inclusion $\iota : \Delta \hookrightarrow \Delta_+$, i.e. "the simplicial part of the augmented simplicial object $g_\star$".

Definition 8.12. A generalized element $a : D \to A$ is called *fu-cocomplete* if the diagram

$$\begin{tikzcd}
 & & & A \arrow[d, "\mathrm{c}_\square"] \\
X \arrow[r, "a"'] & A \arrow[r, "g_\star"'] \arrow[urr, bend left, equal] & A^{\Delta_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "A^{\iota^{\mathrm{op}}}"'] \arrow[ur, phantom, "\Nearrow A^{v^{\mathrm{op}}}"] & A^{\Delta^{\mathrm{op}}}
\end{tikzcd},$$

where the natural transformation $v : \mathrm{c}_{[-1]} \Rightarrow \iota$ is induced by the initiality of $[-1]$ in Δ_+, is an absolute left lifting diagram.

Roughly speaking, a is fu-complete if the natural morphism $a \to \operatorname{colim} g_\bullet a$ is an equivalence.

In our result about fully faithfulness we will consider a "local" variant:

Definition 8.13. Given a functor $H : C \to Z$ between ∞-categories and an element $c : 1 \to C$, H is called *fully faithful at c* if the induced functor $(c \downarrow C) \to (Hc \downarrow H)$ is an equivalence.

This notion can be thought of as a generalization of the condition that the induced map $C(c, c') \to Z(Hc, Hc')$ on Hom-spaces is an equivalence.

Theorem 8.14. *$K : A \to X^t$ is a fully faithful at an element $a : 1 \to A$ if and only if a is fu-cocomplete.*

Finally, we state an analogue of Beck's monadicity theorem in our context:

Theorem 8.15. *$K : A \to X^t$ is the right adjoint of an adjoint equivalence if*

i) A admits colimits of u-split simplicial objects,

ii) $u : A \to X$ preserves colimits of u-split simplicial objects,

iii) $u : A \to X$ is conservative.

9 Cartesian Fibrations (Paul)

10 Two-sided fibrations and modules (Daniel)

10.1 Actual modules

Let's start with some modules over actual rings.

Definition 10.1. A unital ring is an object of an **Ab**-enriched category.

Definition 10.2. Given R, S two unital rings, a bimodule is a functor of the form

$$M : R \otimes S^{\mathrm{op}} \to \mathbf{Ab}\,.$$

We call this a profunctor.

Theorem 10.3. *There is an equivalence of categories between the bimodules and the discrete two-sided fibrations.*

We will come back to what a discrete two-sided fibration is. Don't worry about that.

10.2 Two-sided fibrations

The idea of a two-sided fibration is something like this:

$$A \overset{p}{\twoheadleftarrow} X \overset{q}{\twoheadrightarrow} B$$

where p is cartesian and q is cocartesian.

Definition 10.4. A span $A \overset{p}{\longleftarrow} E \overset{q}{\longrightarrow} B$ is cocartesian on the left if one of these conditions is true:

1. q is a cocartesian fibration and whiskering q-cocartesian cells with p gives invertible cells.
2. The functor in the slice category $(p, q) : E \longrightarrow A \times B$

$$\begin{tikzcd} E \arrow[rr, "(p,q)"] \arrow[dr, "p"'] & & A \times B \arrow[dl, "\pi_B"] \\ & B & \end{tikzcd}$$

   is a cocartesian fibration in K over B.

3. The functor in the slice category $(p, q) : E \longrightarrow A \times B$

$$\begin{tikzcd} E \arrow[rr, "(p,q)"] \arrow[dr, "p"'] & & A \times B \arrow[dl, "\pi_A"] \\ & A & \end{tikzcd}$$

   is a cocartesian functor in K over A.

And what is a cocartesian functor?

Definition 10.5. A (co)cartesian functor is a functor that preserves (co)cartesian cells.

Remark 10.6. In the case of K over A (see the definition of a cocartesian span), the cocartesian functor makes q-cocartesian cells into π_A-cocartesian cells.

Notation 10.7. We will denote the category of cocartesian functors in K by $\mathrm{cocart}(K)$, and the category of cocartesian functors in K over A by $\mathrm{cocart}(K)/A$.

Lemma 10.8. *A span $A \xleftarrow{p} E \xrightarrow{q} B$ is cocartesian on the left and cartesian on the right if the functor in the slice category $(p,q) : E \longrightarrow A \times B$*

$$\begin{tikzcd} E \arrow[rr, "(p,q)"] \arrow[dr, "p"'] & & A \times B \arrow[dl, "\pi_A"] \\ & A & \end{tikzcd}$$

is a cocartesian functor in K over A and a cartesian fibration in K over A.

Definition 10.9. A span $A \xleftarrow{p} E \xrightarrow{q} B$ is a two-sided fibration if the functor in the slice category $(p,q) : E \longrightarrow A \times B$ is a cocartesian functor in K over A and a cartesian fibration in $\mathrm{cocart}(K)$ over A. That means that whenever we have the following diagram

$$\begin{tikzcd} A & E \arrow[l, "q"'] \arrow[r, "p"] & B \\ & X \arrow[ul, "a"] \arrow[u, "e"] \arrow[ur, "b"'] \arrow[ul, phantom, shift left=3, "\overset{\alpha}{\Leftarrow}"] \arrow[ur, phantom, shift right=3, "\overset{\beta}{\Leftarrow}"] & \end{tikzcd},$$

if we form the p-Cartesian lifts $\alpha_* e$ and $\beta^* e$

$$\begin{tikzcd} A & E \arrow[l, "q"'] \arrow[r, "p"] & B \\ & X \arrow[ul, bend left, "\alpha_* e"] \arrow[u, "e"] \arrow[ur, "b"'] \arrow[ul, phantom, shift left=3, "\overset{\beta}{\Leftarrow}"] \arrow[ul, phantom, shift right=3, "\Leftarrow"] & \end{tikzcd} \qquad \begin{tikzcd} A & E \arrow[l, "q"'] \arrow[r, "p"] & B \\ & X \arrow[ul, "a"] \arrow[u, "e"] \arrow[ur, bend left, "\beta^* e"] \arrow[ul, phantom, shift left=3, "\overset{\alpha}{\Leftarrow}"] \arrow[ur, phantom, shift right=3, "\Leftarrow"] & \end{tikzcd}$$

we have that $\beta^* \alpha_* e \cong \alpha_* \beta^* e$.

Example 10.10. The span

$$\begin{tikzcd} & A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & A \end{tikzcd}$$

is a two-sided fibration.

Definition 10.11. We say that $f : E \to F$ is a cartesian functor of two-sided fibrations if it is a cocartesian functor for r, s and a cartesian functor for p, q.

$$\begin{tikzcd}
 & E \arrow[dl, "r"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\
A & & B \\
 & F \arrow[ul, "s"] \arrow[ur, "q"'] &
\end{tikzcd}$$

Remark 10.12. $\mathrm{cocart}(\mathrm{cart}(K)/B)_{A\times B\to B} \cong \mathrm{cart}(\mathrm{cocart}(K)/A)_{A\times B\to A}$ is the cosmos of two-sided fibrations from A to B.

Notation 10.13. The cosmos of two-sided fibrations from A to B is denoted by $A \setminus Fib(K)/B$.

Proposition 10.14. *The functor $A \setminus Fib(K)/B \hookrightarrow K/A \times B$ creates the cosmos structure.*

Lemma 10.15. *Two-sided fibrations are pullback stable. In particular, comma ∞-categories with projections are two-sided fibrations.*

Lemma 10.16. *If we have $A \xleftarrow{p} E \xrightarrow{q} B$ a two-sided fibration, $A \to A'$ cocartesian and $B \to B'$ cartesian, then the composed span $A' \leftarrow E \longrightarrow B'$ obtained like this*

$$\begin{tikzcd}
 & & E \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl] & & B \arrow[dr] & \\
A' & & & & B'
\end{tikzcd}$$

is still a two-sided fibration.

