Talbot 2016:
Equivariant homotopy theory and the Kervaire invariant one problem

(notes from talks given at the workshop)

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Disclaimer

These are notes I took during the 2016 Talbot Workshop. I, not the speakers, bear responsibility for mistakes. If you do find any errors, please report them to:

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Contents

Notation

$\pi^u X$	nonequivariant (underlying) homotopy of X
$i_H^* X$	restriction of the G-spectrum X to an H-spectrum
$\mathcal{S}$	category of (orthogonal) spectra (9.2)
$\mathcal{S}^G$	category of (orthogonal) G-spectra (9.7)
$\mathscr{J}$	indexing category for orthogonal spectra (9.3)
$\mathscr{J}_G$	Mandell-May indexing category for orthogonal G-spectra (9.6)
$MU_{\mathbb{R}}$	see beginning of Talk 15
$MU^{((G))}$	see beginning of Talk 15
$\mathcal{T}^G$	objects are based G-spaces, morphisms are equivariant continuous maps
$\mathcal{T}_G$	objects are based G-spaces, morphisms are all continuous maps

Talk 1: Introduction (Doug Ravenel)

See Doug Ravenel's slides.

Talk 2: Sketch of the proof (Mike Hill)

Theorem 2.1. *There are manifolds of Kervaire invariant one only in dimensions* $2, 6, 14, 30, 62$ *and possibly 126.*

The first step was to take Browder's work with the Adams spectral sequence (ASS) $\mathrm{Ext}_A^{**}(\mathbb{F}_2, \mathbb{F}_2) \implies \pi_*^s$; Browder's theorem says that $h_j^2 \in \mathrm{Ext}^{2,2^{j+1}}$ is a permanent cycle iff there is a manifold of Kervaire invariant $2^j - 2$. (Really it's a coset of manifolds – it's modulo Adams filtration.) There is a map from the Adams-Novikov spectral sequence $\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*)$ to the ASS sending $\beta_{2^{j+1}/2^{j-1}} + \text{noise} \mapsto h_j^2$. This is detected in Adams filtration 0, 1, or 2. But filtration 0 just has rational information which is known, and filtration 1 is just the image of J, which is also known. The only possibility is filtration 2. The question is whether there are any $\beta_{2^{j+1}/2^{j-1}}$ that are permanent cycles.

MU_*MU is recording all possible isomorphisms between formal groups. Given a chosen formal group law, together with a collection of automorphisms, then we get a map from Ext over all possible automorphisms to a smaller Ext, which records just the formal groups that we care about, in our case $H^2(C_8; \pi_{2^{j+1}} R)$. This is the E_2 term for the homotopy fixed point spectral sequence for $\pi_*(R^{hC_8})$. This style of argument (except for the ASS part) is what Doug did for an odd prime. You have classes $\beta_{2^{j+1}/2^{j-1}}$ to the homotopy fixed point spectral sequence that takes the kernel to 0. There's a refinement due to Hopkins and Miller, namely the Lubin-Tate spectrum E_4; Hopkins-Miller says that C_8 acts on this, and $H^2(C_8; \pi_{2^{j+1}} R)$ is the corresponding ASS E_2 page. The problem is that it's too big.

That was too hard, so we made it harder by adding more structure; there are fewer things that can go wrong if there's more structure.

Instead, we compute homotopy groups of actual fixed points. The strategy is four-fold:

(1) (Detection theorem) We produce a C_8-spectrum $\Omega_{\mathbb{O}}$ such that that Kervaire classes are detected in $\pi_*\Omega_{\mathbb{O}}^{hC_8}$. (So I can use Ω in place of R, and everything I said goes through.)

(2) (Gap theorem) $\pi_{-2}\Omega_{\mathbb{O}}^{C_8} = 0$

(3) (Periodicity theorem) $\pi_{k+256}\Omega_{\mathbb{O}}^{hC_8} \cong \pi_k\Omega_{\mathbb{O}}^{hC_8}$

(4) (Homotopy fixed point theorem) $\Omega_{\mathbb{O}}^{C_8} \xrightarrow{\simeq} \Omega_{\mathbb{O}}^{hC_8}$

The first theorem is classical – it's similar to what Doug did in the odd primary case. The other theorems come from the use of a new tool, the slice spectral sequence.

2.1. Extremely crash course in equivariant homotopy. We'll see some models as the week progresses, but I need the following thing. We want to have a notion of G-spectra (we'll stick with finite G) with the following properties:

(1) It should be like the usual category of spectra (cofiber sequences are fiber sequences, etc.)

(2) If X, Y are finite G-CW complexes[1] then

$$[\Sigma_+^\infty X, \Sigma_+^\infty Y] \cong \varinjlim_{\substack{V \text{ a f.d.} \\ \text{rep of } G}} [\Sigma_+^V X, \Sigma_+^Y Y]^G$$

Declare these to be the hom objects. Everything in sight is an abelian group, cofiber sequences are fiber sequences, finite wedges are finite products, etc. You have to close things up under various limit and colimit constructions. The downside is that Alexander duality doesn't work like it does in the classical case, where you use that to get from the Spanier Whitehead category to spectra.

(3) Finite G-sets are self-dual:

$$[G/H_+ \wedge E, F] \cong [E, G/H_+ \wedge F].$$

(4) The category is a closed symmetric monoidal category under $- \wedge -$. Closed means smashing with a fixed spectrum has a right adjoint (internal hom).

(5) Representation spheres are invertible: $S^V \wedge S^{-V} \simeq S^0$. This is what Doug talked about when he said we had an $RO(G)$-grading. Invertible objects are now exactly the representation spheres.

(6) If $H \subset G$ have a restriction $i_H^* : \mathcal{S}^G \to \mathcal{S}^H$ and this has both adjoints, induction $G_+ \wedge_H -$ and coinduction $F_H(G_+, -)$, and we want $G_+ \wedge_H X \xrightarrow{\sim} F_H(G_+, X)$.

3, 5, and 6 are equivalent. In spaces, I have the map in 6 but it's almost never an equivalence: on the right, the group acts by moving the factors around and the fixed points are the diagonal; on the left, there are no fixed points (it's just a bunch of copies of X stacked). This is saying something essential about stability.

Write this as $\bigvee_{G/H} X \xrightarrow{\sim} \prod_{G/H} X$. So in the stable world, finite wedges are finite products.

[1]it has cells of the form $G/H \times D^n$, where the group acts in the obvious way on the left and trivially on the disc

Here are some consequences. $[E,F]$ extends to a pair of functors $(\mathrm{Set}^G)^{op} : T \mapsto [T_+ \wedge E, F]$ and $\mathrm{Set}^G : T \mapsto [E, T_+ \wedge F]$. Self-duality says that the values are naturally isomorphic. So these agree on objects but I have really different maps: the first are called *restriction maps*, and the second are called *transfers*.

Why restriction? Look at $G/H \to *$ and $E = S^0$. Associated to this I have a map $[*_+ \wedge S^0, F] \to [G/H_+ \wedge S^0, F]$; these are equivariant maps from $S^0 \to F$, and S^0 has no action. So the LHS is $[S^0, F^G]$ (where F^G means G-fixed points) and the RHS is $[G_+ \wedge_H S^0, F]$. Using the adjoint, this is $[S^0, F^H]$. So this map is $[S^0, F^G] \to [S^0, F^H]$: G-fixed points restrict to H-fixed points.

We have a second kind of homotopy group: for $V \in RO(G)$, $\underline{\pi}_V(E) = [S^V, E]$. (Note that I'm only writing $[-,-]^G$ in the unstable case.) We have a multiplicative version of induction – the norm. For intuition's sake, if the transfer is summing all the cosets, think of the norm as multiplying all the cosets, and it's a universal Hom for a multiplication (just like tensor product of a module is universal Hom for multiplying). The norm is a functor $N_H^G : \mathcal{S}^G \to \mathcal{S}^G$ satisfying

- $N_H^G S^V \simeq S^{\mathrm{Ind}_H^G V}$ for $V \in RO(H)$
- N_H^G commutes with sifted colimits. If I write something as the geometric realization of a simplicial thing, I can just take the norm of the simplicial levels.
- N_H^G is a strong symmetric monoidal functor for the smash product: $N_H^G(E \wedge F) = N_H^G(E) \wedge N_H^G(F)$. In particular, it's lax monoidal – it takes commutative monoids under $\wedge$ to commutative monoids under $\wedge$.
- So it's the left adjoint to the forgetful functor from $\mathrm{Comm}^G \to \mathrm{Comm}^H$ (where Comm^G means G-equivariant commutative monoids).
- $\Phi N_H^G \simeq \Phi^H E$ (here I really mean just a homotopical statement, whereas the previous $\cong$ is point-set level). "The failure of the norm to be additive is exactly what Φ is destroying."

2.2. What is $\Omega_{\mathbb{O}}$? First take $N_{C_2}^{C_8} MU_{\mathbb{R}}$; the underlying spectrum is $MU^{\wedge 4}$, and the underlying C_2-spectrum is $MU_{\mathbb{R}}^{\wedge 4}$. We know that $\pi_*^8(M_{C_2}^{C_8} MU_{\mathbb{R}}) \cong \pi_*(MU^{\wedge 4}) \cong L^{\otimes 4}$ (where L is the Lazard ring). (But note that the last $\cong$ is not a canonical isomorphism.) The action of C_8 permutes the factors, and when you come back around you use complex conjugation, which corresponds to the $[-1]$ series. That is, $(a,b,c,d) \mapsto (\overline{d}, a, b, c)$.

We can write $\pi_*(MU^{\wedge 4})$ as $\mathbb{Z}[r_1, \gamma r_1, \gamma^2 r_1, \gamma^3 r_1, r_2, \gamma r_2, \dots]$ where $|r_i| = 2_i$ and γ is a generator of C_8, and the C_8-action is:

$$\gamma \cdot (\gamma^j r_k) = \begin{cases} \gamma^{j+1} r_k & j+1 \le 3 \\ (-1)^k r_k & j+1 = 4 \end{cases}$$

Given a monomial $p \in (\pi_* MU^4) \otimes \mathbb{Z}/2$, let $H_p = \mathrm{Stab}(p)$ and $\|p\| = \frac{|p|}{|H_p|} \cdot \rho_{H_p}$. Mod 2, the (-1) above goes away, and C_2 fixes every generator, so $C_2 \subset H_p \subset C_8$.

Let $\Omega_{\mathbb{O}} = \overline{D}^{-1} N_{C_2}^{C_8} MU_{\mathbb{R}}$. I'm not going to say what $\overline{D}$ is, because it's what works. This element is in $\pi_{19\rho_8}^{C_8} N_{C_2}^{C_8} MU_{\mathbb{R}}$. Regular representations are a nice family: they always restrict and induce to other regular representations.

Slice filtration. The spaces in MU are Thom spaces of Grassmannians $BU(n)$. $GL_n(\mathbb{C})$ have a Schubert decomposition describing its cell structure. If I evaluate on $\mathbb{C}$, I have cells that look like $\mathbb{C}^k$, and this is true equivariantly. In particular, C_2 acts on $\mathbb{C}$ by complex conjugation, and turns $\mathbb{C}^k$ into $k\rho_2$. So we have a filtration of $MU(n)$ with filtration quotients $S^{k\rho_2}$.

The norm commutes with sifted colimits; a cell structure is a sort of sifted colimit. So $N_{C_2}^{C_8} MU$ has a cell structure with "cells" $C_{8+} \wedge_H D(k\rho_H)$ for $C_2 \subset H \subset C_8$. I attach disks along the boundary spheres; this is the slice filtration.

The slice filtration is the filtration induced by the collections

$$\{G_+ \wedge_H S^{k\rho_H - \varepsilon} \; : \; k \cdot |H| - \varepsilon \geq n, \; \varepsilon = 0, 1\}.$$

So you should think of the -1 as coming from attaching maps of Schubert cells. After the fact, it turned out that you didn't need to do that. (If you ignore the equivariance, this is the Postnikov filtration.)

The big theorem that makes everything else run is the slice theorem.

Theorem 2.2 (Slice theorem)**.** *The slice associated graded of $N_{C_2}^{C_8} MU_{\mathbb{R}}$ ($\Sigma^{k\rho_8} N_{C_2}^{C_8} MU_{\mathbb{R}}$) is of the form*

$$\Bigg(\bigvee_{\substack{p \text{ monomial in} \\ (\pi_*(MU^{\wedge 4}) \otimes \mathbb{Z}/2)}} C_{8+} \wedge_{H_p} S^{\|p\|} \Bigg) \wedge H\underline{\mathbb{Z}}$$

Here $H\underline{\mathbb{Z}}$ is the spectrum computing Bredon homology with coefficients in $\underline{\mathbb{Z}}$.

Geometric fixed points are so named because it corresponds with your geometric intuition for fixed points: it commutes with Thom spectra and suspensions. In particular, $\Phi^{C_2} BU(n) = BO(n)$. Leveraging this allows you to get all the differentials you need in the slice spectral sequence.

Fact: $(H\underline{\mathbb{Z}})(V) = \mathbb{Z}\{S^V\}/\mathbb{Z} \cdot *$. $H\underline{\mathbb{Z}}$ is the 0^{th} Postnikov section of MU.

Talk 3: The odd-primary Arf invariant (Foling Zou)

(Note: this talk is about Ravenel's paper "The non-existence of odd primary Arf invariant elements in stable homotopy", 1978.) The methods used to prove the non-survival of the Arf invariant element for $p > 3$ is the idea behind the HHR detection theorem.

Recall the Adams spectral sequence $E_2^{s,t} = \mathrm{Ext}_{A_p}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \implies \pi_{t-s}(S^0) \otimes \mathbb{Z}_p$. (We write A_p for the mod-p Steenrod algebra.) The E_r page has differentials $d_r : E^{s,t} \to E^{s+r, t+r-1}$. The diagram is bigraded $(s, t-s)$ such that differential d_r maps 1 leftwards and r upwards. s is called the filtration and $r - s$ is called the stem.

In the 0-line (i.e. $s = 0$), there is just one element (at $(0, 0)$). For $p = 2$, the 1-line consists of elements $h_i \in \mathrm{Ext}_{A_2}^{1, 2^i}$ which correspond to the Steenrod operations Sq^{2^i}. The survival of h_i is equivalent to the existence of Hopf invariant 1 map in suitable degree. By a theorem of

Adams that, for $i \geq 4$, h_i does not survive, and the differential that kills it is $d_2 h_i = h_0 h_{i-1}^2$. For $p > 2$, the 1-line consists of elements $h_i \in \mathrm{Ext}_{A_p}^{1,qp^i}$ for $q = 2(p-1)$, which correspond to the odd Steenrod operations P^{p^i}, and $a_0 \in \mathrm{Ext}_{A_p}^{1,1}$, which corresponds to the Bockstein β. For $i \geq 1$, there is a differential $d_2 h_i = a_0 b_{i-1}$. You should think of the b_i's as analogous to the h_{i-1}^2 elements for $p = 2$.

In the 2-line, there are Arf invariant elements. In the $p = 2$ case, they are h_i^2. The name comes from a theorem of Browder which says that h_i^2 is a permanent cycle iff there is a manifold (of suitable dimension) of Arf invariant 1. It is known that for $i \leq 5$ they survive. The main HHR theorem is that for $i \geq 7$ they don't. And $i = 6$ is still open.

In the $p > 2$ case, let $b_i = -\langle h_i, \dots, h_i \rangle$ be the p-fold Massey product. This is just h_i^2 for $p = 2$.

Aside: The analogy between h_i^2 and b_i can also be seen in the following example. For $p > 2$, $H^*(\mathbb{Z}/p; \mathbb{Z}/p) = E[h] \otimes P[b]$ where $b = \langle h, \dots, h \rangle$ and for $p = 2$, $H^*(\mathbb{Z}/2; \mathbb{Z}/2) = P[h]$ (there's no b here because "b" $= h^2$).

b_i corresponds the an Adem relation involving $P^{(p-1)p^i} P^{p^i}$. b_0 survives. The main result of the paper is to prove that for $p > 3$, b_i does not survive for $i \geq 1$. For $p = 3$, one step of the proof doesn't work, and it turns out that b_1 does not survive, b_2 survives, and it is open for $i \geq 3$.

The first result in this direction is by Toda.

Theorem 3.1 (Toda)**.** $d_{2p-1} b_1 = h_0 b_0^p$

This introduces the hope of nontrivial differentials $d_{2p-1} b_i = h_0 b_{i-1}^p$, but this hope was discouraged by a calculation of May, who showed that for $p = 3$, $h_0 b_1^3 = 0$.

The paper uses the ANSS instead of the ASS to get over this obstacle.

Recall the Brown-Peterson spectrum BP, which has coefficient ring $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ with $|v_i| = 2(p^i - 1)$, and $BP_*BP = BP_*[t_1, t_2, \dots]$ with $|t_i| = 2(p^i - 1)$. This gives the Adams-Novikov spectral sequence

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*) \implies \pi_{t-s}(S^0) \otimes \mathbb{Z}_p.$$

There is a spectrum map $BP \to H\mathbb{Z}/p$ which gives a map from the ANSS to the ASS; it happens that they converge to the same thing outside the 0-stem.

We want to show:

Theorem 3.2. *If $p > 3$ and $i \geq 1$, then b_i does not survive the ASS.*

and we break it into two steps.

Step 1. Find a representative of the preimage of this b_i in the ANSS. In the cobar construction,

$$b_i = -\sum_{0<j<p^{i+1}} \frac{1}{p}\binom{p^{i+1}}{j}[t_1^j | t_1^{p^i - j}] \qquad \text{and} \qquad h_0 = -[t_1].$$

Theorem 3.3. *For $p \geq 3$, $d_{2p-1}b_{i+1} \neq 0$ for all $i \geq 0$ in the ANSS.*

Step 1.1 By induction on Toda's theorem, and the relation $h_{i+1}b_1^p = h_{i+2}b_0^p$ obtained by applying Steenrod operation to another relation from cobar construction, one can show that $d_{2p-1}b_{i+1} = h_0 b_i^p \bmod \ker b_0^{a_i}$, where $a_i = \frac{p(p^i-1)}{p-1}$.

Step 1.2. The nontrivial step is to show that $h_0 b_0^{i_0} \cdot \dots \cdot b_k^{i_k} \neq 0$, $b_0^{i_0} \cdot \dots \cdot b_k^{i_k} \neq 0$ in $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, so that the differential is indeed nontrivial.

