2017 TALBOT WORKSHOP: OBSTRUCTION THEORY FOR STRUCTURED RING SPECTRA

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Prerequisites

We will assume some working knowledge of a "modern" category of spectra, i.e. familiarity with a symmetric monoidal category of spectra with product that descends to the smash product in the stable homotopy category [EKMM97], [HSS00]. We will also use freely the language of model category theory [DS95], [Hov99], [Hir03]. Participants should know about k-invariants and Postnikov towers [GJ99, Chapter VI]. Familiarity with the rudiments of spectral sequences will also be assumed [Wei94, Chapter 5], [McC01].

Detailed List of Talks

1 **Overview talk:** [mentors] What are structured ring spectra, why do we care? Historical perspective.

2 **Operads:** Background [May72], [Val12], Stasheff's associahedron and A_∞ operads [Sta63], E_n-operads: the Boardman-Vogt little n-cubes operad, E_∞-operads: the Barratt-Eccles operad, the linear isometries operad [BV73, May09c]. Simplicial spectra over simplicial operads [GH, section 1.3]

3 **Examples of structured ring spectra:** Brief reminder of additive infinite loop space theory [May77, Seg74, Tho95]. Multiplicative infinite loop space theory, connective E_∞-ring spectra from bipermutative categories, examples arising this way [EM06, May09a]. Thom spectra [May09b], [ABG$^+$14], [Rud98], [CM15].

4 **Robinson's obstruction theory I:** The A_∞ case. Description of strategy. Cooperation algebras and Hochschild cohomology. Obstructions to existence and uniqueness. Applications: Morava K-theory, completed Johnson-Wilson spectra. [Rob89, Bak91, Rob04, Ang08].

5 **Robinson's obstruction theory II:** The E_∞ case and Gamma cohomology. Spaces of trees and Lie(n), as ingredients of an E_∞-operad. Notion of n-stage for an E_∞-structure via filtration of this operad. The Ξ bicomplex and definition of Γ-cohomology. Obstructions to existence of extension of the underlying $n-1$-stage of a given n-stage to an $n+1$-stage. Obstructions to uniqueness. [RW96, RW02, Rob03, Rob04, Ric06].

6 **Gamma cohomology I: properties, calculations** Transitivity long exact sequence. Flat base change. Free E_∞-algebras. Vanishing for étale algebras [RW02]. Calculations for commutative group algebras, polynomial algebras, truncated polynomial algebras, smooth algebras. Application to Lubin-Tate spectra [RR04].

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7 **Gamma cohomology II: more calculations and applications.** Calculations for rings of integer-valued polynomials; applications to uniqueness of the E_∞ structures on KU and KO; existence and uniqueness for the Adams summand $E(1)$ of $KU_{(p)}$ and for the I_n-adic completions of Johnson-Wilson spectra $E(n)$ [BR05]. A lower bound for coherences on BP: at a prime p, BP has a $(2p^2+2p-2)$-stage structure; lower bound for coherences on (localized) Johnson-Wilson spectra [Ric06].

8 **Quillen (co)homology:** Homology as total left derived functors of abelianization [Qui67]. The classical theory for associative and commutative rings: André-Quillen (co)homology, the cotangent complex and derivations [Qui70]. The case of simplicial algebras over a simplicial operad [GH04, section 4]

9 **TAQ I: Construction and properties.** Quillen cohomology in the context of spectra. The E_∞ case [Bas99], [BGR08]. The A_∞ case [Laz04b]. Are there differences between the algebraic and topological versions? Do they agree for Eilenberg-MacLane spectra? Relation to ordinary spectra cohomology. Analogue of Hurewicz theorem. Tools for calculating TAQ [Bas99, section 5]. Universal coefficients spectral sequences [BM13].

10 **TAQ II: Cohomology for operadic algebras.** Derived indecomposables as stabilization [BM05]. Application to suspension spectra of E_∞ ring spaces. The case of E_n R-algebras over H for connective commutative S-algebras R and H [BM13, section 2]. The iterated bar construction computes the derived indecomposables. Related spectral sequences and their multiplicative structure [BM11].

11 **Obstruction theory for connective spectra:** Constructing Postnikov Towers [Bas99, DS06]. Calculations and applications. Lazarev's work [Laz01], [Laz04a]. Morava K-theory [Ang11]. E_n genera [CM15].

12 **Application to the Brown-Peterson spectrum** [BM13]. The speaker should coordinate with the person doing talk 10 to discuss the action of Dyer-Lashof operations on the relevant spectral sequence. Discuss existence of E_4 structure. In what sense is this unique? What goes wrong for E_5? Coordinate with person doing talk 11 on the structure of the Quillen idempotent.

13 **Goerss-Hopkins' obstruction theory I:** Overview of the strategy and theory; input data as E_*-coalgebra in E_*E-comodules; output information on the homotopy type of the moduli space of realizations. Some key ingredients: simplicial spectra over simplicial operads, resolutions, the spiral exact sequence [GH, GH04, DK84, DKS95]. The speaker should coordinate with the person doing talk 2 for the part of simplicial operads.

14 **Goerss-Hopkins' obstruction theory II:** Relevant André-Quillen cohomology. The Bousfield-Kan spectral sequence for the homotopy groups of the space of structured maps. Obstructions to realizing maps. Postnikov towers for simplicial algebras, n-stages and constructing realizations inductively. Decomposition of moduli spaces. Obstructions to realization and uniqueness [GH, GH04].

15 **Applications:** Lifting the algebraic theory of deformations of height n formal groups to structured ring spectra: Lubin-Tate spectra, the Hopkins-Miller Theorem [Rez98], Goerss-Hopkins E_∞ version [GH04]. Generalized truncated Brown-Peterson spectra of height 2, $BP\langle 2\rangle$ [LN12].

16 **Comparison results I:** Gamma homology as stable homotopy of Γ-modules [PR00]. Gamma homology as TAQ of Eilenberg-MacLane spectra (in the flat case) [BM02]. André-Quillen cohomology for E_∞ differential graded algebras and simplicial E_∞ algebras and relation to TAQ [Man03]. The Goerss-Hopkins obstruction groups are isomorphic to gamma cohomology groups [BR04].

17 **Comparison results II:** Stable homotopy of algebraic theories [Sch01]. For commutative augmented algebras, this is isomorphic to stable homotopy of Γ-modules. Isomorphism of Atiyah-Hirzebruch spectral sequences.[BR04].

18/19 **Related Topics:** We have left two lectures for some related topics, to be chosen by those giving these talks. There are connections to many areas of active research. A few possibilities are listed below. Alternatively, participants could propose their own topic to the mentors.

- Negative results: While the question of whether BP has an E_∞ ring spectrum structure is open, there are results which show that there is no such structure with certain good properties one might hope for. In particular, the natural orientation from MU to BP is not an E_∞ map, at least for small primes [JN10]; BP is not an E_∞ core of MU [HKM01].
- Functor Calculus: TAQ as the derivative of the forgetful functor from a category of commutative algebras to a category of modules [BM02]. Also relevant [Ric01, Kuh06, Chi05]
- Algebraic formulation of E_n-homology [Fre10, Fre11]; as functor homology [LR11].
- Factorization homology, [Fra13]
- Homotopy completion [HH13]; TAQ via circle product of operadic algebras and bimodules; associated filtrations [KP].

20 **Future directions:** [mentors, discussion]

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