Lemma 10.17. *Two-sided fibrations compose as follows. Let $A \leftarrow E \longrightarrow B$ and $B \leftarrow F \longrightarrow C$ be two-sided fibrations. If we take the pullback*

$$\begin{tikzcd}
 & & E \times_B F \arrow[dl] \arrow[dr] & & \\
 & E \arrow[dl] \arrow[dr] & & F \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

, then $A \leftarrow E \times_B F \longrightarrow C$ is still a two-sided fibration.

10.3 Modules

Definition 10.18. A module is a discrete two-sided fibration in $K/A \times B$.

Definition 10.19. An object $E \in K$ is called discrete if every two-cell with codomain E is invertible.

Lemma 10.20. *An isofibration $A \overset{p}{\twoheadrightarrow} B$ is discrete in K/B iff for any $X \in K$ and any two-cell α between maps from X to A,*

$$\begin{tikzcd} X \arrow[r, bend left=50, ""{name=U}] \arrow[r, bend right=50, ""{name=D}] & A \arrow[d, "p"] \\ & B \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

if $p\alpha$ is invertible, then α is invertible.

Definition 10.21. We say that a span $A \leftarrow E \rightarrow B$ is a module iff it has the following properties:

1. It is cocartesian on the left.
2. It is cartesian on the right.
3. It is discrete in $K/A \times B$.

Notation 10.22. We will write modules not as $A \leftarrow E \rightarrow B$, but as $E \twoheadrightarrow A \times B$.

Lemma 10.23. *Modules are pullback stable.*

Remark 10.24. Attention!! This last lemma can be misleading. If $A \leftarrow E \rightarrow B$ and $B \leftarrow F \rightarrow C$ are modules, that doesn't mean that the pullback

$$\begin{tikzcd} & & E \times_B F \arrow[dl] \arrow[dr] & & \\ & E \arrow[dl] \arrow[dr] & & F \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

is still a module (actually, it isn't necesarily one). What does the lemma mean, then? It means that if $E \twoheadrightarrow A \times B$ is a module and we take the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] \\ A' \times B' \arrow[r] & A \times B \end{tikzcd}$$

, then $X \twoheadrightarrow A' \times B'$ is still a module.

Example 10.25. The span

$$\begin{tikzcd} & A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & A \end{tikzcd}$$

is a module.

11 The Calculus of Modules (Matt)

11.1 Definitions

Definition 11.1. A virtual double category has the following data:

- a category of vertical morphisms;
- a class of horizontal morphisms (these don't have composition, but are between the same objects as in the vertical category);
- 2-cells of the following shape:

$$\begin{tikzcd} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, phantom, "\Downarrow"] & \arrow[d] \\ \arrow[rr] & & \end{tikzcd}$$

  We can also have two cells whose "domain" is the "empty" horizontal arrow and this will be denoted:

$$\begin{tikzcd} & \arrow[d, equal] & \\ \arrow[d] & \arrow[d, phantom, "\Downarrow"] & \arrow[d] \\ & \arrow[r] & \end{tikzcd}$$

- For any horizontal arrow we have an identity 2-cell with convenient cancellation properties:

$$\begin{tikzcd} & \arrow[r] & \\ \arrow[d, equal] & & \arrow[d, equal] \\ & \arrow[r] & \end{tikzcd}$$

- These 2-cells 'compose' in the following associative way:

$$\begin{tikzcd} \arrow[r] \arrow[d] & \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \arrow[r] \arrow[d] & \arrow[r] \arrow[d, phantom, "\Downarrow"] & \arrow[r] \arrow[d] & \\ \arrow[r] \arrow[d] & \arrow[rr] \arrow[d, phantom, "\Downarrow"] & & \arrow[rr] & & \arrow[d] \\ \arrow[rrr] & & & & & \end{tikzcd} = \begin{tikzcd} \arrow[r] \arrow[d] & \arrow[r] & \arrow[r] & \arrow[r] & \arrow[r] & \\ \arrow[rrr] \arrow[d] & & \arrow[d, phantom, "\Downarrow"] & & & \arrow[d] \\ & & & & & \end{tikzcd}$$

Example 11.2. Modules over rings can be interpreted in this context: the objects are rings, horizontal arrows are modules, vertical arrows are ring homomorphisms, and the 2-cells are module maps.

Example 11.3. If C is a category with all pullbacks, then $Span(C)$ is a virtual double category: the objects are the objects of C, the horizontal arrows are spans, the vertical arrows are maps in C, and 2-cells are diagrams of the form:

$$
\begin{tikzcd}
 & & B \times_{A'} B' \arrow[dl] \arrow[dr] & & \\
 & B \arrow[dl] \arrow[dr] & & B' \arrow[dl] \arrow[dr] & \\
A \arrow[d] & & A' & & A'' \arrow[d] \\
C & & D \arrow[ll] \arrow[rr] & & C
\end{tikzcd}
$$

How would one recover an idea of composition of horizontal arrows in this context? We will want to use a 'mix' of these two example styles. This leads us to the notions of cocartesian cells:

Definition 11.4. A cocartesian cell in a virtual double category is a 2-cell as below:

$$
\begin{tikzcd}
 & \arrow[r, "f_1"] & \cdots \arrow[r, "f_n"] & \arrow[d, equal] \\
\arrow[d, equal] & \arrow[rr, "g"'] \arrow[d, phantom, "\Downarrow\alpha"] & & \\
 & & &
\end{tikzcd}
$$

which is initial; meaning that any other 2-cell "containing the the top horizontal string as a substring", factors uniquely through it:

$$
\begin{tikzcd}
 & \arrow[r] & \cdots \arrow[r] & \arrow[r, "f_1"] & \cdots \arrow[r, "f_n"] & \arrow[r] & \cdots \arrow[r] & \\
\arrow[d, "h"] & & & \arrow[rrrr, "j"'] \arrow[d, phantom, "\Downarrow"] & & & & \arrow[d, "k"]
\end{tikzcd}
=
\begin{tikzcd}
 & \arrow[r, dashed] & \arrow[r, "f_1"] & \cdots \arrow[r, "f_n"] & \arrow[r, dashed] & \arrow[d, equal] \\
\arrow[d, equal] & \arrow[d, equal] & \arrow[rrr, "g"'] \arrow[d, phantom, "\Downarrow\alpha"] & & & \\
\arrow[d, "h"] & \arrow[d, equal] & \arrow[rrr, "j"'] \arrow[d, phantom, "\Downarrow\exists!"] & & & \arrow[d, "k"] \\
 & \arrow[r, dashed] & & & \arrow[r, dashed] &
\end{tikzcd}
$$

A unit 2-cell is a cocartesian 2-cell of the form:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, equal] \\
A \arrow[r, "u_A"'] \arrow[ur, phantom, "\Downarrow e_A"] & A
\end{tikzcd}
$$

For any A, e_A and u_A are unique up to isomorphism (because of the cocartesian property).

Lemma 11.5. *Compositions with units exist.*

Proof. Given a 2-cell (as on the left), we can reinterpret it as follows, using the cocartesian property of the unit to get the factorization:

$$
\begin{tikzcd}
A \arrow[r, "p"'] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[r, "p"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[rr, "p"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[d, phantom, "\Downarrow e_B"] & B \arrow[d, equal] \\
A \arrow[r, "p"'] \arrow[d, equal] & B \arrow[r, "u_B"'] \arrow[d, phantom, "!\beta \Downarrow"] & B \arrow[d, equal] \\
A \arrow[rr, "p"'] & & B
\end{tikzcd}
$$

β is cocartesian because of a "2-out-of-3" property of cocartesian cells: because the overall diagram and the cells in the top row are all cocartesian. $\square$

We also have the idea of *base change*. As an inspiration, consider the modules case:

Example 11.6. Given a module ${}_RM_S$ and ring maps $f : R' \to R, g : S' \to S$, we can consider the module $R' \otimes_R M \otimes_S S'$.