The idea is that, for $n = p-1$, the construction of a map:

$$\begin{tikzcd}
\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \arrow[r] & \mathrm{Ext}_{\mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n})}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}_{\mathrm{Hom}(\mathbb{F}_{p^n}[\mathbb{Z}/p], \mathbb{F}_{p^n})}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[d, "\cong"] \\
& \mathrm{Ext}_{\mathbb{F}_{p^n}[S_n]}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[r] \arrow[d, equal] & \mathrm{Ext}_{\mathbb{F}_{p^n}[\mathbb{Z}/p]}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[d, equal] \\
& H_c^*(S_n; \mathbb{F}_{p^n}) \arrow[r] & H^*(\mathbb{Z}/p; \mathbb{F}_{p^n}) = E[h] \otimes P[b]
\end{tikzcd} \tag{3.1}$$

takes $h_0 \mapsto -ch$ and $b_i \mapsto -c^{p^{i+1}}b$ for a nontrivial constant c. Here S_n is the Morava stabilizer group that will be explained later, and when $n = p-1$ it has a subgroup of order p.

Step 2. For $p > 3$, b_{i+1} in the ASS does not survive. (Theorem 3.3)

Since there is a map of spectral sequences ANSS to ASS, if $x = b_{i+1}$ survives in ASS, then there is some $\widetilde{x}$ in the ANSS which survives to the same thing. It has to have filtration $s \leq 2$, but either 0 or 1 is not possible because of sparseness in the E_2 page of ANSS and stem calculations. So $\widetilde{x}$ has to have filtration 2, and be a preimage of x under the map.

Miller-Ravenel-Wilson showed that

Lemma 3.4. $\mathrm{Ext}^{2, qp^{i+2}}_{BP_*BP}(BP_*, BP_*)$ *is generated by $\beta_{a_{i,j}/p^{i+3-2j}}$ for $j = 1, 2, \dots, \lfloor \frac{i+3}{2} \rfloor$, where $a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1)$. If $j = 1$, then $\beta_{p^{i+1}/p^{i+1}} = b_{i+1}$. The map of SS takes the $j > 1$ elements to 0.*

Then $\widetilde{x}$ looks like $b_{i+1} + y$ where y is the "noise" generated by the beta elements with index $j > 1$. Our goal is to filter these noise out by a detection map.

Lemma 3.5. *Under the natural map*

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \to \mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_3)$$

where $I_3 = (p, v_1, v_2)$, $\beta_{a_{i,j}/p^{i+3-2j}} (j > 1)$ maps to 0.

Theorem 3.6 (Smith)**.** *At $p \geq 5$, there is a spectrum $V(2)$ with $BP_*(V(2)) = BP_*/I_3$ and there is a map $f : S^0 \to V(2)$ which is the obvious map on $BP_*(-)$.*

So f induces a map of ANSS, while on E_2 page it's the natural map as in Lemma 3.5.

Look at $g_* : \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \to \mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$ for $n = p-1$; this is the detection map we want. In other words, we want to show that

$$g_*(d_{2p-1}(b_{i+1} + y)) = g_*d_{2p-1}(b_{i+1}) + g_*d_{2p-1}(y)$$

is nonzero.

Notice that g_* factors through f_*. We have the second part $f_*(d_{2p-1}(y)) = d_{2p-1}(f_*(y)) = 0$ by Lemma 3.4.

$$\begin{tikzcd}
\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & & \mathrm{Ext}(BP_*, v_n^{-1}BP_*/I_n) \\
& \mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_3) \arrow[ur] &
\end{tikzcd}$$

For the first part, g_* is indeed half way through the map in Step 1.2:

$$\begin{tikzcd}
\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \arrow[r, "g_*"] & \dots \arrow[r] & \mathrm{Ext}_{\mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n],\mathbb{F}_{p^n})}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}_{\mathrm{Hom}(\mathbb{F}_{p^n}[\mathbb{Z}/p],\mathbb{F}_{p^n})}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[d, "\cong"] \\
& & \mathrm{Ext}_{\mathbb{F}_{p^n}[S_n]}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[r] \arrow[d, equal] & \mathrm{Ext}_{\mathbb{F}_{p^n}[\mathbb{Z}/p]}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \arrow[d, equal] \\
& & H^*_c(S_n; \mathbb{F}_{p^n}) \arrow[r] & H^*(\mathbb{Z}/p; \mathbb{F}_{p^n})
\end{tikzcd}$$

As $d_{2p-1}(b_{i+1})$ is mapped non trivially to the end, it follows that $g_*(d_{2p-1}(b_{i+1})) \neq 0$.

Revisit of Step 1.2. Now we are left with basically showing $h_0 \mapsto -ch$ and $b_i \mapsto -c^{p^{i+1}}b$. It suffices to show that the image of h_0 at the beginning is some nontrivial multiple of h at the end (see (3.1)), since in the cobar complex, h_0 is represented by $[t_1]$, h_i by $[t_1^{p^i}]$. If $h_0 \mapsto -ch$, then it follows that $h_i \mapsto -c^{p^i}h$ and $b_i \mapsto -c^{p^{i+1}}b$. (Recall $b_i = -\langle h_i, \dots, h_i \rangle$.)

Define the Witt ring $W(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\xi_n] = \{a_0 + a_1p + a_2p^2 + \dots : a_i^{p^n} = a_i\}$, where ξ_n is the $(p^n-1)^{st}$ root of unity.

Define

$$E_n = W(\mathbb{F}_{p^n}) \langle s \rangle /(s^n - p) = \{x_0 + x_1s + \dots + x_{n-1}s^{n-1} : x_i \in W(\mathbb{F}_{p^n})\} = \{\sum_{i\geq 0} e_i s^i : e_i^{p^n} = e_i\}.$$

Note that s does not commute with $x \in W(\mathbb{F}_{p^n})$. Rather, $sx = x^\sigma s$ where σ is the lifting of the Frobenius automorphism on $\mathbb{F}_{p^n}$.

S_n is the units of E_n congruent to 1 mod (s). In other words, there is an exact sequence $1 \to S_n \to E_n^\times \to \mathbb{F}_{p^n}^\times \to 0$.

The map in (3.1)

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \to \mathrm{Ext}_{\mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n})}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \to \mathrm{Ext}_{\mathrm{Hom}(\mathbb{F}_{p^n}[\mathbb{Z}/p], \mathbb{F}_{p^n})}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n})$$

is induced by a map of Hopf algebroid

$$(BP_*, BP_*BP) \to (\mathbb{F}_{p^n}, \mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n})) \to (\mathbb{F}_{p^n}, \mathrm{Hom}(\mathbb{F}_{p^n}[\mathbb{Z}/p], \mathbb{F}_{p^n})).$$

The latter two are indeed Hopf algebras. More precisely, the first map is

$$\begin{array}{ccl} BP_*BP & \to & \mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n})) \\ t_i & \mapsto & (t_i : 1 + \sum_{i>0} e_i s^i \mapsto \overline{e_i}), \quad \overline{e_i} \text{ is the mod-}p \text{ reduction of } e_i. \end{array}$$

and the second map is by restricting to an order p subgroup of S_n.

First we show that t_1 is not zero in $\mathrm{Hom}(\mathbb{F}_{p^n}[\mathbb{Z}/p], \mathbb{F}_{p^n})$. Take the generator x of $\mathbb{Z}/p$ in S_n, such that $x^p = 1$. Write $x = 1 + \sum_{i>0} e_i s^n$. We want to show $\bar{e}_1 \neq 0$. Notice that mod-p reduction being 0 is equivalent to being 0 itself. If e_1 were zero, then we can expand $(1 + \sum_{i>0} e_i s^n)^p = 1$ and mod it out by ideals generated by some power of s. Inductively this would imply all the e_i's are zero, showing the x is of order 1, contradicting the assumption that x is of order p.

Second, we have $\mathrm{Hom}(\mathbb{F}_{p^n}[\mathbb{Z}/p], \mathbb{F}_{p^n}) = \mathbb{F}_{p^n}[t]/(t^p - t)$ where t is primitive. It is a fact that t_1 is primitive in $\mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n}))$. Since it is nonzero in the image, it has to be mapped to a nontrivial multiple of t.

As h_0 is represented by $[t_1]$ and h by $[t]$, it follows that h_0 is mapped to a nontrivial multiple of h in (3.1). This completes the proof.

Talk 4: Review of category theory (Yexin Qu)

This talk is about Kan extensions, enriched categories, and Day convolutions.

4.1. Kan extension. Idea: suppose you have a functor $F : \mathcal{C}' \to \mathcal{E}$ and a functor $\mathcal{C}' \to \mathcal{C}$ that you should think of as "inclusion", and you want to extend F to be a functor out of $\mathcal{C}$. The Kan extension is the universal way to do this.

The definition is built up layer-by-layer. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories, and let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors. If there exists $L : \mathcal{D} \to \mathcal{E}$ equipped with a natural transformation $\eta : F \implies L \circ K$ such that for all $G : \mathcal{D} \to \mathcal{E}$ equipped with a natural transformation to $\gamma = F \implies G \circ K$ there exists a unique $\alpha : L \implies G$ such that $\gamma = \alpha \circ \eta$. Call L the *left Kan extension* of F along K, and write $\mathrm{Lan}_K F$.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "L"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Downarrow\alpha"] &
\end{tikzcd}$$

To get the right Kan extension, reverse all the arrows.

There is a right adjoint $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ sending $h \mapsto h \circ K$, $\alpha \mapsto \gamma = \alpha \circ \eta$. So there is an isomorphism

$$\mathcal{E}^D(\underline{\mathrm{Lan}}_K -, -) \cong \mathcal{E}^{\mathcal{C}}(-, K^* -).$$

This is a *local Kan extension.* To define global Kan extensions, for all $F \in \mathcal{E}^{\mathcal{C}}$, define $K_!$ to be the left adjoint to K^*; this is the left Kan extension functor $\mathrm{Lan}_K -$. The right adjoint K_* to K^* is called the right Kan extension functor and denoted $\mathrm{Ran}_K -$.

A functor $F : \mathcal{C} \to \mathrm{Set}$ is called *corepresentable* if there exists a natural isomorphism $F \cong h^{\mathcal{C}} : \mathcal{C} \to \mathrm{Set}$ where $h : c' \mapsto \mathcal{C}(c, c')$ for some c.

Example 4.1. The left Kan extension of

$$\begin{tikzcd} \begin{smallmatrix} a & \to & b \\ \downarrow & & \\ c & & \end{smallmatrix} \arrow[r, "X"] \arrow[d, hook, "i"'] & \mathcal{E} \\ \begin{smallmatrix} a & \to & b \\ \downarrow & & \downarrow \\ c & \to & d \end{smallmatrix} \arrow[ur, dotted, "\mathrm{Lan}_i X"'] & \end{tikzcd}$$

is the pushout of the diagram X.

A left Kan extension is a *pointwise left Kan extension* if it is preserved by all corepresentable functors: given corepresentable G in:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \arrow[r, "G"] & \mathrm{Set} \\ & \mathcal{D} \arrow[ur, "\mathrm{Lan}_K F"] \arrow[urr, dotted, bend right] & & \end{tikzcd}$$

one has $G \circ \mathrm{Lan}_K F = \mathrm{Lan}_K G \circ F$.

4.2. Ends and coends. Let $H : \mathcal{J}^{op} \times \mathcal{J} \to \mathcal{C}$ be a functor where $\mathcal{J}$ is small and $\mathcal{C}$ is complete. For all $f \in \mathrm{Mor}(\mathcal{J})$, $f : j \to j'$ we have functors $f_* : \mathrm{Hom}(j, j) \to H(j, j')$ and $f^* : \mathrm{Hom}(j', j') \to \mathrm{Hom}(j, j')$. We can generalize this by defining $\varphi_* : \mathrm{Hom}(j, j) \to \prod_{\substack{f \in \mathrm{Mor}\,\mathcal{J} \\ \mathrm{dom}(f) = j}} H(j, \mathrm{codomain}(f))$; then you get morphisms $\prod_{j \in \mathcal{J}} H(j, j) \xrightarrow{\varphi_*} \prod_{f \in \mathrm{Mor}\,\mathcal{J}} (H, \mathrm{dom}(f), \mathrm{cod} f)$ and $\prod_{j' \in \mathcal{J}} H(j', j') \xrightarrow{\varphi^*} \prod_{f \in \mathrm{Mor}\,\mathcal{J}} H(\mathrm{dom} f, \mathrm{cod} f)$.

Definition 4.2. An end $\int_{\mathcal{J}} H(j, j)$ is the equalizer of

$$\int_{\mathcal{J}} H(j, j) \to \prod_{j \in \mathcal{J}} H(j, j) \underset{\varphi_*}{\overset{\varphi^*}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\,\mathcal{J}} H(\mathrm{dom} f, \mathrm{cod} f)$$

if $\mathcal{J}$ is small and $\mathcal{C}$ is cocomplete. A *coend* $\int^{\mathcal{J}} H(j,j)$ is the coequalizer of

$$\coprod_{f\in \mathrm{Mor}(\mathcal{J})} H(\mathrm{dom} f, \mathrm{cod} f) \underset{\varphi_*}{\overset{\varphi^*}{\rightrightarrows}} \coprod_{j\in\mathcal{J}} H(j,j) \to \int^{\mathcal{J}} H(j,j).$$

Let $\mathcal{C}$ be small and $\mathcal{E}$ cocomplete. Then

$$\mathrm{Lan}_K F(d) = \int^{\mathcal{J}} \mathcal{D}(K(c),d)\cdot F(c).$$

Here $S\cdot b = \coprod_{s\in S} b$.

4.3. Monoidal categories.

Definition 4.3. A category $\mathcal{C}$ is *monoidal* if it has a binary operation $\otimes$, a unit object 1, and natural isomorphisms

$$\begin{aligned}\alpha_{X,Y,Z} &: (X\otimes Y)\otimes Z \cong X\otimes (Y\otimes Z)\\ \lambda &: 1\otimes X\cong X\\ \rho &: X\otimes 1\cong X\end{aligned}$$

that satisfies some properties.

Definition 4.4. If a monoidal category $\mathcal{C}$ is *symmetric*, we also need a natural isomorphism $T_{XY}: X\otimes Y\cong Y\otimes X$.

Definition 4.5. A monoidal category is called *closed* if for all $X\in\mathcal{C}$, $-\otimes X$ has a right adjoint. This is called the *internal Hom*, and written $\underline{\mathcal{C}}(X,-)$.

For example, take the category of finite vector spaces with product as $\oplus$, unit object 0, and the required morphisms as embeddings. This is *not* a closed monoidal category, because $(0,\mathcal{C}(X,Y))\ncong\mathcal{C}(X,Y)$.

If you don't have an internal Hom, you might want to consider enriched categories.

4.4. Enriched categories. Let $V=(V_\bullet,\otimes,1)$ be a symmetric monoidal category. We say that $\mathcal{C}$ is a V-category, or a category enriched over V, if

(1) there is a collection of objects in $\mathcal{C}$;
(2) for any $X,Y\in\mathrm{ob}\,\mathcal{C}$, an object $V_0\in\mathcal{C}(X,Y)$, and $1\to\mathcal{C}(X,X)\in\mathrm{Mor}\,V_0$;
(3) for all $X,Y,Z\in\mathrm{ob}\,\mathcal{C}$, there is a composition law

$$\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y)\to\mathcal{C}(X,Z).$$

4.5. Enriched functors. $F:\mathcal{D}\to\mathcal{C}$ is an enriched functor if $F:\mathrm{ob}\,\mathcal{D}\to\mathrm{ob}\,\mathcal{C}$, and $\mathcal{D}(X,Y)\to\mathcal{C}(F(X),F(Y))$.

Definition 4.6. A V-natural transformation $T : F \implies G$ if it assigns to an object X a morphism $T_X : 1 \to \mathcal{C}(F(X), G(X))$ making the diagram commute:

$$\begin{tikzcd}
\mathcal{D}(X,Y) \arrow[r, "T_Y \otimes F"] \arrow[d, "G \otimes T_X"'] & \mathcal{C}(F(X),G(X)) \otimes \mathcal{C}(F(X),F(Y)) \arrow[d] \\
\mathcal{C}(G(X),G(Y)) \otimes \mathcal{C}(F(X),G(Y)) \arrow[r] & \mathcal{C}(F(X),G(Y))
\end{tikzcd}$$

4.6. Day convolution. Let $\mathcal{D} = (\mathcal{D}_0, \oplus, 0)$ be a small, symmetric monoidal category enriched over $V = (V_0, \otimes, 1)$ a cocomplete closed symmetric monoidal category. Let $X, Y \in [\mathcal{D}, V]$ (enriched functors $\mathcal{D} \to V$). Let $X \square Y$ be the left Kan extension of the composition of $\otimes, X \times Y$ along $\oplus$:

$$\begin{tikzcd}
\mathcal{D} \times \mathcal{D} \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & V \times V \arrow[r, "\otimes"] & V \\
& \mathcal{D} \arrow[ur, "X \square Y"'] &
\end{tikzcd}$$

This is a binary operation in $[D, V]$. So $([D,V], \square, I)$ where $I : \mathcal{D} \to V$ sending $D \mapsto \mathrm{Hom}(0_{\mathcal{D}}, D)$.

Let $\mathcal{D} = \mathscr{J}_G$ be the category of finite-dimensional orthogonal representations (but the morphisms require some explanation), this is a symmetric monoidal category with product $\oplus$ and unit 0. Let $V = \mathcal{T}_G$ be topological G-spaces where the maps are not required to be equivariant. Applying this to that diagram with $X, Y : \mathscr{J}_G \to \mathcal{T}_G$, we have

$$\begin{tikzcd}
\mathscr{J}_G \times \mathscr{J}_G \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & \mathcal{T}_G \times \mathcal{T}_G \arrow[r, "\wedge"] & \mathcal{T}_G \\
& \mathscr{J}_G \arrow[ur, "X \wedge Y"'] &
\end{tikzcd}$$

Talk 5: Equivariant homotopy theory I (J.D. Quigley)

RECALL: To build a CW complex, start by knowing how to add a cell by forming the pushout:

$$\begin{tikzcd}
\partial D^n \simeq S^{n-1} \arrow[r, "f_n"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
D^n \arrow[r] & X^n
\end{tikzcd}$$

Maybe we want to add more than one cell, so we look at

$$\begin{tikzcd}
K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
K_n \times D^n \arrow[r] & X^n
\end{tikzcd}$$

So here, f_n is the attaching map, and K_n is a discrete set.

We then have a sequence of inclusions $K_0 = X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \ldots \hookrightarrow \operatorname{colim}_n X^n = X$.

Definition 5.1. A *G-CW complex* is a CW complex as above where each K_n is a G-set, and each f_n is G-equivariant. Use the trivial action of G on S^{n-1} and D^n.

