Talbot 2018: Model-independent theory of ∞-categories

Exercises

Talk 2

1. Exercise 1.1.iv in [RV](to understand the homotopy coherent isomorphism $\mathbb{I}$).
2. Prove from axioms (i) and (ii) in the definition of ∞-cosmos that every trivial fibration has a section.
3. Exercise 1.2.ii in [RV] on enriched products.
4. For $A \in \text{qCat}$ describe explicitly $A^{\Delta[1]}(= A^2) \in \mathbf{qCat}$.

Talk 3

1. Prove that a class of maps given by a right lifting property is closed under limits of towers.
2. What is the homotopy 2-category of the ∞-cosmos **Cat**?
3. Find new examples of ∞-cosmoi (and tell Emily and Dom what they are).

Talk 4

1. Prove that adjunctions compose.
2. Prove:
   (a) $f \dashv u$ and $f' \dashv u$ implies $f \cong f'$.
   (b) $f \dashv u$ and $f \cong f'$ implies $f' \dashv u$.
3. Formulate "the notion of adjunction between ∞-categories is equivalence invariant" and prove it.
4. Exercise 2.4.i in [RV] (proving "cheap RAPL" using adjunction facts).
5. Exercise 2.4.iii in [RV] (proving RAPL in the usual way).

Talk 5

1. Use the isomorphism $\text{Fun}(X, A^2) \cong \text{Fun}(X, A)^2$ that defines the simplicial cotensor to define

$$\begin{tikzcd} A^{2} \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd} \qquad \text{in } \mathrm{h}\mathcal{K}$$

2. Explain how $X \to A^{2}$ encodes a 2-cell

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd} \qquad \text{in } \mathrm{h}\mathcal{K}$$

3. If the 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\cong\Downarrow\gamma"] & A^{2} \end{tikzcd}$$

   is such that domγ and codγ are the identity (here dom and cod are the maps from problem 1 in Talk 5), prove that f and g represent the same 2-cell

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

   Hint: horizontal composition.

Talk 6

1. For $1 \xrightarrow{b} B$ prove $B \downarrow b$ has a terminal element.

2. Prove that two terminal elements $\mathbf{1} \overset{t}{\underset{s}{\rightrightarrows}} A$ are isomorphic in $\mathrm{ho}(A) := \mathrm{hFun}(\mathbf{1}, A)$.

Talk 7

1. Use the graphical calculus to prove that **Adj** contains an adjunction and **Mnd** contains a monad.

2. Exercise 8.1.vi in [RV] expressing **Adj** as a hammock localization.

Talk 8

1. Given a homotopy coherent adjunction, build a u-split augmented simplicial object in A.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B^{\Delta_{\top}} \arrow[d, "\mathrm{res}"] \\ A^{\Delta_{+}^{op}} \arrow[r, "u"'] & B^{\Delta_{+}^{op}} \end{tikzcd}$$

2. Explain why the weighted limit definition of the ∞-category of T-algebras computes the right Kan extension.

$$\begin{tikzcd}
\mathbf{Mnd} \arrow[rr, "T"] \arrow[dr, hook] & \arrow[d, phantom, "\cong\Uparrow"] & \mathcal{K} \\
& \mathbf{Adj} \arrow[ur, dashed, "\mathrm{Ran}"'] &
\end{tikzcd}$$

Talk 9

1. If p is cartersian, the 2-cell ϕ admits a p-cartesian lift as below

$$\begin{tikzcd}
B \downarrow p \arrow[r, two heads, "p_1"] \arrow[dr, bend right, two heads, "p_0"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\nearrow\phi"] \\
& B
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, bend left, two heads, "p_1"] \arrow[r, bend right, dashed, "r"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\
& B
\end{tikzcd}$$

and thus

$$\begin{tikzcd}
B \downarrow p \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow\chi"] & E
\end{tikzcd}
=
\begin{tikzcd}
B \downarrow p \arrow[r, dashed, "\bar{r}"] & E^2 \arrow[r, bend left, "p_1"] \arrow[r, bend right, "p_0"'] \arrow[r, phantom, "\Uparrow\kappa"] & E
\end{tikzcd}$$

Prove that $K \dashv \bar{r}$ with ε an isomorphism.

Talk 10

1. Prove that $A^2 \xrightarrow{(p_1,p_0)} A \times A$ is discrete in $\mathcal{K}_{/A\times A}$.

Talk 11

1. Prove that modules $A \overset{E}{\not\to} B$ and $A \overset{E'}{\not\to} B$ are equivalent over $A \times B$ if and only if they are vertically isomorphic in $\mathbf{Mod}(\mathcal{K})$.

Talk 12

1. Define the limit of a diagram $A \xrightarrow{d} E$ between ∞-categories in a not-necessarily-cartesian closed cosmos.

2. Prove directly from the definition above that right adjoints preserve limits.

Talk 13

1. Pick one of the things that cosmological functors preserve (e.g. adjunctions, limits) and prove it.

2. Suppose that $\mathcal{K} \underset{f}{\xrightarrow{\sim}} \mathbf{1}$ are biequivalent.

   (a) Prove that $A \xrightarrow{u} B \in \mathcal{K}$ has a left adjoint if and only if $FA \xrightarrow{Fu} FB \in \mathbf{1}$ has a left adjoint.

   (b) Prove that $\mathbf{1} \xrightarrow{d} C^J \in \mathcal{L}$ has a limit if and only if any corresponding diagram in $\mathcal{K}$ does.

Talk 14

1. If $E \overset{p}{\twoheadrightarrow} B$ is cocartesian, define the functor E_f.

$$
\begin{tikzcd}
E_x \arrow[rr] \arrow[dr, dashed, "E_f"] \arrow[dd] & & E \arrow[dd, two heads, "p"] \\
& E_y \arrow[ur] \arrow[dl] \arrow[ur, phantom, "\chi \Downarrow"] & \\
1 \arrow[rr, bend left, "x"] \arrow[rr, bend right, "y"'] \arrow[rr, phantom, "\Downarrow f"] & & B
\end{tikzcd}
$$

Talk 15

1. Compute the homotopy coherent n-simplex $\mathrm{FU}_\bullet[n] \in \mathbf{sSet} - \mathbf{Cat}$.

2. If A has limits of shape I and J and products, prove that A has limits of shape $I \coprod J$.

Talk 16

1. Is the cosmological functor $\mathbf{qCat} \xrightarrow{\sim} \mathbf{CSS}$ Toën's Quillen equivalence? Emily said she wasn't sure.

References

[RV] Emily Riehl and Dominic Verity. ∞-categories for the working mathematician (version 6/1/2018). Preliminary draft version available from `www.math.jhu.edu/~eriehl/ICWM.pdf`.