Example 11.7. Another example for this can be seen using profunctors: Consider a profunctor $Hom_C : C \times C^{op} \to \mathbf{Set}$ as a horizontal arrow. If we have maps $f : D \to C$ and $g : D' \to C$, then we get new profunctor given by:

$$
\begin{tikzcd}
D \arrow[r, "Hom(g,f)"] \arrow[d, "f"] & D' \arrow[d, "g"] \\
C \arrow[r, "Hom_C"] & C
\end{tikzcd}
$$

Definition 11.8. A cartesian cell in a virtual double category is a 2-cell of the form:

$$
\begin{tikzcd}
\arrow[r, "p(f,g)"] \arrow[d, "f"'] & \arrow[d, "g"] \\
\arrow[r, "p"'] &
\end{tikzcd}
$$

such that for any 2-cell of the form below left, we get a unique factorization as described below right:

$$
\begin{tikzcd}
\arrow[r] & \arrow[r] & \cdots \arrow[r] & \arrow[d, "h'"] \\
\arrow[d, "h"'] & & \arrow[d, phantom, "\Downarrow \rho"] & \arrow[d, "g"] \\
\arrow[d, "f"'] & & & \\
\arrow[rrr, "p"'] & & &
\end{tikzcd}
=
\begin{tikzcd}
\arrow[r] & \arrow[r] & \cdots \arrow[r] & \arrow[d, "h'"] \\
\arrow[d, "h"'] & \arrow[rrr, "p(f,g)"] \arrow[d, phantom, "\Downarrow \exists!"] & & \\
\arrow[d, "f"'] & \arrow[rrr, "p"'] \arrow[d, phantom, "\Downarrow \rho"] & & \arrow[d, "g"] \\
\arrow[rrr, "p"'] & & &
\end{tikzcd}
$$

When a cartisian cell exists it is called a 'restriction of p along g and f.'

Remark that, from the universal properties, cocartesian, cartesian and unit cells are unique up to isomorphism.

Definition 11.9. A virtual equipment is a virtual double category with all units and such that for all diagrams of the form:

$$\begin{tikzcd} {} \arrow[d, "f"'] \arrow[r, "p"'] & {} \arrow[d, "g"] \end{tikzcd}$$

there is a cartesian cell filling this in.

11.2 Theorems

We now discuss various types of constructions and results which can be obtained using virtual equipments.

Example 11.10. We start constructing what can be thought of as a "horizontal unit". From a vertical arrow, $f : A \to B$, consider the following:

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\ B \arrow[r, "u_B"'] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} {} \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "u_B"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "u_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\exists!"] & {} \arrow[d, "f"] \\ {} \arrow[r, "u_B"'] & {} \end{tikzcd}$$

We call this unique 2-cell u_f.

Note that given a 2-cell

$$\begin{tikzcd} {} \arrow[r, "u_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow u_f"] & {} \arrow[d, "f"] \\ {} \arrow[r, "u_B"'] & {} \end{tikzcd}$$

it can be factorized in two different ways:

$$\begin{tikzcd} {} \arrow[r, "u_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow u_f"] & {} \arrow[d, "f"] \\ {} \arrow[r, "u_B"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "u_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\exists!"] & {} \arrow[d, "f"] \\ {} \arrow[r, "B(1,f)"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\rho"] & {} \arrow[d, equal] \\ {} \arrow[r, "u_B"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "u_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\exists!"] & {} \arrow[d, equal] \\ {} \arrow[r, "B(f,1)"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho'"] & {} \arrow[d, "f"] \\ {} \arrow[r, "u_B"'] & {} \end{tikzcd}$$

Lemma 11.11. *These units behave as follows:*

$$\begin{tikzcd}
\cdot \arrow[r, "u_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & \cdot \arrow[r, "B(1,f)"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\rho"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "B(1,f)"'] \arrow[rr, bend right, "B(1,f)"'] & \cdot \arrow[r, "u_B"'] & \cdot
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "u_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[r, "B(1,f)"] & \cdot \arrow[d, equal] \\
\cdot \arrow[rr, "B(1,f)"'] & & \cdot
\end{tikzcd}$$

And similarly for the second diagram.

Given some 2-cell with one arrow on the top:

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "q"'] & D
\end{tikzcd}$$

this gives rise to another 2-cell:

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow u_A"] & \cdot \arrow[r, "p"] \arrow[d, equal] & \cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow u_C"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "B(1,f)"'] \arrow[d] \arrow[dr, phantom, "\Downarrow\kappa"] & \cdot \arrow[r, "q"'] \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, "D(g,1)"'] \arrow[d] \arrow[dr, phantom, "\Downarrow\lambda"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\exists!"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "B(1,f)\circ q\circ D(g,1)"'] & \cdot
\end{tikzcd}$$

We obtain a similar thing for

$$\begin{tikzcd}
\cdot \arrow[r, "B(f,1)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\exists!"] & \cdot \arrow[r, "p"] & \cdot \arrow[r, "D(1,g)"] & \cdot \arrow[d, equal] \\
\cdot \arrow[rrr, "q"'] & & & \cdot
\end{tikzcd}$$

Given a virtual equipment $\mathbf{X}$, we can form a 2-category $\upsilon\mathbf{X}$, such that

- objects in $\upsilon\mathbf{X}$ are the objects of $\mathbf{X}$
- 1-morphisms in $\upsilon\mathbf{X}$ are the vertical morphisms

- 2-morphisms $\bullet \underset{g}{\overset{f}{\Downarrow}} \bullet$ are given by squares $\begin{tikzcd} \bullet \arrow[r, "u_A"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u_B"'] & \bullet \end{tikzcd}$

This is extremely useful and, perhaps the main example or result to be taken from it is: $\mathcal{M}od(\mathcal{K})$ is a virtual equipment and $v(\mathcal{M}od(\mathcal{K}))$ is $h\mathcal{K}$.

Theorem 11.12. *$\mathcal{M}od(\mathcal{K})$ is a virtual equipment and $v(\mathcal{M}od(\mathcal{K}))$ is $h\mathcal{K}$.*

Sketch: $\mathcal{M}od(\mathcal{K})$ has:

- objects: the same objects as $\mathcal{K}$
- vertical morphisms: functors
- horizontal morphisms: modules
- 2-cells: (similar to the 'spans' example) diagrams of the form:

$$\begin{tikzcd} & & M \times_B M' \arrow[dl, two heads] \arrow[dr, two heads] & & \\ & M \arrow[dl, two heads] \arrow[dr, two heads] & & M' \arrow[dl, two heads] \arrow[dr, two heads] & \\ A \arrow[d] & & B & & C \arrow[d] \\ E & & N \arrow[ll, two heads] \arrow[rr, two heads] & & F \end{tikzcd}$$

Note that $M \times_B M'$ is not a module. The point of the virtual equipments formality is to have a way of 'pretending it is a module.' For the 2-cell, we need a functor $M \times_B M' \to N$ that makes everything commute.

To see that this has indeed the structure of a virtual equipmet, we can see that restrictions will simply be pullbacks

$$\begin{tikzcd} M(d,c) \arrow[r] \arrow[d, two heads] & M \arrow[d, two heads] \\ 1 \arrow[r, "(c,d)"'] & A \times B \end{tikzcd}$$

and units can be identified by the "free living arrow" defined by

$$\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, equal] \arrow[dd] & \\
A & & A \\
 & A^2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\square$

12 Pointwise Kan extensions (Kevin)

MacLane: *all concepts are Kan extensions*.