Since K_n is a G-set, we can express it $K_n = \bigsqcup_{i \in I} G/H_i$ where $H_i < G$ are defined up to conjugacy. We say that $G/H_i \times D^n$ is an *n-dimensional G-cell.*

Example 5.2. Let $G = C_2$. Does S^2 with the antipodal action have a G-CW structure? Think of this as compactified $\mathbb{C}$ with the sign representation, and first think of the CW complex structure with one point and a 2-disc attached. (This is *not* the regular representation.) This is G acting on a CW complex but is not a G-CW complex. (This is actually a problem for a different reason – it doesn't send the basepoint to itself.)

But we can build it up with a few more cells – use two 0-cells, two 1-cells between them, and two 2-cells between those. As a G-CW complex, this has $K_0 = K_1 = K_2 = G$, and C_2 acts by permuting cells.

Theorem 5.3 (Bredon)**.** *A G-equivariant map $f : X \to Y$ between G-CW complexes X, Y is an equivariant homotopy equivalence iff $f^H : X^H \to Y^H$ is an ordinary homotopy equivalence for all $H \leq G$.*

Remark/ Definition 5.4. H-equivariant homotopy of a G-space X is $\pi_*^H(X) = \pi_*(X^H)$.

Example 5.5 (another G-CW complex)**.** Let $\mathcal{P}$ be a family of proper (closed) subgroups of G. Define $E\mathcal{P}$, the "universal space" for $\mathcal{P}$, as the G-space (unique up to equivariant equivalence) satisfying $E\mathcal{P}^G = \emptyset$ and $E\mathcal{P}^H$ is weakly contractible for all $H \in \mathcal{P}$.

For example, if $G = C_p$, then attach G-cells $G/H \times D^n$ for all $H \in \mathcal{P}$ to make $E\mathcal{P}^H$ weakly contractible. (This is described in a paper by Lück called "Transformation groups and algebraic K-theory".)

RECALL: A finite-dimensional orthogonal real representation of G is a group homomorphism $\pi : G \to O(V)$ where V is a finite-dimensional Euclidean real vector space.

Example 5.6 (Sign representation)**.** Let $G = C_{p^n} \subset \Sigma_{p^n}$. Then $\pi(g)(v) = \operatorname{sgn}(g) \cdot v$. This is a 1-dimensional representation. If p is odd, this is trivial.

Example 5.7 (Regular representation)**.** Let G be finite, and let $V = \mathbb{R}[G]$. Then $\pi(g)(v) = \pi(g)(\sum_{g \in G} r_i g_i) = \sum_{g_i \in G} r_i g g_i$.

Example 5.8 (Reduced regular representation)**.** Use $V = \mathbb{R}[G]/\langle g_1 + g_2 + \cdots + g_k \rangle$ where $G = \{g_1, \ldots, g_k\}$. You could also define it as a subspace of $\mathbb{R}[G]$ where the coordinates sum to 0.

Example 5.9. Let $G = C_{2^n}$ have generator γ. Consider $\pi : C_{2^n} \to O(V)$. This is determined by $\pi(\gamma) \in O(V)$, i.e. a choice of orthogonal matrix A with $A^{2^n} = I$. If you thought about this more (or looked it up on wikipedia) you'd believe that this can be written as a diagonal matrix with $R_1, \ldots, R_k, \pm 1, \ldots, \pm 1$ on the diagonal, where $R_i = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Pick θ such that $R_i^{2^n} = I_2$. (Here $\theta \equiv 0 \pmod{\frac{2\pi}{2^n}}$ but not π.)

We can put a partial ordering on these: say $V_1 < V_2$ if for every irreducible representation U, we have $\dim \mathrm{Hom}^G(U, V_1) < \dim \mathrm{Hom}^G(U, V_2) - 1$. Ignoring the -1, this says that V_1 embeds equivariantly into V_2; with the -1, it says that $O(V_1, V_2)^G$ is connected, i.e. all equivariant orthogonal embeddings are homotopic. (Here $O(V_1, V_2)^G$ means G-equivariant orthogonal embeddings.)

Let V be a G-representation. Its *representation sphere* S^V has underlying space the one-point compactification of V with basepoint ∞, and the action is that G acts on V by the representation and acts trivially on ∞.

Let V, V' be G-representations, $H < G$, W an H-representation.

- $S^{V \oplus V'} \cong S^V \wedge S^{V'}$
- $i_H^G S^V = S^{\mathrm{Res}_H^G V}$ where i_H^G : G-spaces $\to$ H-spaces is the forgetful functor.
- $S^{\mathrm{Ind}_H^G(W)} = \mathcal{T}^H(G_+, S^W) = H$-equivariant maps $G_+ \to S^W$ where $\mathrm{Ind}_H^G(W) = \mathbb{R}[G] \otimes_{\mathbb{R}[H]} W$. H acts on G_+ by left-multiplication, G acts on G_+ by right-multiplication, and G acts on $\mathcal{T}^H(G_+, S^W)$ by left-multiplication on the source.

Let $G = C_{p^n}$. We can give S^V the structure of a G-CW complex as follows. Let $G^{(i)}$ to be the index-p^i subgroup of G. With this definition, we get a series of inclusions $S^{V^G} = S^{V^{G^{(0)}}} \hookrightarrow S^{V^{G^{(1)}}} \hookrightarrow \ldots \hookrightarrow S^{V^{G^{(n)}}} = S^{V|_{\{e\}}} = S^V$. G acts on S^{V^G} trivially, so start by attaching a single $|V^G|$-cell.

Get $S^{V^{G^{(i)}}}$ from $S^{V^{G^{(i-1)}}}$by attaching $G/G^{(i)} \times D^n$ for each $|V^{G^{(i-1)}} < n \leq |V^{G^{(i)}}|$.

Example 5.10. Let $V = \mathrm{sgn} \oplus \mathrm{sgn}$ and $G = C_2$. Then $|V^{C_2}| = |\{0\}| = 0$ and $|V^{C_2}| = 0 < n \leq 2$. This is the same decomposition we had before (two each of 0-cells, 1-cells, 2-cells), but G acts trivially on the 0-cells.

If X is a G-space and V is a G-representation, $\pi_V^H X = \pi_0^H \Omega^V X = [S^V]^H$.

Talk 6: Introduction to equivariant homotopy theory II
(Fei Xie)

Let V be a finite-dimensional real orthogonal representation of a finite group G (this is what we mean when we say "representation of G").

Definition 6.1. An *orthogonal G-spectrum* X is a collection of based G-spaces X_V indexed by representations V of G with a non-equivariant action of $O(V)$. For an orthogonal inclusion $t : V \to W$ there is a structure map $\sigma : S^{W-tV} \wedge X_V \to X_W$ compatible with the orthogonal action and the G-action.

Say that X is an Ω-spectrum if the adjoint maps $\widetilde{\sigma} : X_V \to \Omega^{W-tV} X_W$ are all homeomorphisms. (These are the fibrant objects.)

Notation 6.2. Let $\mathcal{S}^G$ denote the category of G-spectra with equivariant maps.

Definition 6.3. For $H \leq G$, the H-equivariant k^{th} stable homotopy group of X is

$$\pi_k^H(X) = \varinjlim_{V > -k} \pi_{V+k}^H(X_V)$$

(this uses the partial ordering in the last talk).

Definition 6.4. A stable weak equivalence (which we call just a weak equivalence) is a map $X \to Y$ in $\mathcal{S}^G$ inducing isomorphisms of $\pi_k^H(-)$ for $k \in \mathbb{Z}$, $H \leq G$.

Recall there is a universal functor $\mathcal{S}^G \to \operatorname{ho}\mathcal{S}^G$ sending weak equivalences to isomorphisms. The functor $\pi_k^H(-) : \mathcal{S}^G \to \mathrm{Ab}$ takes weak equivalences to isomorphisms so it descends to a functor $\pi_k^H(-) = [S^k, -]^H : \operatorname{ho}\mathcal{S}^G \to \mathrm{Ab}$.

Theorem 6.5. *Let X_f be a fibrant replacement for X. Then $\pi_k^H(X) = \pi_k((X_f)^H)$.*

6.1. Isotropy separation sequence. Let $\mathcal{F}$ be a nonempty collection of subgroups of G closed under passing to subgroups and conjugates. Such a collection is called a *family of subgroups of G*. There is a universal unbased G-space $E\mathcal{F}$ satisfying

$$(E\mathcal{F})^H = \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F}. \end{cases}$$

$E\mathcal{F}_+$ can be given a G-CW complex structure with cells of the form $(G/H)_+ \wedge D^n$ for $H \in \mathcal{F}$.

Let $\widetilde{E}\mathcal{F}$ be the cofiber:

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}.$$

This satisfies:

$$(\widetilde{E}\mathcal{F})^H = \begin{cases} * & H \in \mathcal{F} \\ S^0 & H \notin \mathcal{F}. \end{cases}$$

Let $\mathcal{P}$ be the family of all proper subgroups of G. Then the isotropy separation sequence is

$$E\mathcal{P}_+ \wedge X \to X \to \widetilde{E}\mathcal{P} \wedge X.$$

We know that $(G/H)_+ \wedge -$ is adjoint to the restriction, so the LHS is determined by the action of proper H. This can be used to prove things by induction on the order of the group.

Definition 6.6. The geometric fixed point functor is

$$\Phi^G(X) = ((\widetilde{E}\mathcal{P} \wedge X)_f)^G$$

where $(-)_f$ is fibrant replacement.

Remark 6.7. If $X \to Y$ is a map of cofibrant G-spectra $\widetilde{E}\mathcal{P} \wedge X \to \widetilde{E}\mathcal{P} \wedge Y$ is a weak equivalence iff $\Phi^G(X) \to \Phi^G(Y)$ is a weak equivalence. As an H-spectrum $\widetilde{E}\mathcal{P} \wedge X$ is contractible, so if $H \leq G$ is proper, then $\pi_*^H(\widetilde{E}\mathcal{P} \wedge X) = 0 = \pi_*^H(\widetilde{E}\mathcal{P} \wedge Y)$. (Note: $\widetilde{E}\mathcal{P}$ is contractible as an H-space and $- \wedge X$ preserves homotopy equivalences as X is cofibrant, so $\widetilde{E}\mathcal{P} \wedge X$ is also contractible as an H-space. Similarly for Y.)

Example 6.8. Let $G = C_{2^n}$. The space $E\mathcal{P} = EC_2 \simeq S^\infty$ with the antipodal action. The G-action is through the epimorphism $G \twoheadrightarrow C_2$. Then $\widetilde{E}\mathcal{P} = \lim_{n\to\infty} S^{n\sigma}$ where $S^{n\sigma}$ is the one-point compactification of n copies of the sign representation. Let's check this: if H is proper, then $(S^\infty)^H = (S^\infty)^* = S^\infty \simeq *$, and $(S^\infty)^G = (S^\infty)^{C_2} = \emptyset$. Here $S^\infty = \lim_{n\to\infty} S(n\sigma)$.

6.2. Mackey functors.

Definition 6.9. A *Mackey functor* consists of a pair of functors $\underline{M} = (\underline{M}_*, \underline{M}^*)$ from the category of finite G-sets to Ab such that:

- they have the same object function
- $\underline{M}_*$ is covariant
- $\underline{M}^*$ is contravariant
- they take disjoint unions to sums
- for a pullback diagram of finite G-sets

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[d, "\gamma"] & A \arrow[d, "\alpha"] \\ T \arrow[r, "\beta"] & B \end{tikzcd}$$

  it gives a commutative diagram

$$\begin{tikzcd} \underline{M}(S) \arrow[r, "\delta_*"] & \underline{M}(A) \\ \underline{M}(T) \arrow[r, "\beta_*"] \arrow[u, "\gamma^*"] & \underline{M}(B) \arrow[u, "\alpha^*"'] \end{tikzcd}$$

Given a morphism $A \xrightarrow{f} B$, call $\underline{M}^*(f)$ the restriction map and $\underline{M}_*(f)$ the transfer map.

Example 6.10. For a finite G-set B and $X \in \mathcal{S}^G$, homotopy groups

$$(\underline{\pi}_n(X))^*(B) = [S^n \wedge B_+, X]^G$$
$$(\underline{\pi}_n(X))_*(B) = [S^n, B_+ \wedge X]^G$$

form a Mackey functor. They have the same object function because finite G-sets are self-dual. It is enough to know this for $B = G/H$. In this case,

$$\underline{\pi}_n(X)(G/H) = [S^n \wedge G/H_+, X]^G = [S^n, X]^H = \pi_n^H(X)$$

is the homotopy groups we defined earlier.

Given a Mackey functor $\underline{M}$, there is an equivariant Eilenberg-Maclane spectrum $H\underline{M}$ such that

$$\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Definition 6.11. Equivariant homology is defined as

$$H_k^G(X; \underline{M}) = \pi_k^G(H\underline{M} \wedge X)$$

and cohomology is defined as

$$H_G^k(X; \underline{M}) = [X, \Sigma^k H\underline{M}]^G.$$

6.3. Constant and permutation Mackey functors. Let $\underline{\mathbb{Z}}$ denote the Mackey functor represented by $\mathbb{Z}$ with trivial action. So $\underline{\mathbb{Z}}(B) = \mathrm{Hom}^G(B, \mathbb{Z}) = \mathrm{Hom}(B/G, \mathbb{Z})$. This is the constant Mackey functor.

Let S be a G-set, and $\mathbb{Z}\{S\}$ be the free abelian group generated by S. Then the permutation Mackey functor on S is the Mackey functor represented by this, i.e. $\mathbb{Z}\{S\}(B) = \mathrm{Hom}^G(B, \mathbb{Z}\{S\})$. Here restriction maps $(-)^*$ are given by pre-composition, and transfer maps $(-)_*$ are given by summing over fibers. (I.e. let $g : A \to B$, for transfers, f gets sent to $g_*(f)(b) = \sum_{x \in g^{-1}(b)} f(x)$.)

Let B be a G-set. Consider a free G-set $G \times B$ (where the action is trivial on B, and left translation on G). Then the original action map $G \times B \to B$ is equivariant. Automorphisms of $G \times B$ over B are of the form $(g, b) \overset{x}{\mapsto} (gx, x^{-1}b)$ for $x \in G$. It induces a G-action on $\underline{M}(G \times B)$ by $(x^{-1})^*$.

Lemma 6.12. *Let B be a finite G-set, and $\underline{M}$ the permutation Mackey functor (on some G-set S). Then:*

(1) Given an epimorphism $B' \to B$ of finite G-sets, then $\underline{M}(B) \to \underline{M}(B') \rightrightarrows \underline{M}(B' \times_B B')$ is an equalizer.

(2) The action map $G \times B \to B$ induces isomorphism $\underline{M}(B) \to \underline{M}(G \times B)^G$.

(3) Given $G \to G/H$, there is an isomorphism $\underline{M}(G/H) \to \underline{M}(G)^H$.

(4) A map $\underline{M} \to \underline{M}'$ of permutation Mackey functors is an isomorphism iff $\underline{M}(G) \to \underline{M}'(G)$ is an isomorphism (which is automatically G-equivariant).

Talk 7: Model categories I (Ugur Yigit)

Definition 7.1. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

a lifting is a map $h : B \to X$ such that the resulting triangles are commutative.

If such lift exists for any f, g, say that i has the left lifting property w.r.t. p, and p has the right lifting property w.r.t. i.

Definition 7.2. A model category is a category $\mathcal{C}$ with the classes of maps:

(i) weak equivalences (w.e.)

(ii) fibrations (fib)

(iii) cofibrations (cof)

A map which is both a fibration [cofibration] and a weak equivalence is called an acyclic (or trivial) fibration [cofibration].

We require the following axioms:

(MC1) $\mathcal{C}$ has all small limits and colimits.

(MC2) If f and g are maps such that fg is defined and if two of $\{f, g, fg\}$ are weak equivalences, then so is the third.

(MC3) If f is a retract of g, and g is a weak equivalence, fibration, or cofibration, then so is f.

(MC4) Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

a lifting exists in the following two cases:

(i) i is a cofibration and p is an acyclic fibration

(ii) i is an acyclic cofibration and p is a fibration

(MC5) Any map $f : X \to Y$ can be factored in two ways:

(i) $f = pi$ where i is a cofibration and p is an acyclic fibration

(ii) $f = pi$ where i is an acyclic cofibration and p is a fibration

and these factorizations are functorial.[2]

Using the axioms, you can prove that all three classes of maps contain all identities, and that they are closed under transfinite compositions. In a model category, we automatically have a terminal object $*$ and an initial object $\emptyset$.

[2] Functoriality was not required by Quillen, but is typically required now, and all the interesting cases satisfy this. All factorizations gotten by the small object argument are functorial.

Definition 7.3. An object $A \in \mathcal{C}$ is called cofibrant if the unique map $\emptyset \to A$ is a cofibration, and fibrant if $A \to *$ is a fibrant.

Example 7.4. The category Top of topologial spaces can be given a model category structure by defining $f : X \to Y$ to be:

(i) a weak equivalence if it is a weak homotopy equivalence

(ii) a fibration if it is a Serre fibration[3]

(iii) a cofibration if it has the left lifting property w.r.t. all acyclic fibrations.

Note that the above definition of model categories is a bit over-determined: if you know the weak equivalences and cofibrations (or fibrations), you get everything else.

Proposition 7.5. *Let $\mathcal{C}$ be a model category.*

(1) The map $i : A \to B$ is a cofibration iff it has LLP w.r.t. all acyclic fibrations;

(2) the map $i : A \to B$ is an acyclic cofibration iff it has LLP w.r.t. all fibrations;

(3) the map $p : X \to Y$ is a fibration iff it has RLP w.r.t. all acyclic cofibrations;

(4) the map $p : X \to Y$ is an acyclic fibration iff it has RLP w.r.t. all cofibrations.

Be careful, though – if you define the weak equivalences and fibrations, there's only one possible choice of collection of cofibrations, but the resulting structure might not be a model category.

Proof. (1) $\implies$ By definition

$\impliedby$ By MC5, we can factor $i = pj$ where p is an acyclic fibration and j is a cofibration. So you get a diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ B \arrow[r, "1"] \arrow[ur, dotted] & B \end{tikzcd}$$

By assumption, this has a lift q. The diagram

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "i"'] & A \arrow[r, "1"] \arrow[d, "j"] & A \arrow[d, "i"] \\ B \arrow[r, "q"] & Z \arrow[r, "p"] & B \end{tikzcd}$$

shows that i is a retract of j. Now use MC3.

The other statements are similar. □

[3] A map $p : X \to Y$ is a Serre fibration if for any CW complex A, p has the right lifting property w.r.t. all inclusions $A \times \{0\} \to A \times [0,1]$. Equivalently, if it has RLP for $D^n \times \{0\} \to D^n \times [0,1]$, or equivalently, if it has RLP w.r.t. $X \times \{0\} \cup A \times [0,1] \to X \times [0,1]$ for pair (X, A).

Proposition 7.6. *Let $\mathcal{C}$ be a model category. Two of the classes of {weak equivalences, cofibrations, fibrations} determine the third.*

From (3) in the previous proposition, if we know the acyclic cofibrations, then we know the fibrations. By (4), if you know the cofibrations, you get the acyclic fibrations. A map is a weak equivalence iff it can be factored as an acyclic cofibration followed by an acyclic fibration. So it suffices to specify the acyclic cofibrations and cofibrations. (You can make the same statement for acyclic fibrations and fibrations but we like cofibrations.)

Instead of considering all [acyclic] cofibrations, we can consider *generating* [acyclic] cofibrations instead. The small object argument is a pain; I will concentrate on compactly generated model categories.

From now on, we are working in a topological category *(enriched in* Top*?)*.

Definition 7.7. An object $A \in \mathcal{C}$ is compact if given any $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \dots$, we have

$$\operatorname{colim}_n \mathcal{C}(A, X_n) \cong \mathcal{C}(A, \operatorname{colim}_n X_n).$$

Definition 7.8. We say that a class of maps J *permits the small object argument* in $\mathcal{C}$ if $\mathcal{C}$ is cocomplete and the domains of the morphisms in J are compact.

Definition 7.9. A cofibrantly generated model category is a model category such that:

(1) There is set I of maps called generating cofibrations that permits the small object argument and such that a map is an acyclic fibration iff it has the right lifting property w.r.t. all elements of I.

(2) There is a set J of maps called generating acyclic cofibrations that permits the small object argument and such that a map is a fibration iff it has the right lifting property w.r.t. J.

Example 7.10. In Top, we can take $I = \{i_n : n \geq 0\}$ where i_n is the natural inclusion $S^{n-1} = \partial D^{n-1} \to D^n$, and $J = \{j_n : n \geq 0\}$ where j_n is the map $I^n \to I^{n+1}$ *that is inclusion of the bottom face*.

Let $\mathcal{T}^G$ be the category of pointed topological G-spaces and equivariant maps.

Definition 7.11. An equivariant map $f : X \to Y$ is a *naïve Serre fibration* [weak equivalence] if f is a Serre fibration [weak equivalence] in $\mathcal{T}$.

It is a *genuine Serre fibration* [weak equivalence] if $f^H : X^H \to Y^H$ is a Serre fibration [weak equivalence] in $\mathcal{T}$.

Theorem 7.12. *In the naïve case, the sets*

$$I'_G = \{i_{n+} \wedge G_+ : n \geq 0\}$$
$$J'_G = \{j_{n+} \wedge G_+ : n \geq 0\}.$$

form a cofibrantly generated model category.

In the genuine case, use:

$$I_G = \{i_{n+} \wedge G/H_+ \,:\, n \geq 0, H \subset G\}$$
$$J_G = \{j_{n+} \wedge G/H_+ \,:\, n \geq 0, H \subset G\}.$$

You can also define model structures in between these by using the families in Fei's talk.

Cofibrations are retracts of relative CW complexes.

This satisfies the pushout-product axiom w.r.t. the smash product.

These are monoidal model structures.

Talk 8: Model categories II (Alex Yarosh)

I'm going to tell you when a model category can fulfill its dream and be a cofibrantly generated model category.

Definition 8.1. A category $\mathcal{C}$ is a homotopical category if it has a wide[4] subcategory W whose objects satisfy the 2 our of 6 property: if we have $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ where fg and hg are in W ("are equivalences") then f, g, h, and hgf are in W.

This implies the 2 out of 3 property, by taking one of the maps to be the identity. It can be shown that the underlying category of a model category is a homotopical category.

Theorem 8.2 (Kan Recognition Theorem)**.** *Let M be a bicomplete[5] homotopical category with sets of morphisms I, J that satisfy:*

(1) I, J permit the small object argument.

(2) $\mathrm{LLP}(\mathrm{RLP}(J)) \subset \mathrm{LLP}(\mathrm{RLP}(I)) \cap W$ *where RLP(I) is the set of all maps that have the right lifting property w.r.t. all maps in I. (*$\mathrm{RLP}(J)$ *is the class of all fibrations; LLP of that is acyclic cofibrations. Similarly,* $\mathrm{LLP}(\mathrm{RLP}(I))$ *is cofibrations.)*

(3) $\mathrm{RLP}(I) \subset \mathrm{RLP}(J) \cap W$*.*

(4) One of (2) or (3) is an equality.

Then M admits a cofibrantly generated model structure with I as the generating cofibrations and J as the generating acyclic cofibrations.

Theorem 8.3 (Kan Transfer Theorem)**.** *Let M be a cofibrantly generated model category with I and J the generating cofibrations and acyclic cofibrations and N a bicomplete category. Also assume that there is an adjunction $F : M \rightleftarrows N : U$. Then if:*

(1) FI, FJ both permit the small object argument

[4] W is wide if all objects are in W

[5] has all limits and colimits

(2) U takes relative FJ-cell complexes to weak equivalences

then N admits a cofibrantly generated model structure with FI and FJ as the generating cofibrations and acyclic cofibrations, respectively. The weak equivalences in N are the maps f such that Uf is a weak equivalence in M.

Definition 8.4. For a set of morphisms I, the subcategory of relative I-cell complexes is a subcategory of transfinite compositions of pushouts of maps in I: i.e.

$$X_0 \to X_1 \to \cdots \to X_\beta \to X_{\beta+1} \to \dots$$

where each of these morphisms is a pushout of maps in I.

As the notation in the transfer theorem suggests, this is often used where U is a forgetful functor and F is a free functor. So you can use this to put a model structure on a category with more structure.

New question: what if I have a model category but I don't like it because the weak equivalences are not suitable for my purposes – for example, we might want to enlarge the class of weak equivalences. This is what Bousfield localization is for.

From now on my model category M will be simplicial or topological, so I can talk about homotopy types of mapping spaces.

Definition 8.5. Let $\mathcal{C}$ be a class of morphisms. A fibrant object W is $\mathcal{C}$-local if for any map $f : A \to B$ in $\mathcal{C}$, the induced map $M(B, W) \to M(A, W)$ is a weak equivalence. A map $g : X \to Y$ is a $\mathcal{C}$-local equivalence if for any $\mathcal{C}$-local W, $M(Y, W) \to M(X, W)$ is a weak equivalence.

Definition 8.6. The (left) Bousfield localization of M at a class of morphisms $\mathcal{C}$ is a new model structure on M such that

(1) the weak equivalences are the $\mathcal{C}$-local equivalences;

(2) the cofibrations remain the same;

(3) fibrations are defined by the right lifting property along trivial cofibrations.

Enlarging the weak equivalences and keeping the cofibrations the same means that there are fewer fibrations. Note that this isn't a guarantee that this exists!

Sometimes "Bousfield localization" is the name of a functor $L_\mathcal{C}$ from M with the old model structure to M with the new model structure. We also have a natural transformation $\eta : \mathbb{1} \to L_\mathcal{C}$ such that $L_\mathcal{C}(X)$ is $\mathcal{C}$-local and $X \to L_\mathcal{C}X$ is a $\mathcal{C}$-equivalence.

Example 8.7 (Localization of spaces w.r.t. a homology theory h_*)**.** If $h_* = H(-, \mathbb{Z}/p)$ then localization w.r.t. h_* is p-completion. If $h_* = H(-, \mathbb{Z}_{(p)})$ then localization w.r.t. h_* is p-localization.

Definition 8.8. We define the *strict model structure on simplicial spectra* as follows:

- $f : X \to Y$ is a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence.
- $f : X \to Y$ is a fibration if $f_n : X_n \to Y_n$ is a fibration.
- $f : X \to Y$ is a cofibration if $f_0 : X_0 \to Y_0$ and induced maps $X_{n+1} \coprod_{\Sigma X_n} \Sigma Y_n \to Y_{n+1}$ for $n \geq 0$ are cofibrations.

Definition 8.9. We define the *stable model structure on simplicial spectra* as follows. Assume there's a functor $Q : sSp \to sSp$ and a natural transformation $\eta : \mathbb{1} \to Q$ such that QX is an Ω-spectrum for all X and $QX \to QQX$ is a weak equivalence. (For example, $(QX)_n = \varinjlim \operatorname{Sing} \Omega^i |X_{n+i}|$.)

- $f : X \to Y$ is a weak equivalence if $f_* : \pi_n X \to \pi_n Y$ is an isomorphism
- f is a stable cofibration if it's a strict cofibration.
- f is a stable fibration if:
  - f is a strict fibration
  - the diagram
$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & (QX)_n \arrow[d] \\ Y_n \arrow[r] & (QY)_n \end{tikzcd}$$
    is a homotopy pullback.

Example 8.10. Localize Top_* w.r.t. $\{f : S^{n+1} \to *\}$ *for a fixed n?*. Local objects are those X that have $\Omega^{n+1} X$ contractible. (Or equivalently, $\pi_k(X) = 0$ for $k \geq n+1$.)

Then the localization of X is the n^{th} Postnikov section of X.

Theorem 8.11. *If S is a set and M is either left proper and cellular*[6] *or left proper*[7] *and combinatorial*[8]*, then the Bousfield localization exists.*

You can construct the localization functor by fibrant replacement.

Talk 9: The Mandell-May definition of G-spectra (Renee Hoekzema)

9.1. Motivation. A classical spectrum is a sequence of spaces E_n with structure maps $\sigma : \Sigma E_n \to E_{n+1}$. Let's rephrase this in a way that's useful: let $\mathbb{N}$ be the category that has objects $n \in \mathbb{N}$ and morphisms only identities. This is a symmetric monoidal category: the product sends $(n, m) \mapsto n + m$ and on morphisms, $(\mathbb{1}_n, \mathbb{1}_m) \mapsto \mathbb{1}_{n+m}$. Consider $\mathbb{S}_{\mathbb{N}} \in \mathrm{Fun}(\mathbb{N}, \mathrm{Top}_*)$ sending $n \mapsto S^n$. I can trivially view this as a topologically-enriched category, where n is a point. This has a monoidal structure with Day convolution.

[6] "like Top"
[7] "like sSet"
[8] a model category is left proper if the pushout of a weak equivalence along a cofibration is a weak equivalence

Definition 9.1. A spectrum is a functor $E : \mathbb{N} \to \mathrm{Top}_*$ that is a module for $\mathbb{S}_{\mathbb{N}}$.

The problem is that $\mathbb{S}_{\mathbb{N}}$ is not a commutative monoid, and so $\mathbb{S}_{\mathbb{N}}$-modules do not form a symmetric monoidal category. By "modules for $\mathbb{S}_{\mathbb{N}}$", I mean it has an action $E_n \wedge S^m \to E_{n+m}$ such that the following diagram commutes:

$$\begin{tikzcd}
E_n \wedge S^k \wedge S^\ell \arrow[r] \arrow[d, "\sigma"] & E_n \wedge S^{k+\ell} \arrow[d, "\sigma"] \\
E_{n+k} \wedge S^k \arrow[r] & E_{n+k+\ell}
\end{tikzcd}$$

Commutativity is commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{S}_{\mathbb{N}} \otimes \mathbb{S}_{\mathbb{N}} \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\
& \mathbb{S}_{\mathbb{N}} \\
\mathbb{S}_{\mathbb{N}} \otimes \mathbb{S}_{\mathbb{N}} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

On objects, this is

$$\begin{tikzcd}
S^n \wedge S^m \arrow[r] \arrow[d, "\tau"] & S^{n+m} \\
S^m \wedge S^n \arrow[ur] &
\end{tikzcd}$$

In the top map, the n basis vectors in the $\mathbb{R}^n$ that is compactified to form S^n get sent to the first n basis vectors from S^{n+m}. Going the other way around, the n basis vectors from S^n are sent to the last n basis vectors from S^{n+m}. This doesn't commute!

Obviously, these things are related by a basis permutation map $S^{n+m} \to S^{n+m}$.

Actually, you only need commutativity up to isomorphism. So the diagram you really want to commute is

$$\begin{tikzcd}
\mathbb{S}_{\mathbb{N}} \otimes \mathbb{S}_{\mathbb{N}} \arrow[r, "\mu"] \arrow[d, "\tau"'] & \mathbb{S}_{\mathbb{N}} \arrow[d, "\cong"] \\
\mathbb{S}_{\mathbb{N}} \otimes \mathbb{S}_{\mathbb{N}} \arrow[r, "\mu"'] & \mathbb{S}_{\mathbb{N}}
\end{tikzcd}$$

and the problem is that in this category we don't actually have the requisite nontrivial automorphism of $\mathbb{S}_{\mathbb{N}}$ (due, of course, to lack of good automorphisms of $\mathbb{N}$ in this category).

9.2. Orthogonal spectra. Let $\mathcal{O}$ be the category whose objects are $n \in \mathbb{N}$ and whose morphisms are given by $\mathrm{Hom}(n, n) = O(n)$ and $\mathrm{Hom}(n, m) = \emptyset$ for $n \neq m$. This is symmetric monoidal: again, the map on objects is $(n, m) \mapsto n + m$, and on morphisms, the pair (f, g) for $f \in O(n)$ and $g \in O(m)$ gets sent to the transformation $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in O(n + m)$.

Again consider $\mathbb{S}_{\mathcal{O}} \in \mathrm{Fun}(\mathcal{O}, \mathrm{Top}_*)$ sending $n \mapsto S^n$. Now, the symmetric structure on $\mathrm{Fun}(\mathcal{O}, \mathrm{Top}_*)$ by Day convolution incorporates both $X \wedge Y \leftrightarrow Y \wedge X$, and switching f and g in $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$.

What does this Day convolution do? Remember it's the dotted arrow in

$$\begin{tikzcd} \mathcal{O} \times \mathcal{O} \arrow[r] \arrow[d, "+"] & \mathrm{Top}_* \times \mathrm{Top}_* \arrow[r, "- \wedge -"] & \mathrm{Top}_* \\ \mathcal{O} \arrow[urr, dotted] & & \end{tikzcd}$$

Definition 9.2. An orthogonal spectrum is a functor $E : \mathcal{O} \to \mathrm{Top}_*$ that is a module for $\mathbb{S}_{\mathcal{O}}$.

Definition 9.3. We define a category $\mathscr{J}$ whose objects are vector spaces V with inner product. The morphisms are... a little complicated. First consider $\mathcal{O}(V, W)$, the Stiefel manifold of orthogonal embeddings $V \hookrightarrow W$. Let $\varphi \in \mathcal{O}(V, W)$. Take the orthogonal complement $W - \varphi(V)$ for each φ; this defines a vector bundle on $O(V, W)$, where over every embedding sits the orthogonal complement of that embedding. Now define $\mathscr{J}(V, W)$ to be the Thom space of this vector bundle.

This is symmetric monoidal: on objects, the product sends $(V, W) \mapsto V \oplus W$ and on morphisms there is a composition $\mathscr{J}(V, W) \wedge \mathscr{J}(V', W') \to \mathscr{J}(V \oplus V', W \oplus W')$ and a composition $\mathscr{J}(V, W) \wedge \mathscr{J}(U, V) \to \mathscr{J}(U, W)$ that I won't define. Also, it's enriched over Top_*.

Let's re-define orthogonal spectra.

Definition 9.4. An orthogonal spectrum is a functor $E : \mathscr{J} \to \mathrm{Top}_*$ sending $V \mapsto E_V$. On morphisms, there's a map $\mathscr{J}(V, W) \to \mathrm{Hom}(E_V, E_W)$. The Hom-tensor adjunction in Top_* gives structure maps $\varepsilon_{V,W} : \mathscr{J}(V, W) \wedge E_V \to E_W$.

For example, $\mathscr{J}(\mathbb{R}^n, \mathbb{R}^{n+1})$ is the Thom space of the bundle $E \to O(n, n+1)$ with fiber $\mathbb{R}^1$, and that is $\bigvee_{O(n,n+1)} S^1$. So $\mathscr{J}(R^n, \mathbb{R}^{n+1}) \wedge E_{\mathbb{R}^n} = \bigvee_{\varphi} \Sigma_{\varphi} E_{\mathbb{R}^n} \to E_{\mathbb{R}^{n+1}}$.

9.3. The sphere spectrum and Yoneda spectra. (I write $\mathbb{S}$ instead of S to emphasize that it's a spectrum.) The sphere spectrum $\mathbb{S}^{-0} : V \mapsto S^V$ is the Thom space of $V \to *$ which is $\mathscr{J}(0, V)$. More generally, $\mathbb{S}^{-V} : W \mapsto \mathscr{J}(V, W)$. These are the representable functors – the image of the enriched Yoneda embedding $\mathscr{J}^{op} \hookrightarrow \mathrm{Fun}(\mathscr{J}, \mathrm{Top}_*)$ sending $V \mapsto \mathrm{Hom}_{\mathscr{J}}(-, V) = \mathscr{J}(V, -)$.

Lemma 9.5 (Enriched Yoneda lemma). $\mathrm{Hom}_{Sp}(\mathbb{S}^{-V}, E) = E_V$

9.4. Towards G-spectra. Let G be a finite group. Let $\mathcal{T}^G$ be the category whose objects are based G-spaces and whose morphisms are equivariant continuous maps.

Let $\mathcal{T}_G$ be the category whose objects are based G-spaces and whose morphisms are *all* continuous maps. This is the category we'll be interested in. It has a G-action: given some $f : X \to Y$ then we'll define the action of γ as the composition:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\gamma"] \\ X \arrow[u, "\gamma^{-1}"] & Y \end{tikzcd}$$

Now I'm going to adapt my indexing category to also have something to do with G.

Definition 9.6. The Mandell-May category is the category $\mathscr{J}_G$ whose objects are orthogonal representations of G. To define the morphisms, do the same thing as before; note that $\mathcal{O}(V, W)$ has all orthogonal embeddings $V \hookrightarrow W$, not just equivariant ones.

Definition 9.7. An (orthogonal) G-spectrum is a functor $E : \mathscr{J}_G \to \mathcal{T}_G$, sending $V \mapsto E_V$.

9.5. Naïve vs. genuine G-spectra. There is an embedding $\mathscr{J} \overset{i}{\hookrightarrow} \mathscr{J}_G$ by taking trivial representations – whatever G is, every vector space is present here because it has a trivial representation. It's also a full subcategory. So if I have a functor $\mathscr{J}_G \xrightarrow{E} \mathcal{T}_G$ then you compose to get a functor $\mathscr{J} \to \mathcal{T}_G$. Such a functor is called a *naïve G-spectrum.* It is just an orthogonal spectrum with a G-action.