12.1 Introduction

Definition 12.1. In a 2-category T, a diagram

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[dr, "d"'] & K \arrow[d, "u_* d"] \arrow[dl, phantom, "\overset{\varepsilon}{\swarrow}"] \\ & A \end{tikzcd}$$

is a **right extension** of d along u if for any

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[dr, "d"'] & K \arrow[d, "f"] \arrow[dl, phantom, "\overset{\xi}{\swarrow}"] \\ & A \end{tikzcd}$$

there exists a unique $\xi' : f \Rightarrow u_* d$ such that $\xi = \xi' \star \varepsilon$.

Kan extensions are an extremely useful tool in category theory. For instance, if $K = 1$, a right Kan extension is a limit for d. Our goal is to introduce the concept of Kan extensions for ∞-categories.

12.2 Naïve definition

A first approach for this definition, which doesn't actually works, would be to define the Kan extension using limits, as defined earlier. A right Kan extension of $d : 1 \to A^J$ along $u : J \to K$ could be an absolute right lifting

$$\begin{tikzcd} & A^K \arrow[d, "A^u"] \\ 1 \arrow[ur, "u_* d"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\overset{\varepsilon}{\swarrow}"] & A^J \end{tikzcd} \tag{6}$$

We will discuss why this concept is not the appropriate one and how to fix this.

Definition 12.2. An ∞-cosmos $\mathcal{K}$ is **cartesian closed** if $\times : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ has Δ-right adjoints in each variable.

$$\mathcal{K}(J \times K, L) \simeq \mathcal{K}(K, L^J) \simeq \mathcal{K}(J, L^K)$$

Let's now assume that we are in a cartesian cosmos. For 6 to be an absolute lifting diagram we must have a bijection:

$$\left\{\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\xi"] & A^K \arrow[d, "A^u"] \\ 1 \arrow[r] & A^J \end{tikzcd}\right\} \longleftrightarrow \left\{\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\xi'"] & A^K \arrow[d, "A^u"] \\ 1 \arrow[r] \arrow[ur] \arrow[ur, phantom, shift right=3ex, "\swarrow\varepsilon"] & A^J \end{tikzcd}\right\}$$

Using cartesianess, we see that this happens if and only if the following diagram in $h\mathcal{K}$

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[dr, "d"'] & K \arrow[d, "u_*d"] \arrow[dl, phantom, "\swarrow\varepsilon"] \\ & A \end{tikzcd}$$

is an right extension which remains so upon pasting with every square

$$\begin{tikzcd} X \times J \arrow[r] \arrow[d] & X \times K \arrow[d] \\ J \arrow[r] & K. \end{tikzcd}$$

The problem is that, in our context, pasting with these squares is not enough.

12.3 Pointwise Kan extensions in category theory

We now introduce a new concept, inspired by a motto by Kelly: *All important concepts are pointwise Kan extensions*. So let's go back and look at how this works in ordinary category theory.

Suppose the diagram below in **Cat** is a right extension:

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[dr, "d"'] & K \arrow[d, "u_*d"] \arrow[dl, phantom, "\swarrow\varepsilon"] \\ & A. \end{tikzcd}$$

Can we compute $u_*d(k)$?

For any $k \to d(j)$, we get

$$\begin{tikzcd} u_*d(k) \arrow[r] \arrow[dr] & (u_*d)u(j) \arrow[d, "\varepsilon"] \\ & d(j). \end{tikzcd}$$

And these maps assemble into $k \xrightarrow{p_k} \lim_{k\downarrow u} d(j)$.

Definition 12.3. We say $u_* d$ is a **pointwise Kan extension** if p_k is an isomorphism for all k, i.e. $u_* d(k) = \lim_{k\downarrow u} d$

Example 12.4. An example of a Kan extension which is not pointwise is

$$\begin{tikzcd} \mathbb{1} \arrow[r, "0"] \arrow[dr, "x"'] & \mathbb{2} \arrow[d, "cst_x"] \arrow[dl, phantom, "\overset{\mathrm{id}}{\swarrow}"] \\ & \mathbb{1} \amalg \mathbb{1}. \end{tikzcd}$$

For it to be pointwise, we would need $0_* x(1) = \lim_{1\downarrow 0} x = \lim_{\varnothing} x$ to be terminal in $\mathbb{1} \amalg \mathbb{1}$.

Remark 12.5. The diagram in **Cat** considered above is a pointwise Kan extension in **Cat** if and only if it remains a right Kan extension upon pasting with comma squares

$$\begin{tikzcd} 1 \arrow[r, "k"] & K \\ (k \downarrow u) \arrow[u] \arrow[r] \arrow[ur, phantom, "\Searrow"] & J \arrow[u, "u"'] \end{tikzcd}$$

for all $k \in K$.

12.4 Pointwise Kan extensions in ∞-cosmoi

A better proposal to import this concepts into ∞-categories would then be to define a diagram in $h\mathcal{K}$

$$\begin{tikzcd} J \arrow[r] \arrow[dr] & K \arrow[d] \arrow[dl, phantom, "\overset{\varepsilon}{\swarrow}"] \\ & A. \end{tikzcd}$$

to be a **pointwise right Kan extension** if it is a right extension that is stable under pasting with comma squares

$$\begin{tikzcd} L \arrow[r, "k"] & K \\ (L \downarrow u) \arrow[u] \arrow[r] \arrow[ur, phantom, "\Searrow"] & J. \arrow[u, "u"'] \end{tikzcd}$$

This is a good definition, but we know that a pointwise Kan extension should be stable under pasting with other convenient squares. So we have a quest now to define the "exact squares", i.e. the class of squares for which pasting with preserves pointwise right Kan extensions.

Definition 12.6. A diagram

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[d, equal] & K \arrow[r, "u_*D"] \arrow[d, Rightarrow, "\varepsilon"] & A \arrow[d, equal] \\ J \arrow[r, "D"'] & & A \end{tikzcd}$$

in $Mod(\mathcal{K})$ is a **right extension** if ε induces a bijection:

$$\left\{ \begin{tikzcd} J \arrow[r, "u"] \arrow[d, equal] & K \arrow[r, "E_1"] & \cdots \arrow[r, "E_n"] \arrow[d, Rightarrow, "\xi"] & A \arrow[d, equal] \\ J \arrow[rrr] & & & A \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{tikzcd} J \arrow[r, "u"] \arrow[d, equal] & K \arrow[r, "E_1"] \arrow[d, equal] & \cdots \arrow[r, "E_n"] \arrow[d, Rightarrow, "\exists!\xi'"] & A \arrow[d, equal] \\ J \arrow[r, "u"'] \arrow[d, equal] & K \arrow[rr, "u_*D"'] \arrow[d, Rightarrow, "\varepsilon"] & & A \arrow[d, equal] \\ J \arrow[rrr, "D"'] & & & A \end{tikzcd} \right\}.$$

In this context, we get:

Theorem 12.7. *A diagram*

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[dr, "d"'] & K \arrow[d, "u_*d"] \arrow[dl, phantom, "\varepsilon \swarrow"] \\ & A \end{tikzcd}$$

is a pointwise Kan extension in $h\mathcal{K}$ if and only if one of the following holds

- *It is a right extension, stable under pasting with comma squares;*
- *It gives a right extension*

$$\begin{tikzcd} J \arrow[r, "K(1,u)"] \arrow[d, equal] & K \arrow[r, "A(1,u_*d)"] \arrow[d, Rightarrow, "\varepsilon"] & A \arrow[d, equal] \\ J \arrow[r, "A(1,d)"'] & & A \end{tikzcd}$$

  in $Mod(\mathcal{K})$;

- *It is a right extension in $h\mathcal{K}$, stable under pasting with exact squares.*

Definition 12.8. A diagram

$$\begin{tikzcd} & D \arrow[dl, "h"'] \arrow[dr, "k"] & \\ C \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

is an **exact square** if $\mathrm{Hom}_C(h, C) \times \mathrm{Hom}_B(B, k) \to \mathrm{Hom}_A(f, g)$ is a composite in $Mod(\mathcal{K})$.