A functor from $\mathscr{J}_G$ is completely determined by its value on $\mathscr{J}$. The categories are equivalent as Top_*-enriched categories, but the natural model structure you put on them is different. Why are they equivalent? Consider the structure maps $\varepsilon_{V,W} : \mathscr{J}(V, W) \wedge E_V \to E_W$, which factors over $\mathscr{J}_G(V, W) \wedge_{O(V)} E_V$. If $\dim V = \dim W$, then $\mathscr{J}(V, W) \wedge_{O(V)} E_V = O(V, W) \cong E_W$. So the functor is completely determined by what it does on the trivial representations.

An equivariant map that is an underlying homeomorphism is an equivariant homeomorphism.

9.6. Spectra and spaces. Given a spectrum F and a G-space X, define $E \wedge X : V \mapsto E_V \wedge X$. If $X = S^W$ then $E \wedge S^W$ is denoted $\Sigma^W E$. Define $F_G(X, E) : V \mapsto \mathrm{Hom}_{\mathcal{T}^G}(X, E_V)$. If $X = S^W$, then $F_G(S^W, E)$ is denoted $\Omega^W E$.

9.7. Tautological presentation. Any spectrum E is the reflexive coequalizer of (i.e. colimit of the diagram)

$$\bigvee_{V,W} \mathbb{S}^{-W} \wedge \mathscr{J}_G \wedge E_V \rightleftarrows \bigvee_V \mathbb{S}^{-V} \wedge E_V$$

The forward maps are $j_{V,W} \wedge E_V$ and $\mathbb{S}^{-W} \wedge \varepsilon_{V,W}$ and the backwards map is the obvious inclusion. Abbreviate this as follows: $\varinjlim_{V} \mathbb{S}^{-V} \wedge E_V$.

9.8. Smash product of spectra. We want to define the smash product given the Day convolution, and the Day convolution was the Kan extension of the following diagram

$$\begin{tikzcd} \mathscr{I}_G \times \mathscr{I}_G \arrow[r, "E\times F"] \arrow[d, "\oplus"'] & \mathcal{T}_G \times \mathcal{T}_G \arrow[r] & \mathcal{T}_G \\ \mathscr{I}_G \arrow[urr, dotted] & & \end{tikzcd}$$

$E \wedge F = \varinjlim_{V,W'} \mathbb{S}^{-V\oplus V'} \wedge E_V \wedge F_{V'}$, i.e. the reflexive coequalizer of

$$\bigvee_{V,V',W,W'} \mathbb{S}^{-W\oplus W'} \wedge \mathscr{I}_G(V,W) \wedge \mathscr{I}(V',W') \wedge E_V \wedge F_V \rightleftarrows \bigvee \mathbb{S}^{-V\oplus V'} \wedge E_V \wedge F_{V'}$$

The category of G-spaces and all maps is a closed monoidal category, but the category of G-spaces and equivariant maps is not closed.

Talk 10: The homotopy category of $\mathcal{S}^G$ (Allen Yuan)

A homotopical category is a category $\mathcal{C}$ plus a class of weak equivalences W (satisfying the two-of-six condition). There is a functor $\mathcal{C} \to \operatorname{ho}\mathcal{C}$, where $\operatorname{ho}\mathcal{C} = \mathcal{C}[W^{-1}]$ is just inverting the weak equivalences. This talk will be about $\operatorname{ho}\mathcal{C}$, and what we can say about it without actually putting a model structure on it.

Definition 10.1. For any category $\mathcal{A}$, a homotopy functor $\mathcal{C} \to \mathcal{A}$ is one that takes weak equivalences to isomorphisms. By the universal property of localization, this means it factors through the homotopy category

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \operatorname{ho}\mathcal{C} \arrow[d, dotted] \\ & \mathcal{A}. \end{tikzcd}$$

The analogous notion for functors between two homotopical categories is:

Definition 10.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is homotopical if it takes weak equivalences to weak equivalences. As such, it induces the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\ \operatorname{ho}\mathcal{C} \arrow[r, dotted, "\widetilde{F}"] & \operatorname{ho}\mathcal{D}. \end{tikzcd}$$

Thus, the study of functors on the homotopy category is equivalent to finding and studying homotopical functors upstairs.

Goals:

(1) Determine if functors are homotopical.

(2) If not, determine a homotopically wide[9] subcategory $\widetilde{\mathcal{C}} \subset \mathcal{C}$

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{ho}\,\widetilde{\mathcal{C}} \arrow[r, "\sim"] & \mathrm{ho}\,\mathcal{C} \end{tikzcd}$$

on which the functor is homotopical.

Some motivation:

Definition 10.3. The Spanier-Whitehead category SW^G is the category whose objects are finite G-CW complexes and whose morphisms are $SW^G(X,Y) = \mathrm{colim}_V [S^V \wedge X, S^V \wedge Y]^G$.

Pros:

- the objects are pretty easy to work with; it's easy to get a fuss-free smash product. So it's symmetric monoidal.
- can add formal desuspensions.
- Spanier-Whitehead duality.

Cons:

- It's too small: no small limits and colimits. Analogy: think about finite-dimensional vector spaces – they are the fully dualizable objects in the category of all vector spaces, but there are no small colimits. This isn't good enough. For example, you can't hope to get a model structure.

Our goal is to produce an embedding of SW^G into $\mathrm{ho}\,\mathcal{S}^G$ that is fully faithful and a symmetric monoidal embedding. There are some things we want this to satisfy:

- (Additivity) We need $\vee$ and finite $\prod$ to be homotopical functors, and we want $\bigvee_{i\in I} X_i \simeq \prod_{i\in I} X_i$ if I is a finite set. (For example, $S^n \times S^n = S^n \vee S^n \cup S^{2n}$ and stably $S^{2n} \to S^\infty$ which is contractible, so $S \times S \simeq S \vee S$.)
- (Stability) We want $- \wedge S^V$ and $- \wedge S^{-V}$ to be inverse equivalences on $\mathrm{ho}\,\mathcal{S}^G$.
- (Monoidal structure) We want $\mathrm{ho}\,\mathcal{S}^G$ to be symmetric monoidal under $\wedge$.

10.1. Additivity. We want to show $\pi_*(X \vee Y) = \pi_* X \oplus \pi_* Y$. Normally, you get a LES of cofiber sequences, and get a splitting of that. The key to getting a LES here is that the (co)tensor on $\mathcal{S}^G$ is levelwise. So you can define $F := X \times_Y PY$ (fiber) and $Y \cup CX$ (cofiber) levelwise.

Proposition 10.4. *For all $H \subset G$, get a long exact sequence:*

$$\cdots \to \pi_k^H F \to \pi_k^H X \to \pi_k^H Y \to \pi_{k-1}^H F \to \dots$$

[9] $\widetilde{\mathcal{C}} \subset \mathcal{C}$ is homotopically wide if every object in $\mathcal{C}$ is weakly equivalent to something in $\widetilde{\mathcal{C}}$.

Proof. Fiber sequences are defined levelwise and stuff commutes with colimits.

$$\mathrm{colim}_V(\pi^H_{k+V}F_V \to \pi^H_{k+V}X_V \to \dots)$$

□

For a cofiber sequence we want a similar LES.

Remark 10.5. If $Z \in \mathcal{S}^G$ there exists a fiber sequence $Z^{Y\cup CX} \to Z^Y \to Z^X$. Taking homotopy groups gives you the usual long exact sequence associated to a cofiber sequence. However, we want one with mapping INTO a cofiber sequence - this is the usual property of a stable category.

Proposition 10.6. *For $Z \in \mathcal{S}^G$, get a long exact sequence*

$$\dots \to [Z,X] \to [Z,Y] \to [Z, Y\cup CX] \to \dots$$

Proof. You have

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\beta"] & CZ \arrow[d] \\ Y \arrow[r] & Y\cup CX \end{tikzcd}$$

and you want to get the map α in

$$\begin{tikzcd} Z \arrow[r, "="] \arrow[d, dotted, "\alpha"] & Z \arrow[r] \arrow[d, "\beta"] & CZ \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Y\cup CX \end{tikzcd}$$

Extend this to

$$\begin{tikzcd} Z \arrow[r] \arrow[d, dotted, "\alpha"] & Z \arrow[r] \arrow[d] & CZ \arrow[r] \arrow[d] & \Sigma Z \arrow[r] \arrow[d, "\varphi"] & \Sigma Z \arrow[d, "\Sigma\beta"] \\ X \arrow[r] & Y \arrow[r] & Y\cup CX \arrow[r] & \Sigma X \arrow[r] & \Sigma Y \end{tikzcd}$$

Then claim that $\varphi = \Sigma\alpha$. □

Lemma 10.7. *There is a natural isomorphism $\pi^H_* X \cong \pi^H_{k+1}\Sigma X$.*

We want $Y \cup CX \simeq Y/X$. In spaces, this happens for h-cofibrations, and those participate in the Hurewicz model structure.

Definition 10.8. An h-cofibration is a map $i : A \hookrightarrow X$ in $\mathcal{S}^G$ such that for all $f : X \to Y$ and homotopy $H : A \wedge I_+ \to Y$ there exists a homotopy $\widetilde{H} : X \wedge I_+ \to Y$.

Proposition 10.9. *h-cofibrations in $\mathcal{S}^G$ are objectwise closed inclusions.*

Corollary 10.10. *If $f : X \to Y$ is an h-cofibration, then you get a LES*

$$\dots \to \pi^H_k(X) \to \pi^H_k(Y) \to \pi^H_k(Y/X) \to \dots.$$

Corollary 10.11. *For all $H \subset G$:*

(1) For any collection $\{X_\alpha\} \subset \mathcal{S}^G$, $\bigoplus \pi_*^H(X_\alpha) = \pi_*^H(\bigvee X_\alpha)$.

(2) If $\{X_\alpha\}$ is finite, then $\prod \pi_*^H(X_\alpha) = \pi_*^H(\prod X_\alpha)$. This follows from the fact that the unstable homotopy groups commute with products.

(3) If $\{X_\alpha\}$ is finite, then the natural map $\bigvee X_\alpha \to \prod X_\alpha$ is a weak equivalence.

Corollary 10.12. $\operatorname{ho}\mathcal{S}^G$ *is an additive category. Furthermore, the coproduct is computed as $\bigvee$, and finite products are $\prod$.*

How you get this is to use that these are the adjoints of the diagonal map, the diagonal map is homotopical, and things descend properly.

10.2. Showing that $-\wedge S^V$, $-\wedge S^{-V}$ are inverse equivalences on $\operatorname{ho}\mathcal{S}^G$. First, we need to show that these are homotopical. Do this later.

We know how to define $S^{-V} \wedge S^V \wedge X$, and we want to show it's weakly equivalent to X.

Example 10.13. In the non-equivariant case, look at $S^{-1} \wedge S^1$ vs. S^0. These are not the same spectrum: $(S^0)_n = S^n$ but $(S^{-1} \wedge S^1)_n = \mathscr{J}(1, n) \wedge S^1 = \operatorname{Thom}(O(1, n), \mathbb{R}^n \backslash \mathbb{R}) \wedge S^1 = \operatorname{Thom}(T(S^{n-1})) \wedge S^1$. When you add the tangent bundle to the normal bundle, you get a trivial bundle. So this is $\operatorname{Thom}(\mathbb{R}^n \downarrow S^{n-1}) = \Sigma^n(S^{n-1}_+) = S^n \vee S^{2n-1}$ and the second factor dies stably.

If $t : V \hookrightarrow W$ is an embedding, we get a map $S^{-W} \wedge S^W \to S^{-V} \wedge S^V$. The previous argument works in this case, too.

We constructed $\mathcal{S}^G \to \operatorname{ho}\mathcal{S}^G$ but we don't know how to access the morphisms in this homotopy category. Let's approximate it instead. One thing to try is that you can look at $\pi_0\mathcal{S}^G$. This has the advantage that it'll still detect weak equivalences.

But we would want to show that $S^{-V} \wedge S^V \wedge X \simeq X$, so instead, we're going to create a slightly tweaked category $\pi^{st}\mathcal{S}^G$ that is rigged to have this property. $\pi^{st}\mathcal{S}^G$ is the category whose objects are the objects of $\mathcal{S}^G$, but the morphisms are $\pi^{st}\mathcal{S}^G(X, Y) = \operatorname{colim}_V \pi_0\mathcal{S}^G(S^{-V} \wedge S^V \wedge X; Y)$.

Recall: for $H \subset G$, we had $\pi_n^H(X) = \operatorname{colim}_V \pi_{n+V}^H(X_V)$. The stable weak equivalences are maps $f : X \to Y$ inducing isomorphisms on π_n^H for all $H \subset G$, $n \in \mathbb{Z}$.

This approximating category $\pi^{st}\mathcal{S}^G$ still sees weak equivalences:

Proposition 10.14. $\pi^{st}\mathcal{S}^G(G/H_+ \wedge S^k, Y) = \pi_k^H(Y)$

This says that if two things are isomorphic in $\pi^{st}\mathcal{S}^G$ then they have to be weakly equivalent. Maps from $G/H_+ \wedge S^k$ is the same as $\pi^H(-)$. One thing that's really nice about this category is that it's easy to calculate maps in it.

Proof. Using the Yoneda property that $\mathcal{S}^G(S^{-V} \wedge A, Y) = \mathcal{T}^G(A, Y_V)$, we can turn this into a statement about spaces: $\operatorname{colim}_V \pi_0 \mathcal{S}^G(S^{-V} \wedge S^V \wedge G/H_+ \wedge S^k, Y) = \operatorname{colim}_V \pi_0 \mathcal{T}^G(S^V \wedge G/H_+ \wedge S^k, Y_0)$. But this is $\operatorname{colim}_V \pi_0 \mathcal{T}^H(S^V \wedge S^k, Y_V) = \pi_k^H(Y)$. □

(Recall $\mathcal{S}^G$ is the category of G-spectra with equivariant maps, enriched over Top_*.)

Proposition 10.15. *$S^{-V} \wedge S^V \wedge X \to X$ is an equivalence in $\mathrm{ho}\,\mathcal{S}^G$.*

We want to show that $- \wedge S^V$ and $- \wedge S^{-V}$ are inverse equivalences on $\mathrm{ho}\,\mathcal{S}^G$. The only thing we're missing is that they're actually functors on $\mathrm{ho}\,\mathcal{S}^G$, so we want to show that they're homotopical.

Remark 10.16. The fact that these are inverse equivalences over $\pi^{st}\mathcal{S}^G$ gives you a tensor over the Spanier-Whitehead category. We can rewrite $\pi^{st}\mathcal{S}^G(X, Y) = \operatorname{colim}_V \pi_0 \mathcal{S}^G(S^V \wedge X, S^V \wedge Y)$ so for finite G-CW complexes, there is an inclusion $SW^G(K, L) \hookrightarrow \pi^{st}\mathcal{S}^G(K, L)$ and so $\pi^{st}\mathcal{S}^G$ is tensored over SW^G.

As a consequence, you get duality. If J is a finite G-set, $\pi^{st}\mathcal{S}^G(Z, J_+ \wedge X) = \pi^{st}\mathcal{S}^G(J_+ \wedge Z, X) = \pi^{st}\mathcal{S}^G(Z, \prod_G X)$. If you do more work, you can amp this up to arbitrary indexed wedges, but we won't do that.

Now we're back to showing that $- \wedge S^V$ and $- \wedge S^{-V}$ are homotopical. Recall $\pi^{st}\mathcal{S}^G(G/H_+ \wedge S^k, -) = \pi_k^H(-)$ so this is a homotopy functor. The LES gives an argument to build things up with cells, so you can show that $\pi^{st}\mathcal{S}^G(S^\ell \wedge K, -)$ is homotopical, for K a finite G-CW complex and $\ell \in \mathbb{Z}$. In particular, $\pi^{st}\mathcal{S}^G(G/H_+ \wedge S^k \wedge S^V, -)$ is homotopical. By duality, this is the same thing as $\pi^{st}\mathcal{S}^G(G/H_+ \wedge S^k, S^{-V} \wedge -) = \pi_k^H(S^{-V} \wedge -)$ so that is homotopical. This is just the homotopy groups, so that means $S^{-V} \wedge -$ is homotopical. Since S^V and S^{-V} are inverse on the approximating category, $- \wedge S^V$ is also homotopical.

Corollary 10.17. *$- \wedge S^V$ and $- \wedge S^{-V}$ are inverse equivalences on $\mathrm{ho}\,\mathcal{S}^G$.*

Proposition 10.18. *We've shown that $\pi^{st}\mathcal{S}^G(S^{-V} \wedge K, -)$ is homotopical. Duality says that $- \wedge S^{-V} \wedge K$ is homotopical.*

Now we can say something about how good the approximation is.

$\pi^{st}\mathcal{S}^G$ is a homotopical category using the weak equivalences in $\mathcal{S}^G$. This gives an equivalence of categories $\mathrm{ho}\,\mathcal{S}^G \xrightarrow{\sim} \mathrm{ho}\,\pi^{st}\mathcal{S}^G$. The key idea is that, for a homotopical category $\mathcal{C}$, you get a natural transformation $\mathcal{C}(X, -) \to \mathrm{ho}\,\mathcal{C}(X, -)$. The LHS is a homotopy functor and the RHS isn't. But this is the initial natural transformation from the Yoneda functor $\mathcal{C}(X, -)$ to a homotopy functor. So the Yoneda functor in the homotopy category is the closest homotopical functor to $\mathcal{C}(X, -)$ from the right.

This means in particular that if $\mathcal{C}(X, -)$ were homotopical to begin with, then the two functors would be equal. This looks bothersome, but this condition is just never satisfied in e.g. actual G-spectra. But it is satisfied for our approximating category.

So when $\pi^{st}\mathcal{S}^G(X,-)$ is homotopical, we have $\operatorname{ho}\mathcal{S}^G(X,Y) = \operatorname{ho}\pi^{st}\mathcal{S}^G(X,Y) = \pi^{st}\mathcal{S}^G(X,Y)$, and this is good because we know how to calculate the thing on the right.

In particular, when X, Y are finite G-CW complexes, then $\operatorname{ho}\mathcal{S}^G(X,Y) = \operatorname{colim}_V \pi_0\mathcal{S}^G(X^V \wedge X, S^V \wedge Y) = SW^G(X,Y)$. So we've shown:

Corollary 10.19. *$\Sigma^\infty : SW^G \to \operatorname{ho}\mathcal{S}^G$ is fully faithful.*

Remark 10.20. (This is a property of the homotopy category that I'm not going to prove.) Any $X \in \mathcal{S}^G$ has a "canonical homotopy presentation"; that means you can write

$$X = \operatorname{hocolim}(\cdots \to S^{-V_n} \wedge X_{V_n} \to S^{-V_{n+1}} \wedge X_{V_{n+1}} \to \dots)$$

where $\{V_n\}$ is an exhaustive sequence and X_{V_n} is a G-CW complex.