The intuition for this definition can be drawn from the context **qCat**, where a square φ is exact if and only if for all $c \in C$ and $b \in B$, φ induces a weak homotopy equivalence

$$D_{c,b} \to A(fb, gc)$$

where $D_{c,b}$ is the fiber of D over c, b.

Remark 12.9. Any commutative square is exact, products of exact squares are exact, and pullback squares along cartesian/cocartesian maps are exact.

Moreover, if E has all pointwise right Kan extensions and we have an exact square

$$\begin{tikzcd}
 & D \arrow[dl, "h"'] \arrow[dr, "k"] & \\
C \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

then in the following square

$$\begin{tikzcd}
 & E^C \arrow[dl] \arrow[dr] & \\
E^D \arrow[dr] & \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & E^A \arrow[dl] \\
 & E^B &
\end{tikzcd}$$

the cell φ is an isomorphism.

13 Model independence (Jonathan)

13.1 Cosmological functors

Definition 13.1. A cosmological functor $F : \mathscr{K} \to \mathscr{L}$ is a simplicial functor that preserves isofibrations and all simplicial limits specified in the definition of ∞-cosmoi.

Example 13.2. The nerve embedding $N : \mathbf{Cat} \hookrightarrow \mathbf{qCat}$.

Example 13.3. The functors $\mathrm{Fun}(A, -) : \mathscr{K} \to \mathbf{qCat}$ for any A in $\mathscr{K}$, and in particular, the "underlying quasi-category" functor $(-)_0 := \mathrm{Fun}(1, -)$.

Example 13.4. For any morphism $f : A \to B$ in $\mathscr{K}$, we get a cosmological functor $f^* : \mathscr{K}_{/B} \to \mathscr{K}_{/A}$ given by pullback.

Example 13.5. Given a cosmological functor $F : \mathscr{K} \to \mathscr{L}$ and an object A in $\mathscr{K}$, the induced functor $F_{/A} : \mathscr{K}_{/A} \to \mathscr{L}_{/FA}$ is cosmological.

Example 13.6. The inclusion of the subcategory of discrete objects $\mathrm{Disc}(\mathscr{K}) \hookrightarrow \mathscr{K}$.

Example 13.7. The inclusion $\mathrm{Cart}(\mathscr{K}) \hookrightarrow \mathscr{K}^2$ of the category of cartesian fibrations and cartesian functors between them.

Example 13.8. Similarly, the inclusion $\mathrm{Cart}(\mathscr{K})_{/B} \hookrightarrow \mathscr{K}_{/B}$.

Remark 13.9. The homotopy functor $hA = ho(\mathrm{Fun}(1, A))$ is not cosmological.

13.2 Observations about preservation properties of cosmological functors

- Discrete objects are preserved.

  Recall that E is discrete when $\mathrm{Fun}(X, E)$ is a Kan complex, which is equivalent to saying the map $E^{\mathbb{I}} \xrightarrow{\sim} E^{\mathbb{2}}$ induced by the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ is an equivalence.

- Comma objects are preserved.

Proposition 13.10. *Cosmological functors preserve the following:*

1. *adjunctions, right adjoints right inverses, left adjoints right inverses,*
2. *invertible 2-cells and mates,*
3. *homotopy coherent adjunctions and monads,*
4. *absolute right and left lifting diagrams,*
5. *(co)limits indexed by cotensors of simplicial sets,*
6. *(co)cartesian fibrations, and cartesian functors,*
7. *discrete (co)cartesian fibrations,*
8. *two-sided fibrations, and modules.*

13.3 Cosmological biequivalence

Definition 13.11. A cosmological functor $F : \mathscr{K} \to \mathscr{L}$ is a biequivalence when it is

1. essentially surjective, i.e., for all $C \in \mathscr{L}$ there exists $A \in \mathscr{K}$ such that $FA \simeq B$; and
2. a local equivalence of quasi-categories, i.e., for every $A, B \in \mathscr{K}$, the map

$$\mathrm{Fun}(A, B) \xrightarrow{\sim} \mathrm{Fun}(FA, FB)$$

   is an equivalence of quasi-categories.

We will also call such a functor a cosmological biequivalence.

Definition 13.12. Two ∞-cosmoi are biequivalent if there exists a finite zigzag of biequivalences between them.

Example 13.13. The following underlying quasi-category functors

$$\mathbf{CSS} \xrightarrow{(-)_0} \mathbf{qCat} \xleftarrow{(-)_0} \mathbf{Seg}$$

give a biequivalence between **CSS** and **Seg**.

Example 13.14. For any ∞-cosmos $\mathscr{K}$ biequivalent to **qCat**, the underlying quasi-category functor $(-)_0 : \mathscr{K} \to \mathbf{qCat}$ is a biequivalence.

Example 13.15. For any weak equivalence $f : A \xrightarrow{\sim} B$ in $\mathscr{K}$ the induced cosmological functor $f^* : \mathscr{K}_{/B} \to \mathscr{K}_{/A}$ is a biequivalence of ∞-cosmoi.

Example 13.16. If $F : \mathscr{K} \to \mathscr{L}$ is a cosmological biequivalence, then for any $A \in \mathscr{K}$ the induced functor $F_{/A} : \mathscr{K}_{/A} \to \mathscr{L}_{/FA}$ is also a cosmological biequivalence.

Proposition 13.17. *A cosmological biequivalence $F : \mathscr{K} \to \mathscr{L}$ induces a 2-categorical biequivalence of homotopy categories $\mathfrak{h}F : \mathfrak{h}\mathscr{K} \to \mathfrak{h}\mathscr{L}$. This is, the 2-functor $\mathfrak{h}F$ is*

1. *surjective in objects up to equivalence and*
2. *defines a local equivalence of categories* $\mathrm{hFun}(A, B) \xrightarrow{\sim} \mathrm{hFun}(\mathfrak{h}FA, \mathfrak{h}FB)$ *for all $A, B \in \mathscr{K}$.*

Proposition 13.18. *A cosmological biequivalence $F : \mathscr{K} \to \mathscr{L}$:*

1. *preserves and reflects invertibility of 2-cells;*
2. *preserves, reflects, and creates adjunctions between ∞-categories, including right adjoint right inverse adjunctions and left adjoint right inverse adjunctions;*
3. *preserves and reflects discreteness;*
4. *preserves, reflects, and creates absolute right and left lifting diagrams over a given cospan;*
5. *preserves and reflects cartesian and cocartesian fibrations and cartesian functors between them;*
6. *preserves and reflects discrete cartesian fibrations and discrete cocartesian fibrations;*

7. *preserves and reflects two-sided fibrations and cartesian functors between them;*

8. *preserves and reflects modules and induces a bijection on equivalence classes of modules between a fixed pair of ∞-categories;*

9. *preserves and reflects limits or colimits of diagrams indexed by a simplicial set $\boldsymbol{J}$ inside an ∞-category and creates the property of an ∞-category in $\mathscr{K}$ admitting a limit or colimit of a given diagram.*

Corollary 13.19. *If $F : \mathscr{K} \to \mathscr{L}$ is a cosmological biequivalence, then the following induced cosmological functors are biequivalences:*

1. $\mathrm{Disc}(\mathscr{K}) \to \mathrm{Disc}(\mathscr{L})$,

2. $\mathrm{Cart}(\mathscr{K}) \to \mathrm{Cart}(\mathscr{L})$, *and*

3. ${}_{A\backslash}\mathrm{Mod}(\mathscr{K})_{/B} \to {}_{FA\backslash}\mathrm{Mod}(\mathscr{L})_{/FB}$.

13.4 Model independence

There is a way to embed $\mathfrak{h}\mathscr{K}$ into its virtual equipment $\mathrm{Mod}(\mathfrak{h}\mathscr{K})$ given by

$$\begin{array}{ccc} \mathfrak{h}\mathscr{K} & \hookrightarrow & \mathrm{Mod}(\mathfrak{h}\mathscr{K}) \\ (A \xrightarrow{f} B) & \longmapsto & (A \xrightarrow{\mathrm{Hom}_B(B,f)} B) \end{array}$$

Theorem 13.20. *If $F : \mathscr{K} \to \mathscr{L}$ is a cosmological biequivalence, then the induced functor of virtual equipments*

$$F : \mathrm{Mod}(\mathfrak{h}\mathscr{K}) \longrightarrow \mathrm{Mod}(\mathfrak{h}\mathscr{L})$$

defines a biequivalence of virtual equipments, i.e., it is

1. *bijective on equivalence classes of objects;*

2. *locally bijective on isomorphism classes of parallel vertical functors extending the bijection of 1;*

3. *locally bijective on equivalence classes of parallel modules extending the bijection of 2;*

4. *locally bijective on cells extending the bijections of 1, 2, and 3.*

14 Comprehension and the Yoneda embedding (Ze)

The main reference for this talk is [RV-VI] "The Comprehension Construction" and all the numbers of results refer to that.

What is the concept of Comprehension in **Cat**? We know that in this contexts, given a map, we can construct its fibres via a pullback Moreover, if p is is a fibration, these pullbacks are related in a well-behaved way. More explicitly, we get a functor

$$\begin{aligned} B = \mathrm{Fun}(1, B)^{op} &\xrightarrow{c_p} \mathbf{Cat} \\ b &\to E_b \end{aligned}$$

Regardng the Yoneda Embedding, when we think of it in **Cat** we can see it as an ebedding

$$B \hookrightarrow [B^{op}, Set] \hookrightarrow [B^{op}, \mathbf{Cat}] \simeq Cart(\mathbf{Cat})/_B \hookrightarrow \mathbf{Cat}\ /_B$$

We weill get an analogous result in the context of ∞-cosmos and Comprehension will palay an important part in that.

In the centre of this discussion will be cocartesian fibrations, which we now recall.

In **Cat**, a map $E \twoheadrightarrow B$ is called a cocartesian fibration if for any morphsm $pe \xrightarrow{f} b$ in B, there is a lift $e \xrightarrow{\xi(f)} f_* e$.

In the homotopy 2-category of an ∞-cosmos, we define a $E \twoheadrightarrow B$ to be a cocartesian fibration if for any there exstis a lift We can also think of cocartesian fibrations, not in the homotopy 2-category, but n the ∞-cosmos itself. And that is what we will do from now on.

Theorem 14.1. *If $p : E \twoheadrightarrow B$ is a cocartesian fibration and $A \in \mathcal{K}$, we have a map of quasi-categories*

$$\begin{aligned} c :_{p,A} : \mathrm{Fun}_{\mathcal{K}}(A, B) &\to coCart(K)/_A \\ (A \xrightarrow{b} B) &\to diagram. \end{aligned}$$

where $K = Ng_*\mathcal{K}$ is the homotopy coherent nerve of the groupoidal core of the ∞-cosmos $\mathcal{K}$. Such groupoidal core $g_*\mathcal{K}$ is locally Kan and K is then quasi-category. In this context, $coCart(K)/_A \doteq Ng_*(coCart(K)/_A)$.

We can also see what $c_{p,A}$ does on 2-simplices. Namely, it takes

What the functor does on higher simplices can also be described explicitly and the details can be seen in the paper.

This functor $c_{p,A}$ is called the Comprehension functor for p over A.

Example 14.2.

	$\mathcal{K}$	A	$E \xrightarrow{p} B$	$c_{p,A}$
Comprehension	any	1	any	$B_0 = \mathrm{Fun}_{\mathcal{K}}(1, B) \xrightarrow{c_p} K/_1 = K$
Straightening	**QCat**	1	any	$B_0 = B \to \mathbf{QCat}$
Unstraightening	**QCat**	B	$E \in \mathbf{qCat}^*,\ B \in \mathbf{qCat}$	$\mathrm{Fun}(B, \mathbf{qCat}) \to coCart(\mathbf{QCat})/_B$

15 Limits and colimits (Tim)

In this section we will consider homotopy limits in simplicial categories and how they relate to limits in ∞-categories. We start by reviewing some constructions related to simplicial categories.

First, we note that we can obtain simplicial categories from categories with weak equivalences.

Remark 15.1. If $(\mathbf{C}, W)$ is a (1-)category with weak equivalences, then there is a simplicial category $\mathbf{C}[W^{-1}]$, called the *Dwyer–Kan localization of* $(\mathbf{C}, W)$, equipped with a functor $\mathbf{C} \to \mathbf{C}[W^{-1}]$ which maps W to equivalences.

Next, we mention a model structure on the category of simplicial categories which is relevant to the theory of ∞-categories.

Remark 15.2. There is a model structure on **sCat**, called the *Bergner model structure* which has

- simplicial categories which are enriched in Kan complexes as fibrant objects
- simplicial computads (c.f. Remark 15.3) as cofibrant objects and
- Dwyer–Kan equivalences, i.e. simplicial functors which are essentially surjective and Hom-wise weak homotopy equivalences, as weak equivalences.

Remark 15.3. There are multiple ways of thinking of simplicial computads:

- They are simplicial categories which are built by attaching cells of the form $\varnothing \to 1$ or $\Sigma\partial\Delta[n] \to \Sigma\Delta[n]$ where for a simplicial set X, ΣX is the simplicial category with objects $\{0, 1\}$ and morphisms $\mathrm{Hom}(0, 1) = X$ and $\mathrm{Hom}(1, 0) = \varnothing$.
- They are simplicial categories which are levelwise free when seen as a simplicial object in the category of categories (i.e. whose category of n-simplices is a free category on a graph) such that the degeneracies of atomic morphisms are atomic atomic.

Remark 15.4. There is a free-forgetful adjunction $F : \mathbf{RGraph} \leftrightarrows \mathbf{Cat} : U$ between the category of (reflexive) graphs and the category of (small) categories which yields a comonad FU on **Cat**. Given a category $\mathcal{C}$, its comonadic resolution $FU_{\bullet}\mathcal{C}$ is a simplicial computad.

Indeed, the aforementioned Bergner model structure on **sCat** is closely related to the Joyal model structure on **sSet** as follows.

Remark 15.5. There is an adjunction

$$\mathfrak{C} : \mathbf{sCat} \leftrightarrows \mathbf{sSet} : N$$

where $\mathfrak{C}[\Delta[n]]$ is given by a simplicial category which has $\{0, \dots, n\}$ as objects and whose morphisms satisfy $\mathrm{Hom}(i, j) \cong \Delta[1]^{i-j-1}$.

Theorem 15.6. *$\mathfrak{C} \dashv N$ is a Quillen equivalence between* **sCat** *equipped with the Bergner model structure and* **sSet** *equipped with the Joyal model structure.*

Now we turn to homotopy limits in simplicial categories.

Recall that a *flexible weight* is a weight $W : J \to \mathbf{sSet}$ that is projectively cofibrant (w.r.t. the Quillen model structure on **sSet**), i.e. built up from cells of the form $\partial\Delta[n] \times J(j,-) \to \Delta[n] \times J(j,-)$.