10.3. Smash product. $\wedge$ is not known to be homotopical in general. But sometimes it is...

Definition 10.21. X is flat if $- \wedge X$ is homotopical.

We've shown that $S^{-V} \wedge K$ is flat. Now you should believe that this is a big enough class of objects that you can do the small object argument and get flat approximations.

Proposition 10.22. *If $X \in \mathcal{S}^G$, there exists a functorial weak equivalence $\widetilde{X} \xrightarrow{\sim} X$ where $\widetilde{X}$ is flat.*

For example, a cofibrant-fibrant approximation of anything is going to be flat. Cofibrant approximation in any of the model structures Hood will talk about is flat.

Let $\mathcal{S}^G_{\mathrm{fl}} \subset \mathcal{S}^G$ be the full subcategory of flat objects. Then you get an equivalence of categories $\operatorname{ho}\mathcal{S}^G_{\mathrm{fl}} \xrightarrow{\sim} \operatorname{ho}\mathcal{S}^G$. This is the homotopically wide subcategory I mentioned at the beginning.

Proposition 10.23. *Let X, Y be flat, and Z any spectrum. If you have a weak equivalence $X \xrightarrow{\sim} Y$ then $Z \wedge X \to Z \wedge Y$ is a weak equivalence.*

Proof.

$$\begin{tikzcd} X \wedge \widetilde{Z} \arrow[r] \arrow[d] & X \wedge Z \arrow[d] \\ Y \wedge \widetilde{Z} \arrow[r] & Y \wedge Z \end{tikzcd}$$

The top and bottom maps are weak equivalences. Smashing with $\widetilde{Z}$ is homotopical, so the left vertical map is a weak equivalence. □

Corollary 10.24. *$\wedge$ is a homotopical functor $\mathcal{S}^G_{fl} \times \mathcal{S}^G \to \mathcal{S}^G$. This gives you $\wedge : \operatorname{ho}\mathcal{S}^G \times \operatorname{ho}\mathcal{S}^G \to \operatorname{ho}\mathcal{S}^G$.*

Corollary 10.25. $\Sigma^\infty : SW^G \to \operatorname{ho}\mathcal{S}^G$ *is fully faithful and symmetric monoidal.*

Talk 11: The positive complete model structure and why we need it (Hood Chatham)

Allen told us in his talk about what information we can get about the homotopical structure of $\mathcal{S}^G$ directly. In particular, he built the homotopy category by "bare hands". However, we have a large number of functors that we care about, both on $\mathcal{S}^G$, and importantly on the category of rings comm^G. Most of these functors are not homotopical on the whole category of $\mathcal{S}^G$. In order to get these functors to behave well, they need to be derived. Thus, we want to find some appropriate model structure on $\mathcal{S}^G$ for which the various functors we care about are homotopical on maps between cofibrant objects so that we can use cofibrant replacement to derive them. In particular, we want a monoidal cofibrantly generated model structure on $\mathcal{S}^G$ such that:

(1) The weak equivalences are the stable equivalences

(2) The Kan Transfer Theorem applies to the adjunction $\mathcal{S}^G \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{comm}^G$

(3) If f is a cofibration or trivial cofibration then so is $\Phi^H(f)$ for all $H \subseteq G$.

(4) Many other things we don't explicitly have to worry about: cofibrant replacement should derive all of our favorite functors.

I should note at this point that (2) is the condition that is most important and will require the most work. It has nothing in particular to do with equivariant homotopy theory – this talk would be almost the same even if we were only interested in nonequivariant ring spectra. Condition (3) is the source of a minor modification to the model structure that explicitly relates to equivariantness.

Recall that a model category is a category with specified classes of cofibrations, fibrations, and weak equivalences satisfying the following conditions:

(1) All three classes are closed under composition and retracts.

(2) If two of f, g, and gf are weak equivalences, then so is the remaining.

(3) Call a morphism a trivial cofibration (resp triv. fibration) if it is a cofibration (resp. fibration) and a weak equivalence. Given a class C of morphisms, write $\mathrm{LLP}(C)$ for the set of maps with the left lifting property against elements of C and $\mathrm{RLP}(C)$ for the set of maps with the right lifting property against elements of C. We should have:
   (a) RLP(cofibrations) = trivial fibrations
   (b) RLP(trivial cofibrations) = fibrations
   (c) LLP(fibrations) = trivial cofibrations
   (d) LLP(trivial fibrations) = cofibrations

(4) Every map can be factored either as a trivial cofibration followed by a fibration or as a cofibration followed by a trivial fibration.

Note that if you write down two of the three classes of cofibrations, fibrations, and weak equivalences, there is at most one model structure with these classes as specified: for instance, if we have cofibrations and weak equivalences, the fibrations must be RLP(trivial cofibrations).

If cofibrations and fibrations are given, then we know the trivial cofibrations and trivial fibrations too. Any weak equivalence may be factored as a trivial cofibration followed by a fibration, and by two of three the fibration must be a weak equivalence too, so we get the class of weak equivalences by taking compositions of any trivial cofibration followed by any trivial fibration. Note that not any pair of classes satisfying conditions (1) and (2) gives rise to a third collection that makes a model category – condition (3) need not be satisfied.

In general, there can be set theoretic problems in constructing the factorizations for condition (4). The concept of a cofibrantly generated model structure is designed to ensure that our model category can be defined using a set-theoretically small amount of data, which allows us to use the small object argument to show that condition (4) is satisfied, and also allows us to use a large collection of tools that only work on cofibrantly generated model categories, most importantly for this talk, the Kan Transfer theorem.

Recall that a cofibrantly model category is a category with two specified sets, the set $\mathcal{I}$ of generating cofibrations, and the set $\mathcal{J}$ of generating trivial cofibrations. These need to satisfy:

(1) $\mathcal{I}$ and $\mathcal{J}$ admit the small object argument (so there exists some cardinal κ such that all the domains of maps in $\mathcal{I}$ and $\mathcal{J}$ are κ-small)

(2) Write $\operatorname{cofib}(S)$ for $\operatorname{LLP}(\operatorname{RLP}(S))$. Set cofibrations $= \operatorname{cofib}(\mathcal{I})$, fibrations $= \operatorname{RLP}(\mathcal{J})$, trivial cofibrations $= \operatorname{cofib}(\mathcal{J})$, and trivial fibrations $= \operatorname{RLP}(\mathcal{I})$. Set the weak equivalences to be compositions of trivial cofibrations with trivial fibrations. These classes must satisfy condition (3) for model categories, and also we must have that trivial (co)fibrations = (co)fibrations $\cap$ weak equivalences.

These conditions are sufficient for these collections to be the collections of cofibrations, fibrations, and weak equivalences of a model category. This statement is called the Kan Recognition theorem. Note that in our case, we will be dealing with topological model categories, and all the domains of all of the maps in $\mathcal{I}$ and $\mathcal{J}$ will always be compact, so they will always admit the small object argument.

Given a category and a set of maps L, we call a map $f : X \to Y$ a relative L cell complex if f is a possibly transfinite composition of pushout maps $Z \to Z'$ of the form:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

for $i \in L$. In fact, $\operatorname{cofib}(L)$ is the class of relative L-cell complexes. The two cases we care about are of course $L = \mathcal{I}$ and $L = \mathcal{J}$.

In Top, we obtain a cell complex by coning off maps from a sphere, which suggests that we should define the generating cofibrations and generating trivial cofibrations to be

$$\mathcal{I} = \{S^{n-1} \to D^n\}$$
$$\mathcal{J} = \{I^{n-1} \to I^n\}$$

This definition satisfies the Kan recognition theorem and gives a cofibrantly generated model structure on Top. This model structure has the standard cofibrations (relative cell complexes), fibrations (Serre fibrations), and weak equivalences (weak homotopy equivalences).

Now I will make an analogue for $\mathcal{S}^G$; this will be the *complete level model structure*. We set

$$\mathcal{I} = \{G_+ \wedge_H S^{-V} \wedge (S_+^{n-1} \to D_+^n) \ : \ \ H \leq G \text{ and } V \text{ an } H\text{-rep}\}$$
$$\mathcal{J} = \{G_+ \wedge_H S^{-V} \wedge (I_+^{n-1} \to I_+^n) \ : \ H \leq G \text{ and } V \text{ an } H\text{-rep}\}$$

Here "complete" means that H is allowed to range over the complete family of all subgroups of G (rather than some smaller family). The word "level" is a synonym for "strict," which was the term used yesterday. It refers to the weak equivalences – it turns out that the weak equivalences we get for this model structure are the levelwise weak equivalences: they are the maps $f : X \to Y$ of spectra such that $f_V : X_V \to Y_V$ is a weak equivalence of G-spaces for all G.

Let's demonstrate that the levelwise equivalences are the class of weak equivalences we get for the cofibrantly generated model structure on $\mathcal{S}^G$ specified by this $\mathcal{I}$ and $\mathcal{J}$. Note that all elements of $\mathcal{I}$ and $\mathcal{J}$ are levelwise weak equivalences. This implies that all relative $\mathcal{J}$ complexes are levelwise weak equivalences. We just need to check that all trivial fibrations are levelwise weak equivalences too.

The trivial fibrations are given by RLP($\mathcal{I}$). Such a trivial fibration needs to admit a lift for all diagrams of the form:

$$\begin{tikzcd} G_+ \wedge_H S^V \wedge S_+^{n+1} \arrow[r] \arrow[d] & X \arrow[d] \\ G_+ \wedge_H S^{-V} \wedge D_+^n \arrow[r] & Y \end{tikzcd}$$

which is equivalent by the adjunction between $G_+ \wedge -$ and Res^G_H to admitting a lift for all diagrams of the form:

$$\begin{tikzcd} S^{-V} \wedge S_+^{n-1} \arrow[r] \arrow[d] & \mathrm{Res}^G_H X \arrow[d] \\ S^{-V} \wedge D_+^n \arrow[r] & \mathrm{Res}^G_H Y \end{tikzcd}$$

Now using the adjunction between $S^{-V} \wedge : \mathrm{Top}_*^G \to \mathcal{S}^G$ to get S^{-V} to the other side:

$$\begin{tikzcd} S_+^{n-1} \arrow[r] \arrow[d] & (\mathrm{Res}^G_H X)_V \arrow[d] \\ D_+^n \arrow[r] & (\mathrm{Res}^G_H Y)_V \end{tikzcd}$$

Now asking for a map $Z \to W$ of pointed spaces to admit all lifts of this form implies that $Z \to W$ is a trivial fibration. We see that $X_V \to Y_V$ is a weak equivalence for all V and hence $X \to Y$ is a levelwise weak equivalence. Since every weak equivalence is the composition of a trivial cofibration with a trivial fibration and both trivial cofibrations and trivial fibrations are levelwise weak equivalences, we deduce that the weak equivalences are exactly the set of levelwise weak equivalences.

So we get the levelwise model structure. This isn't so nice because we care about stable things. If we can make the maps $e_{V,W} : S^{-(V \oplus W)} \wedge S^V \to S^{-W}$ into weak equivalences, this will get us our stable complete model structure. We have the most direct control over generating trivial cofibrations, which all must be weak equivalences, so our approach will be to add the maps $e_{V,W}$ to the weak equivalences by putting them into the set $\mathcal{J}$. Of course $\mathcal{J}$ can only contain cofibrations (that is, elements of $\mathrm{cofib}(\mathcal{I})$). Since we aren't planning on changing $\mathcal{I}$ and $\mathcal{I}$ admits the small object argument, we can use it to factor a morphism as an element of $\mathrm{cofib}(\mathcal{I})$ followed by an element of $\mathrm{RLP}(\mathcal{I})$ – that is as a cofibration followed by a trivial fibration. Let $\widetilde{e}_{V,W}$ be this cofibrant replacement of $e_{V,W}$. Let $\mathcal{K} = \{G_+ \wedge_H S^{-V} \wedge (I_+^{n-1} \to I_+^n) : H \leq G, V \text{ an } H\text{-rep}\}$ be our old $\mathcal{J}$, and let our new $\mathcal{J}$ be the union $\mathcal{J} = \mathcal{K} \cup \{\widetilde{e}_{V,W}\}$. (This isn't actually exactly what we do, because we want a monoidal cofibrantly generated model category, so we have to take $\mathcal{J} = \mathcal{K} \cup (\mathcal{I} \,\square\, \{\widetilde{e}_{V,W}\})$, but I don't feel like defining the $\square$ product.)

This gives us the stable complete model structure. Later, I'll show that the fibrations in this model structure are relative Ω-spectra.

11.1. The positive (stable) complete model structure. We are interested in studying G-equivariant commutative rings, so naturally we want to get a model structure on them. As is usually the case with categories of algebraic structures in another category, there is a free forgetful adjunction $\mathcal{S}^G \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{comm}^G$, where U is the functor that "forgets" the ring structure from a G-ring to get the underlying G-spectrum, and Sym is the functor $X \mapsto \bigvee_{n \in \mathcal{N}} \mathrm{Sym}^n X$, where $\mathrm{Sym}^n X = S^{\wedge n}/\Sigma_n$ (here the quotient is formed levelwise). Because we're going to frequently use both Sym and the forgetful functor, we want to make sure that the model structure we put on comm^G makes this adjunction to be a Quillen adjunction between model categories.

The Kan transfer theorem solves exactly the problem we're interested in. Given a cofibrantly generated model category $\mathcal{M}$, some other category $\mathcal{N}$ and an adjunction $\mathcal{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{N}$ (where $F : \mathcal{M} \to \mathcal{N}$ is the left adjoint), the Kan Transfer theorem gives us conditions that will allow us to "transfer" the model structure from $\mathcal{M}$ to $\mathcal{N}$ to get a cofibrantly model structure on $\mathcal{N}$ with weak equivalences given by maps that become weak equivalences after applying U ("underlying weak equivalences") and making the adjunction into a Quillen adjunction.

Theorem 11.1 (Kan Transfer Theorem)**.** *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and trivial cofibrations $\mathcal{J}$, let $\mathcal{N}$ be a bicomplete category and let $\mathcal{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{N}$ be an adjunction. Let $F\mathcal{I} = \{Fi : i \in \mathcal{I}\}$ and likewise with $F\mathcal{J}$. Then if:*

(1) both $F\mathcal{I}$ and $F\mathcal{J}$ admit the small object argument and

(2) U takes relative $F\mathcal{J}$-cell complexes to weak equivalences

then there is a cofibrantly generated model structure on $\mathcal{N}$ such that $F\mathcal{I}$ and $F\mathcal{J}$ are the generating cofibrations and trivial cofibrations and the weak equivalences are the maps taken to weak equivalences by U. With respect to this model structure on $\mathcal{N}$, the adjunction (F, U) is a Quillen adjunction.

Condition (1) is harmless – it will be satisfied more or less automatically. Condition (2) is clearly necessary because we must have that all maps in $\mathrm{cofib}(F\mathcal{J})$ ("relative $F\mathcal{J}$-cell complexes") are weak equivalences, which means that Uf must be a weak equivalence for $f \in \mathrm{cofib}(F\mathcal{J})$.

From now on, we'll say that a map $f : X \to Y$ of rings is a weak equivalence if Uf is a weak equivalence. We want to apply the Kan transfer theorem with the adjunction $\mathcal{S}^G \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{comm}^G$. Unfortunately, with the current model structure on $\mathcal{S}^G$, condition (2) fails. Note that because weak equivalences are closed under transfinite compositions, it would suffice for the pushout map $X \to Y$ in the following diagram to be a weak equivalence for all $f : A \to B \in \mathcal{J}$ and for any ring X.

$$\begin{tikzcd} \mathrm{Sym}\, A \arrow[r, "\mathrm{Sym}\, f"] \arrow[d] & \mathrm{Sym}\, B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{11.1}$$

As a special case of this, for any trivial cofibration $A \to B$, we must have $\mathrm{Sym}\, A \to \mathrm{Sym}\, B$ a weak equivalence. By Ken Brown's Lemma, this implies that for any weak equivalence between cofibrant objections $A \to B$, we must have $\mathrm{Sym}\, A \to \mathrm{Sym}\, B$ a weak equivalence. I claim that this is not true. This is a purely non-equivariant concern, so let's take $G = \{e\}$ for the moment (the same problem occurs for any G). Consider the weak equivalence $e_{1,1} : S^{-1} \wedge S^1 \to S^0$. In the complete stable model structure, both $S^{-1} \wedge S^1$ and S^0 are cofibrant, so Ken Brown's Lemma tells us that if condition (2) of the KTT holds then $\mathrm{Sym}\, e_{1,1}$ must be a weak equivalence. I claim that $\mathrm{Sym}(e_{1,1})$ is not a weak equivalence. This will contradict condition (2).

On the one hand, we have $\mathrm{Sym}^n S^0 = (S^0)^{\wedge n}/\Sigma_n = (S^0)/\Sigma_n = S^0$ (since the only Σ_n action on S^0 is trivial). On the other hand, $(S^{-1} \wedge S^1)^{\wedge n} = (S^{-1})^{\wedge n} \wedge S^n = S^{-n} \wedge S^n$. To form Sym^n from this, we take the quotient, so we just need to figure out what the Σ_n action is. Now Σ_n acts on S^n in the expected way – treating S^n as the one-point compactification of $\mathbb{R}^n$, it permutes the n factors of $\mathbb{R}^n$. What about the action on S^{-n}? Recall that $\mathscr{J}(n, -) = \mathrm{Thom}(O(n, -) \downarrow \text{something})$ and this has a free left $O(n)$-action because $O(n)$ acts freely on $O(n, -)$. The action of Σ_n on $S^{-n} = \mathscr{J}(n, -)$ is via the inclusion $\Sigma_n \subset O(n)$, so that's also free. Since the action is free, $(S^{-n} \wedge S^n)/\Sigma_n$ is the same as the homotopy quotient $(S^{-n} \wedge S^n)_{\mathrm{ho}\,\Sigma_n} \simeq (S^0)_{\mathrm{ho}\,\Sigma_n} := (E\Sigma_n)_+ \wedge_{\Sigma_n} S^0 = \Sigma^\infty(B\Sigma_n)_+$.

So we deduce that $\mathrm{Sym}(S^{-1} \wedge S^1) = \bigvee_{n \in \mathbb{N}} \Sigma^\infty_+ B\Sigma_n$. This is not weak equivalent to $\mathrm{Sym}\, S^0 = S^0$, so the Kan Transfer Theorem doesn't apply. (You should think of $\mathrm{Sym}(S^{-1} \wedge S^1)$ as what $\mathrm{Sym}\, S^0$ "should" be – that is, this is the derived version of $\mathrm{Sym}\, S^0$.)

Note that Sym^n is the composition of the n-fold smash power with the quotient. The n-fold smash power operation is homotopical, but the quotient is not. however, that as long as the action of Σ_n is free, then the actual quotient will be homotopy equivalent to the homotopy quotient, and the homotopy quotient is homotopical. This free action happens quite generally: our proof above that S_n acts freely on $(S^{-V})^{\wedge n}$ works as long as $\dim V > 0$. So now let's change the definition of $\mathcal{I}$ and $\mathcal{J}$, replacing "for all H-representations V" to "for all representations V such that $\dim V^H > 0$":

$$\mathcal{I} = \{G_+ \wedge_H S^{-V} \wedge (S^{n-1}_+ \to D^n_+) : \ H \leq G \text{ and } V \text{ an } H\text{-rep with } \dim V^H > 0\}$$
$$\mathcal{K} = \{G_+ \wedge_H S^{-V} \wedge (I^{n-1}_+ \to I^n_+) : H \leq G \text{ and } V \text{ an } H\text{-rep with } \dim V^H > 0\}$$
$$\mathcal{J} = \mathcal{K} \cup \{\widetilde{e}_{V,W}\} \mathbin{\square} \mathcal{I}$$

Requiring $\dim V > 0$ would be sufficient to ensure that Sym condition (2) of the Kan Transfer Theorem is met, but then $\varphi^H(G_+ \wedge_H S^{-V} \wedge (S^{n-1}_+ \to D^n_+)) = S^{-V^H} \wedge (S^{n-1}_+ \to D^n_+))$ and so if $\dim V^H = 0$ then $\varphi^H(f)$ is not positive cofibrant even though f is. So instead we require the mildly stronger condition that $\dim V^H > 0$.

Remark 11.2. This modification of the model structure means that the sphere spectrum is no longer cofibrant. This was necessary because $e_{1,1} : S^{-1} \wedge S^1 \to S^0$ is a weak equivalence and $\mathrm{Sym}(e_{1,1})$ is not a weak equivalence. Condition (2) of the KTT implies (by Ken Brown's Lemma) that Sym takes weak equivalences between cofibrant objects to weak equivalences. Thus, in a model structure where the KTT applies, either $S^{-1} \wedge S^1$ or S^0 must not be cofibrant. The positivity condition keeps $S^{-1} \wedge S^1$ cofibrant and makes S^0 not cofibrant. This was more or less necessary, since for any reasonable change of the generating set, we will have $S^{-n} \wedge S^n$ cofibrant for some $n \gg 0$, and then the same problem will apply to $e_{n,n}$.

11.2. Bonus: Fibrations are relative Ω-spectra. The category $\mathcal{S}^G$ is a topological model category. This means that given $i : A \to B$ a cofibration and $p : X \to Y$ a fibration, one which must be trivial, there is a contractible choice of lifts $A \to X$ for any diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd} \tag{11.2}$$

In particular, the lifting condition for a topological model category $\mathcal{M}$ is as follows: given (i, p) as above, there is a space of pairs $\mathcal{M}(A, X) \times \mathcal{M}(B, Y)$ of maps $B \to Y$ and $A \to X$. We want to add the condition that the diagram commutes. There is a continuous map $\mathcal{M}(B, Y) \to \mathcal{M}(A, Y)$ given by precomposing with i and a map $\mathcal{M}(A, X) \to \mathcal{M}(A, Y)$ given by postcomposing with p. The condition that a pair of maps $f : A \to X$ and $g : B \to Y$ make the diagram commute is that $pf = gi$. Thus, the space of commuting diagrams is given by

$$\mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y).$$

Any lift $l : B \to X$ gives a commuting diagram by taking $f = li$ and $g = pl$. Thus there is a map

$$\mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y).$$

If $\mathcal{M}$ is a topological model category, this map is assumed to be a trivial fibration whenever i is a cofibration, p a fibration, and one of the two is a weak equivalence.

$$\begin{tikzcd} \mathcal{M}(B,X) \arrow[dr, "\text{triv. fib.}"] \arrow[drr, bend left, "i^*"] \arrow[ddr, bend right, "p_*"'] & & \\ & \mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y) \arrow[r] \arrow[d] & \mathcal{M}(A,X) \arrow[d, "p_*"] \\ & \mathcal{M}(B,Y) \arrow[r, "i^*"'] & \mathcal{M}(A,Y) \end{tikzcd}$$

Let's consider what this means for $\mathcal{S}^G$. Consider a map $p : X \to Y$ of spectra. It suffices to check cofibrancy on the generating trivial cofibrations. We showed above that lifting against the maps in $\mathcal{K}$ means that p is a levelwise fibration, except that we checked this before we added the positivity condition. With the extra positivity condition, p_V only has to be a fibration when $\dim V^H > 0$, you can call this a "positive levelwise fibration". Thus, we just need to check what lifting against $e_{V,W} : S^{-(V\oplus W)} \wedge S^V \to S^{-W}$ means. Note that $\mathcal{S}^G(S^{-W}, Z) = Z_W$ and $\mathcal{S}^G(S^{-(V\oplus W)} \wedge S^V, Z) = \Omega^V X_{V\oplus W}$. Instantiating the previous diagram using $i = e_{V,W}$ and using these identities of mapping spaces, we get the following diagram:

$$\begin{tikzcd}
X_W \arrow[dr, "\text{triv. fib.}"] \arrow[drr, bend left] \arrow[ddr, bend right, "p_W"'] & & \\
& \Omega^V X_{V\oplus W} \times_{\Omega^V Y_{V\oplus W}} Y_W \arrow[r] \arrow[d, "i"'] & \Omega^V X_{V\oplus W} \arrow[d, "\Omega^V p_{V\oplus W}"] \\
& Y_W \arrow[r] & \Omega^V Y_{V\oplus W}
\end{tikzcd}$$

So we learn that for all V, W, we have that $X_W \simeq \Omega^V X_{V\oplus W} \times_{\Omega^V Y_{V\oplus W}} Y_W$. When Y is a point, this says that $X_W \simeq \Omega^V X_{V\oplus W}$, which means that X is an Ω-spectrum. Because of this, in the general case we say that $p : X \to Y$ is a relative Ω-spectrum.

Talk 12: The norm construction and geometric fixed points

(Benjamin Böhme)

12.1. The norm construction. We have a restriction functor $\mathrm{Res}^G_H : G\text{-reps} \to H\text{-reps}$. It has a left adjoint $\mathrm{Ind}^G_H : V \mapsto \mathbb{R}G \otimes_{\mathbb{R}H} V$ and a right adjoint $V \mapsto \mathrm{Map}_{\mathbb{R}H}(\mathbb{R}G, V)$ (coinduction). It's a simple fact that these functors are actually isomorphic.

For spectra, there is a restriction $i_H^* : \mathcal{S}^G \to \mathcal{S}^H$ that has a left adjoint $X \mapsto G_+ \wedge_H X$ and a right adjoint $X \mapsto \mathrm{Map}_H(G_+, X)$. These two functors are weakly equivalent. This is the Wirthmüller isomorphism.

Rewrite the left adjoint as $G_+ \wedge_H X = \bigvee_{g_i \in G/H} X_i$ where $X_i = (g_i H)_+ \wedge_H X$. An element $g \in G$ acts by writing $g = g_i h$ for some coset representative g_i and then letting g_i permute the factors and h act on the factor. Dually, write

$$\mathrm{Map}_H(G_+, X) = \prod_{\widetilde{g}_i \in H\backslash G} X^i \text{ for } X^i = \mathrm{Map}_H(Hg_{i+}, X) \cong \prod_{g_i \in G/H} X_i$$

Make a preliminary definition

$$\bigwedge_{i \in G/H} X_i =: N_H^G X.$$

Now we talk about the abstract framework behind this.

Definition 12.1. For J a finite G-set, define a category $B_J G$ whose objects are J and whose morphisms are $\mathrm{Mor}(j, j') = \{g \in G \,:\, g \cdot j = j'\}$.

There's a special case: if J is a point, then $B_J G$ is the usual BG.

Fact 12.2. *For $H \leq G$, the inclusion $BH \hookrightarrow B_{G/H} G$ is an equivalence of categories. Hence there are adjoint functors* $\mathrm{incl} : \mathcal{S}^{BH} \rightleftarrows \mathcal{S}^{B_{G/H}G} : \mathrm{Ind}_*$*. The LHS is equivalent to $\mathcal{S}^H$.*

Definition 12.3. The obvious map $G/H \to *$ induces a functor $p : B_{G/H} \to BG$. Define the norm functor $N_H^G : \mathcal{S}^H \to \mathcal{S}^G$ to be the following composite:

$$\mathcal{S}^H \simeq \mathcal{S}^{BH} \overset{\text{incl}}{\to} \mathcal{S}^{B_{G/H}G} \overset{p_\otimes}{\to} \mathcal{S}^{BG} \simeq \mathcal{S}^G.$$

We're using the fact that there's an equivalence between genuine H-spectra and spectra with objectwise H-action (naïve spectra). This is *just* a statement about categories; they're not equivalent as homotopical categories.

Properties: N_H^G commutes with sifted colimits, is symmetric monoidal, and the induced functor $N_H^G : \mathrm{Comm}^H \to \mathrm{Comm}^G$ is the left adjoint of restriction.

The *canonical homotopy presentation* is

$$X \simeq \mathrm{hocolim}_V\, S^{-V} \wedge X_V.$$

Then $N_H^G X \simeq \mathrm{hocolim}_V\, S^{-\mathrm{Ind}_H^G V} \wedge N_H^G(X_V)$, where $N_H^G(X_V)$ is an analogously defined norm functor on spaces with H-action, applied to X_V.

Proposition 12.4. *$(-)^{\wedge J} : \mathcal{S}^{B_J G} \to \mathcal{S}^G$ taking $X \mapsto \bigwedge_{j\in J} X_j$ preserves all cofibrations and acyclic cofibrations between cofibrant objects.*

Hence there is a left derived functor $(-)^{\wedge^L J} : \mathrm{ho}\,\mathcal{S}^{B_J G} \to \mathrm{ho}\,\mathcal{S}^G$ (it's the left adjoint in an adjunction and that descends to an adjunction of homotopy categories).

Corollary 12.5. $N_H^G : \mathrm{Comm}^H \rightleftarrows \mathrm{Comm}^G : \mathrm{Res}$ *is a Quillen adjunction (adjunction on the level of model categories which descends to the level of homotopy categories where the right adjoint preserves (co)fibrations, etc.).*

Consider

$$\begin{tikzcd}
\mathrm{Comm}^H \arrow[r, "N_H^G"'] \arrow[d] & \mathrm{Comm}^G \arrow[d] \\
\mathcal{S}^H \arrow[r, "N_H^G"'] & \mathcal{S}^G
\end{tikzcd}$$

This commutes up to natural isomorphism.

Question 12.6. Is this true on the homotopy level? I.e. does

$$\begin{tikzcd}
\mathrm{ho}\,\mathrm{Comm}^H \arrow[r] \arrow[d, "N_H^G"'] & \mathrm{ho}\,\mathrm{Comm}^G \arrow[d] \\
\mathrm{ho}\,\mathcal{S}^H \arrow[r, "N_H^G"'] & \mathrm{ho}\,\mathcal{S}^G
\end{tikzcd}$$

Cofibrant commutative rings aren't cofibrant spectra in general. (The forgetful functor is a right adjoint so it needn't preserve cofibrancy.)

We have to check:

- If $R_c \xrightarrow{\sim} R$ is an approximation in spectra then $(R_c)^{\wedge J} \to R^{\wedge J}$ is a weak equivalence.

Proposition 12.7 (HHR B.147)**.** *This is true if R is cofibrant in the category of ring spectra.*

12.2. Genuine fixed point spectra. Consider the $\mathcal{S} \to \mathcal{S}^G$ taking $X \mapsto X$ with the trivial action. This has a right adjoint $(-)^G : X \mapsto (i_0^* X)^G =: X^G$ (i_0 takes it to an ordinary spectrum, so indexed on $\mathbb{Z}$ not representations; levelwise, this still has a G-action, and so you take the fixed points levelwise). This is the genuine fixed points functor.

Properties:

- $\pi_*(X^G) \cong \pi_*^G(X)$ for Ω-G-spectra
- $(-)_f^G$ is not compatible with $- \wedge -$ and $\Sigma^\infty(-)$ (i.e. $(\Sigma^\infty(X))^G \neq \Sigma^\infty(X^G)$)

Remark 12.8 (tom Dieck splitting)**.** If K is a G-space, then

$$((\Sigma^\infty K)_f)^G \simeq \bigvee_{\substack{(H)\\ H \leq G}} \Sigma^\infty(EW_G H_+ \wedge_{W_G H} K^H)$$

where $W_G H = N_G H/H$ is the Weyl group ($N_G =$ the normalizer).

12.3. Geometric fixed points. Recall: a G-space EF is universal for a family $\mathcal{F}$ if

$$(E\mathcal{F})^H \simeq \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F}. \end{cases}$$

For $\mathcal{P}$ the family of all proper subgroups, we get the isotropy separation sequence $E\mathcal{P}_+ \wedge X \to X \to \widetilde{E}\mathcal{P} \wedge X$. This leads to a definition of geometric fixed points.

Definition 12.9. Let $\Phi^G(X) := ((\widetilde{E}\mathcal{P} \wedge X)_f)^G \in \mathcal{S}$ is the geometric fixed point spectrum.

Why do we need another fixed point spectrum? The first one was not compatible with smash product and what comes from the level of spaces. But this one is compatible with smash product.

Properties:

- Φ^G preserves with all weak equivalences
- it commutes with filtered hocolim
- If $A \in \mathrm{Top}^G$ then $\Phi^G(S^{-V} \wedge A) \xrightarrow{\sim} S^{-V^G} \wedge A^G$
- For $X \sim \operatorname{hocolim}_V S^{-V} \wedge X_V$ then it's clear from the previous properties that $\Phi^G X \simeq \operatorname{hocolim} S^{-V^G} \wedge X_V^G$
- $\Phi^G(X \wedge Y) \overset{\sim}{\leftrightarrow} \Phi^G X \wedge \Phi^G Y$ (that's a zigzag of weak equivalences).

Problem 12.10. There's no natural transformation $\Phi^G X \wedge \Phi^G Y \to \Phi^G(X \wedge Y)$.

The solution is to define the monoidal geometric fixed point functor:

$$\Phi^G_M X = \operatorname{coeq}\Big(\bigvee_{V,W} S^{-W^G} \wedge \mathscr{J}_G(V,W)^G \wedge X^G_V \rightrightarrows \bigvee_V S^{-V^G} \wedge X^G_V\Big)$$

Proposition 12.11. *Φ^G is the left derived functor of Φ^G_M i.e. $\Phi^G(X) \xrightarrow{\sim} \Phi^G_M(X)$ for X cofibrant.*

As we'd expect of a functor called the monoidal functor, Φ^G_M is weakly monoidal, which means the following. We have an honest natural transformation $\Phi^G_M X \wedge \Phi^G_M Y \to \Phi^G_M(X \wedge Y)$ which is a weak equivalence on cofibrant X and Y.

12.4. Relationship between N^G_H and $\Phi^G_{(M)}$.

Proposition 12.12. *If $H \le G$ is a subgroup, there is a natural transformation $\Phi^H_M(-) \to \Phi^G_M N^G_H(-)$ which is an isomorphism (not just a weak equivalence) on cofibrant objects. (Both are functors from genuine G-spectra to non-equivariant spectra.)*

Corollary 12.13. *If $N^G_H(X_c) \to N^G_H(X)$ is a weak equivalence for a cofibrant approximation $X_c \to X$, then there is a chain of weak equivalences $\Phi^H X \xleftrightarrow{\sim} \Phi^G N^G_H X$.*

Proof. We get a weak equivalence $\Phi^H X_c \to \Phi^H(X)$ (because of property (i) of Φ^G). We've seen before that there's a zigzag of weak equivalences $\Phi^H X \xleftrightarrow{\sim} \Phi^H_M(X_c)$

$$\begin{tikzcd}
\Phi^H X_c \arrow[r, leftrightarrow, "\sim", "12.11"'] \arrow[d, "\sim"'] & \Phi^H_M(X_c) \arrow[r, "\sim", "12.12"'] & \Phi^G_M N^G_H(X_c) \arrow[r, leftrightarrow, "\sim", "12.11"'] & \Phi^G N^G_H(X_c) \arrow[d, "\sim"', "\text{by assumption}"] \\
\Phi^H(X) & & & \Phi^c N^G_H(X)
\end{tikzcd}$$

□

Application 12.14. If $H = C_2 \le C_{2^n} \le G$ and $X = MU_{\mathbb{R}}$, then we denote $N^G_H X = MU^{((G))}$. The corollary we just proved shows that $\Phi^{C_{2^n}} MU^{((C_{2^n}))} \sim \Phi^{C_2} MU_{\mathbb{R}}$. It follows from the (forthcoming) construction of $MU_{\mathbb{R}}$ that this is $\simeq MO$.

Talk 13: The slice filtration and slice spectral sequence
(Koen van Woerden)

13.1. Definitions. We work in G-spectra, and G will be a finite group. Let $K \subset G$ be a subgroup. Let $\widehat{S}(m,K) = G_+ \wedge_K S^{m\rho_K}$ where ρ_K is the regular representation on K.

Definition 13.1. The set of slice cells is

$$SC = \{\widehat{S}(m,K), \Sigma^{-1}\widehat{S}(m,K) \,:\, K \subset G, m \in \mathbb{Z}\}.$$

A slice cell is:

- *regular* if it is of the form $\widehat{S}(m,K)$
- *induced* if it is of the form $G_+ \wedge_H S^{m\rho_H}$ or $\Sigma^{-1}(G_+ \wedge_H S^{m\rho_H})$ for $H \subsetneq G$
  - *free* if it is induced from $H = \{e\}$
- *isotropic* if it is induced from $\{e\} \neq H$ (but possibly $H = G$)

Definition 13.2. The dimension of a slice cell is the dimension of the underlying sphere. Let $SC_{\geq n}$ be the set of slice cells of dimension $\geq n$. Consider the Bousfield localization at the collection of maps from something in $SC_{\geq n}$ to a point. This localization applied to X is called P^n. If Y is $SC_{\geq n}$-local, write $Y < n$, $Y \leq n-1$, or that Y is n-null.

Say that X is *slice n-positive*, written $X > n$ or $X \geq n+1$, if $\mathcal{S}_G(X,Y) \simeq *$ for all Y such that $Y \leq n$. Equivalently, X is slice n-positive if $P^nX \simeq *$. The full subcategory of $\mathcal{S}^G$ generated by $X > n$ is called $\mathcal{S}^G_{>n}$. Define $\mathcal{S}^G_{\leq n}$ similarly.