Definition 15.7. Let $W : J \to \mathbf{sSet}$ be a weight. Then a morphism $\overline{W} \to W$ is a *flexible resolution* if it is an equivalence and $\overline{W}$ is flexible.

Definition 15.8. Let $\mathcal{C}$ be a Kan-enriched category. Let $W : J \to \mathbf{sSet}$ be a weight and $F : J \to \mathcal{C}$ a functor.

A *W-weighted homotopy limit of F* consists of

- a flexible resolution $\overline{W} \to W$,
- an object L of $\mathcal{C}$ equipped with a natural transformation $\lambda : W \Rightarrow \mathcal{C}(L, F(-))$ such that for all $C \in \mathcal{C}$, the maps $\mathcal{C}(C, L) \xrightarrow{\sim} \{\overline{W}, \mathcal{C}(C, F(-))\}$ induced by λ and $\overline{W} \to W$ are equivalences.

A *homotopy limit of F* is a 1-weighted homotopy limit.

Lemma 15.9. *Up to equivalence, homotopy limits are invariant under equivalences of weights and functors. Moreover, different flexible resolutions of the weight yield equivalent homotopy limits.*

Definition 15.10. Let J be a simplicial set. The *canonical weight on* $\mathfrak{C}[J]$ is given by $W_J(j) = \mathrm{Hom}_{\mathfrak{C}[\Delta[0]\star J]}(\bot, j)$ where $\bot \in \mathfrak{C}[\Delta[0] \star J]$ is the cone point.

Lemma 15.11. *$W_J \to 1$ is a flexible resolution.*

Now we will state some results about the existence of limits in ∞-categories and their relation to homotopy limits in simplicial categories.

Theorem 15.12. *Let $\mathcal{C}$ be a Kan-enriched category, $J \in \mathbf{sSet}$ and $D : \mathfrak{C}[J] \to \mathcal{C}$. Then D has a homotopy limit if and only if its adjoint $\overline{D} : J \to N\mathcal{C}$ has a limit.*

Theorem 15.13. *Let A be an ∞-category in an ∞-cosmos $\mathcal{K}$.*

1. *The homotopy coherent nerve $N\underline{\mathrm{Cart}}^g(\mathcal{K})_{/A}$ of the Kan-enriched category of groupoidal cartesian fibrations in $\mathcal{K}$ with codomain A has all limits.*
2. *A is complete if and only if A has pullbacks and products.*
3. *Let $F : A \to B$ a functor between complete ∞-categories. Then F preserves limits if and only if it preserves pullbacks and products.*

Proof idea. One can construct limits along the skeletal structure of their indexing simplicial sets. For this, one needs to show (in particular) that the limit of a diagram indexed by a homotopy pushout of simplicial sets can be computed as the pullback of the limits of the diagrams indexed by the simplicial sets in the pushout diagram. □

16 Other approaches to model-independence (Nima)

Up until now we focused on the theory of ∞-cosmoi. We discussed how we can use them to study $(\infty, 1)$-categories from a model-independent point of view.

16.1 A walk through history

Initially people were looking at topological spaces and how they can be studied using homotopy invariant tools, such as homotopy groups or homology. Eventually they realized that some of these phenomena also appeared in non-topological environments. One case of that are Kan complexes which also have a notion of homotopy group. Another one were chain complexes. This motivated people to look for a common generalization which allows us to study all of these at once and compare them. In a very vague sense we call any such collection of objects a homotopy theory. So, we say the "homotopy theory of spaces". Let us see some approaches to this notion of a homotopy theory:

- **Simplicial Categories, 1967**: Simplicial categories were at least studied since Quillen introduced model categories.
- **Quasi-Categories, 1973**: Quasi-categories were first introduced by Bordman and Vogt when they were working on algebraic structures on spaces (introduced as restricted Kan complexes. They use it to construct fundamental categories.
- **Relative Categories, 1980**: Relative categories were first considered by Dwyer and Kan as a weakening of model categories with the goal of constructing *simplicial localizations*.
- **Segal Categories, 1989**: The underlying idea of a Segal category can be found in work of Segal where he introduces Γ-spaces. Segal categories as an alternative to simplicial categories were introduced in.
- **Complete Segal Spaces, 2001**: Complete Segal spaces are an alternative way to generalize Γ-spaces that was introduced by Rezk.

Certainly there are also other models but for the purpose of getting somewhere we will focus on these ones. As this is a very diverse group of definitions let us try to find a way to sort them. We can think of the world of $(\infty, 1)$-categories as a common generalization of two different subjects:

1. Category Theory
2. Homotopy Theory

Based on this point of view most generalizations start with one of these two and try to incorporate aspects of the other. Thus we either start with a category and then give it a meaning of homotopy or we start with spaces and give it a sense of direction. From this point of view our models break into the following two groups:

Category with Structure	Simplicial Objects
Simplicial Categories	Quasi-categories
Relative Category	Complete Segal spaces
	Segal categories

All of these different models have many things in common:

1. Objects
2. Morphisms
3. Enrichment over spaces
4. ...

But they also have some differences:

1. The categorical models have strict notion of composition, but the simplicial objects don't.
2. Relative categories don't have the enrichment as part of the definition
3. The simplicial models are ∞-cosmoi but the categorical ones are not.

So the question is what they do have in common and whether we can take a more unified approach to the theory of $(\infty, 1)$-categories. That is the goal we want to move towards.

16.2 A unified approach to the theory of $(\infty, 1)$-categories

First we need to make sense of the following questions:

How are we justified in calling all of these models a theory $(\infty, 1)$*-categories?*

What does it even mean to be a model for a theory of $(\infty, 1)$*-categories?*

A more advanced answer is along the lines of using an ∞-cosmos as a unifying framework. However, we want to take a different look here that includes the non-cosmos models and uses less machinery, namely use the language of model categories.

Fortunately, all the categories above come with a model structure associated to them that we can use here. Here is the list of model structures:

1. **Simplicial Categories**: Bergner Model Structure
2. **Quasi Categories**: Joyal Model Structure

3. **Relative Categories**: B and K ?

4. **Segal Categories**: Hirschowitz and Smith

5. **Complete Segal Spaces**: Complete Segal Space Model Structure - Rezk

One thing we could do then is to show that all of these model structures are Quillen equivalent. That would certainly be a good first start, but wouldn't actually answer the question what we mean by an $(\infty, 1)$-category. We want to understand how to identify the right model structures using inherent properties rather than building equivalences every time.

The way to answer this question comes down to the following fascinating observation about $(\infty, 1)$-categories. If $\mathscr{C}$ is anything worthy of that name, then it should have an underlying maximal subgroupoid $\mathscr{C}^{equiv}$ which itself is a space. Clearly we cannot recover the information of $\mathscr{C}$ from this space as all non-invertible arrows are gone. However, we should have a notion of "category of arrows" $Arr(\mathscr{C})$ and we can then repeat the same step as above to construct $Arr(\mathscr{C})^{equiv}$. This space has the all the information about the arrows but doesn't know anything about composition. However, it comes with arrows

$$\mathscr{C}^{equiv} \underset{s}{\overset{t}{\leftleftarrows\!\!\rightrightarrows}} Arr(\mathscr{C})^{equiv}$$

We can generalize this argument and look at the higher category of 2-morphisms in a higher category, $2 - Mor(\mathscr{C})$.

$$\mathscr{C}^{equiv} \leftleftarrows\!\!\rightrightarrows Arr(\mathscr{C})^{equiv} \leftleftarrows\!\!\rightrightarrows\!\!\leftleftarrows\!\!\rightrightarrows 2 - Mor(\mathscr{C})^{equiv} \leftleftarrows\!\!\rightrightarrows\!\!\leftleftarrows\!\!\rightrightarrows\!\!\leftleftarrows\!\!\rightrightarrows \cdots$$

This clearly suggests a pattern where we can build further and further spaces that keep track of higher dimensional information that assemble into a simplicial space. This simplicial space looks suspciously like a complete Segal space. We want to find an axiomatic way to exactly recover this construction.

It was To en who realized that the key concept is that of an interval. For a model category $\mathscr{M}$ he axiomatically defined an interval object as a cosimplicial objects and used that to make the construction above.

Definition 16.1. An interval object in $\mathscr{M}$ is a cosimplicial object

$$C : \Delta \to \mathscr{M}$$

such that $C(0)$ is contractible and $C(1)$ has a contractible geometric realization.

Example 16.2. In quasi-categories this interval object is the standard one

$$\Delta[0] \underset{d_1 \quad s_0}{\overset{d_0}{\leftleftarrows\!\!\rightrightarrows}} \Delta[1] \underset{d_2}{\overset{d_0}{\leftleftarrows\!\!\rightrightarrows\!\!\leftleftarrows\!\!\rightrightarrows}} \Delta[2] \leftleftarrows\!\!\rightrightarrows\!\!\leftleftarrows\!\!\rightrightarrows\!\!\leftleftarrows\!\!\rightrightarrows \cdots$$

Notice here the geometric realization of $\Delta[1]$ is the isomorphism $I[1]$ which is indeed a contractible quasi-category.

Using an interval object To en introduces seven axioms on a model category and calls any model category $\mathscr{M}$ with cosimplicial object $C : \Delta \to \mathscr{M}$ that satisfies those axioms a theory of $(\infty, 1)$-categories . He uses that prove the following result.

Theorem 16.3. *Let $(\mathscr{M}, C)$ be a theory of $(\infty, 1)$-categories. Then $\mathscr{M}$ is Quillen equivalent to the complete Segal space model structure.*

Proof. The gist of the proof is that for each object $\mathscr{C}$ in $\mathscr{M}$ we can construct a simplicial space

$$Map(C(0), \mathscr{C}) \leftleftarrows Map(C(1), \mathscr{C}) \leftleftarrows\!\!\!\!\rightrightarrows Map(C(2), \mathscr{C}) \leftleftarrows\!\!\!\!\rightrightarrows \cdots$$

and use the axioms on $\mathscr{M}$ to show this is a complete Segal space.

This can be expanded into a hom-tensor adjunction

$$- \otimes C(\bullet) : \mathrm{Fun}(\Delta^{\mathrm{op}}, Spaces) \rightleftarrows \mathscr{M} : \mathrm{Map}(C(\bullet), -)$$

which we can show is a Quillen equivalence. □

Remark 16.4. From a modern perspective what we just did is give a presentation. A presentation for a higher category $\mathscr{P}$ is an adjunction

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, Spaces) \rightleftarrows \mathscr{P}ai$$

where i is an inclusion.

The proof can be translated to saying that the $(\infty, 1)$-category of $(\infty, 1)$-categories is presentable and concretely we have the presentation

$$- \otimes C(\bullet) : \mathrm{Fun}(\Delta^{\mathrm{op}}, Spaces) \rightleftarrows \mathscr{M} : \mathrm{Map}(C(\bullet), -)$$

This is what actually makes the result so useful.

The theorem above gives us the following interesting result.

Theorem 16.5. *There is an equivalence between the space of self-equivalences of a theory of $(\infty, 1)$-categories and Δ, which is both $B(\mathbb{Z}/2)$.*

Remark 16.6. Concretely the only self-equivalence of a theory of $(\infty, 1)$-categories is reversing the direction of the arrow.

16.3 Comparison of models

By now we have a definition of what a homotopy theory should be, but we have not shown whether there are any examples and in particular whether any of the other models are actually examples. Throughout the years an amazing amount of work has been done in this regard. In the the following diagram we can see some of the work that has been done of the years.

$$
\begin{tikzcd}
 & \mathscr{C}\mathrm{at}_\Delta \arrow[dl, bend right, "N"'] \arrow[dl, bend left, "F"] \arrow[d, bend right, "N_\Delta"'] \arrow[d, bend left, "\mathfrak{C}"] \arrow[d, dashed, shift right=4, "N_\Delta^{nec}"'] \arrow[d, dashed, shift left=4, "\mathfrak{C}^{nec}"] \arrow[dr, bend left, "p_1^*"] \arrow[dr, bend right, "i_1^*"'] \\
\mathscr{S}\mathrm{eg}\mathscr{C}\mathrm{at} \arrow[ur, bend left, "d_*"] \arrow[ur, bend right, "d^*"'] \arrow[ur, bend left, "j^*"] \arrow[ur, bend right, "q^*"'] \arrow[rr, bend left, "I"] \arrow[rr, bend right, "R"'] \arrow[rr, phantom, "\perp"] & \mathscr{D}\mathscr{C}\mathrm{at} \arrow[u, bend left, "t_!"] \arrow[u, bend right, "t^!"'] & \mathscr{C}\mathscr{S}\mathscr{S}
\end{tikzcd}
$$

Here are the references for this table:

- $(\mathfrak{C}, N_\Delta)$: Lurie
- $(\mathfrak{C}^{nec}, N_\Delta^{nec})$: D and S (This one is not a Quillen adjunction, but an equivalence of simplicial categories.)
- (p_1^*, i_1^*): Joyal and Tierney
- $(t_!, t^!)$: Joyal and Tierney
- (d^*, d_*): Joyal and Tierney
- (q^*, j^*): Joyal and Tierney
- (I, R): Bergner
- (F, N): Bergner

There are couple important things to notice about this diagram:

1. First of all this is just a sample of the results that exists in the literature.

2. Second notice although the uniticity theorem suggests that all equivalences should go through complete Segal spaces, but we see most connections are through quasi-categories. This is because the historical attention quasi-categories have received over time.

3. The functors $\mathfrak{C}$ and $\mathfrak{C}^{nec}$ both go from quasi-categories to simplicial categories. By the theorem of Toen they should then be equivalent. Turn out they are, however, that does not mean that one is useless. On the contrary, the fact that two completely differently defined functors are equivalent helps us understand each better.

16.4 Pros and cons of uniticity vs. ∞-Cosmoi

Now that we have reviewed the uniticity it's interesting to compare it to our approach this week. Each method has their own pros and cons.

1. The unicity approach is more inclusive. There are less restrictions (in particular we don't need a refinement over quasi-categories) and so we are able to include models that are not ∞-cosmoi.

2. On the other hand the theory of ∞-cosmoi gives us concrete methods to build categorical tools in each one of these models. The model category approach does not tell us how a definition such as adjunctions or limits transfer from one limit to another. This argument might seem theoretical, so let us see one actual example, with left fibrations, which model covariant functors valued in spaces.

   Definition 16.7. A map of simplicial sets $L \to S$ is a *left fibration* if for each n and $0 \leq i < n$ the diagram of the form lifts.

$$\begin{tikzcd} \Lambda[n]_i \arrow[r] \arrow[d] & L \arrow[d] \\ \Delta[n] \arrow[r] & S \end{tikzcd}$$

   Definition 16.8. A map of simplicial spaces $L \to X$ is a left fibration if it is a Reedy fibration and the map

$$L_1 \to L_0 \times_{X_0} X_1$$

   is a weak equivalence of simplicial sets.

   Based on appearences these definitions seem quite different. Turns out we can prove that the data of both definitions is exactly the same. Concretely, both these maps are fibrant-cofibrant objects in some model structures which are Quillen equivalent.

16.5 A similar case for (∞, n)-categories

The results above have already been generalized to the setting of (∞, n)-categories using similar techniques. In particular, they define a theory of (∞, n)-categories and prove that the space of self-equivalences of any such theory is equivalent to $B((\mathbb{Z}/2)^n)$.