There's a functor called acyclization that takes X to the fiber of $X \to P^nX$. Denote this by $P_{n+1}X$. The P^nX are in $\mathcal{S}^G_{\leq n}$ and the $P_{n+1}X$ are in $\mathcal{S}^G_{>n}$, and these are universal with this property.

We have inclusions $SC_{\geq n} \supset SC_{\geq n+1}$ of sets and hence $\mathcal{S}^G_{\leq n-1} \subset \mathcal{S}^G_{\leq n}$ of categories. This gives a natural transformation $P^nX \to P^{n-1}X$.

Definition 13.3. The slice tower of X is $\cdots \to P^nX \to P^{n-1}X \to \ldots$. P^nX is the n^{th} *slice section of* X. Take the fiber

$$P^n_nX \overset{\text{fiber}}{\to} P^nX \to P^{n-1}X.$$

P^n_nX is called the n^{th} slice of X.

Observe that $P^n_nX = P_nP^nX$ so it is in $\mathcal{S}^G_{\geq n}$, but also we have a fiber sequence $P^n_nX \to P^nX \to P^{n-1}$ where $P^nX \leq n$ and $P^{n-1}X \leq n-1$, which shows that $P^n_nX \in \mathcal{S}^G_{\leq n}$.

So the tower looks like

$$\begin{tikzcd}
\vdots \arrow[d] & \\
P^1X & P^1_1X \arrow[l] \\
P^0X \arrow[from=2-1, to=3-1] & P^0_0X \arrow[l] \\
P^{-1}X \arrow[from=3-1, to=4-1] \arrow[d] & P^{-1}_{-1}X \arrow[l] \\
\vdots &
\end{tikzcd}$$

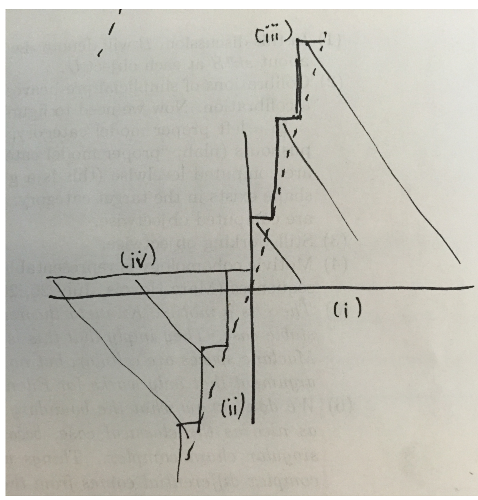
where each
$$\begin{tikzcd} P^n X \arrow[d] & P^n_n \arrow[l] \\ P^{n-1}X \end{tikzcd}$$
is a fiber sequence.

Definition 13.4. The slice spectral sequence is the homotopy spectral sequence of the slice tower:
$$E_1^{s,t} = \pi^G_{t-s} P^t_t X \implies \pi^G_{t-s} X.$$
The differentials go exactly like the Adams spectral sequence.

Doug: It's the E_2 page if you know that the oddly-indexed slices are zero (e.g. in our application).

Mike: This *is* the AHSS where you're cofiltering instead of filtering.

Theorem 13.5. *The spectral sequence converges and there are four vanishing lines (marked (i), (ii), (iii), (iv) in the diagram below).*



The dotted line has slope $|G| - 1$. In particular, if you take $G = \{*\}$ the spectral sequence degenerates. (But that doesn't help because the slices will be Postnikov slices.)

Doug: you could also do a version for H-equivariant homotopy groups; this can be made into a spectral sequence of bigraded Mackey functors rather than abelian groups. There's also a way to make this $RO(G)$-graded: take $RO(G)$-graded homotopy groups, but the index s (the vertical coordinate) is always an integer, as is the index of the slice t, but you can replace the t in the homotopy group by some other representation other than the trivial one.

Proposition 13.6. *Let X be a G-spectrum.*

(1) $X \geq 0$ iff X is (-1)-connected ("connected" means by mapping in with S^V)

(2) $X \geq -1$ iff X is (-2)-connected

(3) $X < 0$ iff X is 0-coconnected[10]

(4) $X < -1$ iff X is (-1)-coconnected

These patterns don't continue; these dimensions are special.

PROOF IDEA. S^0 is a slice cell of dimension 0, S^{-1} is a slice cell of dimension -1. □

Corollary 13.7. *$P^{-1} = \mathrm{Pos}^{-1}$ (where Pos is doing the localization w.r.t. ordinary cells – it's the equivariant Postnikov filtration) and $P^{-2} = \mathrm{Pos}^{-2}$. This shows that $P^{-1}_{-1}X = \Sigma^{-1}H\underline{\pi}_{-1}X$.*

So we can compute P^{-1}_{-1}.

13.2. Subgroups, induction, and restriction.

Proposition 13.8. *Induction and restriction send a slice cell to a (wedge of) slice cells of the same dimension.*

The norm sends a wedge of regular (i.e. not desuspended) H-cells to a wedge of regular G-cells.

Proposition 13.9. *Induction and restriction preserve n-nullness and n-positivity.*

Lemma 13.10. *Suppose we have a fiber sequence of the form $\widetilde{P}_{n+1} \to X \to \widetilde{P}^n$ where $\widetilde{P}_{n+1} > n$ and $\widetilde{P}^n \leq n$. Then the canonical maps $\widetilde{P}_{n+1} \to P_{n+1}X$ and $P^nX \to \widetilde{P}^n$ are weak equivalences.*

"If we have a sequence that looks like the right thing then it is the right thing."

Corollary 13.11. *P^n commutes with restriction and induction.*

This implies that P_n commutes with restriction and induction, because they determine each other, and so does P^n_n.

Remark 13.12. Taking $i^*_{\{e\}}$ shows that the non-equivariant tower underlying the slice tower is the Postnikov tower.

13.3. More examples.

Proposition 13.13.
(1) X is an (-1)-slice iff there exists a Mackey functor $\underline{M}$ such that $X = \Sigma^{-1}H\underline{M}$.

[10] X is n-coconnected if its homotopy groups vanish in dimensions n and above

(2) X is a 0-slice iff there exists a Mackey functor $\underline{M}$ with all restrictions injections such that $X = H\underline{M}$.

Proposition 13.14. *$X > 0$ iff X is (-1)-connected and $\pi_0^u X = 0$.*

Proposition 13.15. *For $n \geq 0$, $G/H_+ \wedge S^n \geq n$.*

For example, when G is the trivial group, the spheres S^n are slice cells of dimension $\geq n$, hence in particular $S^n \geq n$.

Corollary 13.16. *$P_{-1}^{-1} S^{-1} = \Sigma^{-1} H\underline{A}$ where $\underline{A}$ is the Burnside Mackey functor*[11]*, and $P_0^0 S^0 = H\underline{\mathbb{Z}}$.*

Proof. We already saw the first statement.

For the second statement, taking the P^0 localization is the same as $S^0 \to \operatorname{Pos}^0 S^0 \to P^0 S^0 = P^0 \operatorname{Pos}^0 S^0$. (This always happens, not just for S^0.) We know that $\operatorname{Pos}^0 S^0 = H\underline{A}$ and $P^0 \operatorname{Pos}^0 S^0 = P^0 H\underline{A}$. To compute the zeroth slice, we need to localize down further: take the fiber of $P^0 H\underline{A} \to P^{-1} H\underline{A}$ to get $P_0^0 S^0 \to P^0 H\underline{A} \to P^{-1} H\underline{A}$. But the last term is zero, so the first map is an equivalence. Write down a fiber sequence $H\underline{I} \to H\underline{A} \to H\underline{\mathbb{Z}}$. We have that $H\underline{\mathbb{Z}} \leq 0$ because restriction maps are monos, and $H\underline{I} \geq 1$ by Proposition 13.14. Now we conclude by Lemma 13.2 that $P^0 \underline{A} = H\underline{\mathbb{Z}} = P_0^0 S^0$. $\square$

13.4. Multiplicative properties. Smashing with $S^{m\rho_G}$ gives an equivalence $\mathcal{S}^G_{\geq n} \to \mathcal{S}^G_{\geq n+m|G|}$ because it gives an equivalence between slice cells.

The natural maps $S^{m\rho_G} \wedge P_n X \to P_{n+m|G|}(S^{m\rho_G} \wedge X)$ and $S^{m\rho_G} \wedge P^n X \to P^{n+m|G|}(S^{m\rho_G} \wedge X)$ are weak equivalences.

13.5. Even more examples. For $K \subset G$ the $m|K|$-slice of $\widehat{S}(m, K)$ is $H\underline{\mathbb{Z}} \wedge \widehat{S}(m, K)$, and the $(m|K| - 1)$-slice of $\Sigma^{-1}\widehat{S}(m, K)$ is $H\underline{A} \wedge \Sigma^{-1}\widehat{S}(m, K)$.

Proof. $G_+ \wedge (-)$ commutes with slice sections can restrict to $K = G$. Multiplicativity and the previous two examples give the result. $\square$

Talk 14: Dugger's computation for real K-theory (Agnes Beaudry)

Our goal is to compute $\underline{\pi}_*^{C_2} K_{\mathbb{R}}$ (to be defined later). Let V be a C_2-representation, and B a finite C_2-set. Consider Mackey functor homotopy $\underline{\pi}_V^{C_2} K_{\mathbb{R}}(B) = [S^V \wedge B_+, K_{\mathbb{R}}]_{C_2}$.

[11] This is the monoidal unit, and $\underline{\pi}_0 S^0 = \underline{A}$. Alternatively, it sends H to the (Grothendieck group of) the Burnside ring of H, and the Burnside ring of H is just finite H-sets.

The idea is to use this as an example of the slice spectral sequence:

$$\underline{E}_2^{s,t} = \underline{\pi}_{t-s} P_t^t K_{\mathbb{R}} \to \underline{\pi}_{t-s} K_{\mathbb{R}}.$$

Now let's talk about Atiyah's real K-theory. One way to obtain $K_{\mathbb{R}}$ is to "kill off stuff from $MU_{\mathbb{R}}$". But we're going to do it more hands on.

The category of "Atiyah–real" vector bundles has objects complex vector bundles $p : E \to X$, for E and X C_2–spaces and p a C_2–map. Further, we require that the map $E_x \to E_{\gamma(x)}$ be complex anti-linear. This means $\gamma(ce) = \bar{c}\gamma(e)$. The morphisms of this category are C_2-equivariant maps of vector bundles. This is not the same as C_2-complex vector bundles because of the twist we've added.

From now on, we assume that $G = C_2$.

Definition 14.1. Let $K_{\mathbb{R}}(X)$ be the Grothendieck group of the category of isomorphism classes of Atiyah–real vector bundles. (We group-complete with respect to direct sum.)

$K_{\mathbb{R}} : \mathrm{ho}\,\mathrm{Top}^{C_2} \to \mathrm{Ab}$ is a functor and there's a corresponding Bott periodicity which

$$\widetilde{K}_{\mathbb{R}}(X) \cong \widetilde{K}_{\mathbb{R}}(S^\rho \wedge X_+).$$

(Here, $\rho = \rho_2$ is the regular representation for C_2.) We can use this to extend $K_{\mathbb{R}}$ to an $RO(G)$-graded cohomology theory as follows. Recall that any representation can be written as $n + m\sigma$ for $n, m \in \mathbb{Z}$. Given any $s, t \in \mathbb{Z}$ (potentially negative), we let

$$K_{\mathbb{R}}^{s+t\sigma}(X) := K_{\mathbb{R}}^0(S^{m\rho} \wedge S^{-s-t\sigma} \wedge X) = K_{\mathbb{R}}^0(S^{m-s+(m-t)\sigma} \wedge X) \text{ for } m \gg 0.$$

Since $K_{\mathbb{R}}^*$ is an $RO(G)$-graded homology theory, it is represented by a genuine C_2-spectrum $K_{\mathbb{R}}$, which is in fact an Ω-C_2-spectrum.

Because of Bott periodicity, $\underline{\pi}_\star^{C_2} K_{\mathbb{R}}$ is determined by $\underline{\pi}_*^{C_2} K_{\mathbb{R}}$ for $* \in \mathbb{Z}$. (For example, $\underline{\pi}_{m+n\sigma}^{C_2} K_{\mathbb{R}} \cong \underline{\pi}_{m+n\sigma}^{C_2} \Sigma^{n\rho} K_{\mathbb{R}} \cong \underline{\pi}_{m-n}^{C_2} K_{\mathbb{R}}$. In other words, every $RO(G)$-graded homotopy group of $K_{\mathbb{R}}$ is isomorphic to a $\mathbb{Z}$-graded homotopy group of $K_{\mathbb{R}}$.)

One reason $K_{\mathbb{R}}$ is interesting is that it knows about KU (Grothendieck group of isomorphism classes of complex vector bundles over $X \in \mathrm{Top}$) and KO (Grothendieck group of isomorphism classes of *actually* real vector bundles over $X \in \mathrm{Top}$).

Claim 14.2. *If $X^{C_2} = X$ then $K_{\mathbb{R}}(X) \cong KO(X)$.*

Proof. The categories of actually real and Atiyah real vector bundles on X are equivalent. Here is a sketch of why. Given $E \to X$ an actually real vector bundle, letting the C_2 action on $\mathbb{C} \otimes_{\mathbb{R}} E$ be the one coming from complex conjugation on $\mathbb{C}$ gives $\mathbb{C} \otimes_{\mathbb{R}} E$ the structure of Atiyah-real vector bundle. Given an Atiyah-real bundle, we can take fixed points to obtain an actually real vector bundle $E^{C_2} \to X^{C_2} = X$. Since it is an equivalence of categories, the Grothendieck completions are the same. $\square$

In particular, $K_{\mathbb{R}}(*) = KO(*)$. So $\underline{\pi}_* K_{\mathbb{R}}(C_2/C_2) \cong \pi_* KO$. Atiyah's $K_{\mathbb{R}}$ also know about KU. Here is how.

Claim 14.3. $K_{\mathbb{R}}(X \times C_2) \cong KU(X)$

There is no ambiguity about $X \times C_2$ as a C_2-space: The action coming from the diagonal action and the one coming from the trivial action on X give isomorphic spaces.

Proof. Again, this comes from an equivalence of categories between complex vector bundles on X and Atiyah-real bundles on $C_2 \times X$. Given an Atiyah–real bundle $E \to X$, consider the pull-back

$$\begin{tikzcd} E \arrow[d] & i_e^* E \arrow[l] \arrow[d] \\ X & X \times \{e\} \arrow[l, hook'] \end{tikzcd}$$

The right bundle is a complex vector bundle. Further, given a complex vector bundle $E \to X$, let $\overline{E} \to X$ be the conjugate bundle. Then $E \sqcup \overline{E} \to X \times C_2 = X_e \sqcup X_\gamma$ is an Atiyah-real vector bundle on $X \times C_2$. $\square$

Corollary 14.4. $\underline{\pi}_* K_{\mathbb{R}}(C_2/e) \cong \pi_* KU$

Mike talked about the fact that $\Omega_{\mathbb{O}}^{hC_8} \simeq \Omega_{\mathbb{O}}^{C_8}$; this is the fixed point theorem. K-theory also has a fixed point theorem $K_{\mathbb{R}}^{C_2} \cong K_{\mathbb{R}}^{hC_2}$.

Recall the isotropy separation sequence

$$E\mathcal{P}_+ \to S^0 \to \widetilde{E}\mathcal{P}$$

where $\mathcal{P}$ is the family of proper subgroups of G. We have

$$E\mathcal{P}^H = \begin{cases} * & H \subsetneq G \\ \emptyset & H = G \end{cases}$$

$$\widetilde{E}\mathcal{P}^H = \begin{cases} * & H \subsetneq G \\ S^0 & H = G \end{cases}$$

It follows that, for $G = C_2$, $E\mathcal{P}$ is a model for EC_2 since the only proper subgroup of C_2 is the trivial subgroup. Also, we can write $EC_2 = \operatorname{colim}_{n\to\infty} S(n\infty) = S(\infty\sigma)$, where $S(n\sigma)$ is the unit sphere with antipodal action. Then $\widetilde{E}C_2 = \operatorname{colim}_{n\to\infty} S^{n\sigma} = S^{\infty\sigma}$, and $S^{\infty\sigma}$ has 2 fixed points, ∞ and 0.

Mingcong will show in his lecture that if X is an equivariant homotopy ring spectrum and $\widetilde{E}G \wedge X$ is contractible, then $X \to X^{EG_+}$ is a weak equivalence. This implies that $X^G \to (X^{EG_+})^G = X^{hG}$ is equivalence.

Claim 14.5. $\widetilde{E}C_2 \wedge K_{\mathbb{R}} \simeq *$

Proof. $\widetilde{E}C_2 = \mathrm{colim}(S^0 \xrightarrow{a_\sigma} S^\sigma \to S^{2\sigma} \to \dots) = S^0[a_\sigma^{-1}]$. However, there is a factorization

$$\begin{tikzcd} K_{\mathbb{R}} \wedge S^0 \arrow[r, "a_\sigma"] \arrow[dr, "K_{\mathbb{R}}\wedge\Sigma^{-1}\eta"'] & K_{\mathbb{R}} \wedge S^\sigma \\ & K_{\mathbb{R}} \wedge S^{-1} \arrow[u, "\beta"'] \end{tikzcd}$$

Here, where $\eta : S^1 \to S^0$ is the Hopf map and β is induced by the Bott map $S^\rho \to K_{\mathbb{R}}$. Since $\eta^4 = 0$, the colimit inverts a nilpotent element η. $\square$

This gives a homotopy fixed point theorem for $K_{\mathbb{R}}$. Now, consider the homotopy fixed point spectral sequence

$$H^s(C_2, \underline{\pi}_t^{C_2} K_{\mathbb{R}}) \implies \underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}^{hC_2} \cong \underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}^{C_2}.$$

Since $\underline{\pi}_t^{C_2} K_{\mathbb{R}}(C_2) = \pi_t KU$ and $\underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}^{C_2}(C_2) = \pi_{t-s} KO$, we recover the classical homotopy fixed point spectral sequence

$$H^s(C_2, \pi_t KU) \implies \pi_{t-s} KO.$$

Let $v_1 \in \pi_2 KU$ denote the classical Bott class $S^2 \to KU$; this appears in $\pi_* KU = \mathbb{Z}[v_1^{\pm 1}]$. It is the underlying map of the equivariant map $S^\rho \to K_{\mathbb{R}}$. The E_2-term is thus given by $H^*(C_2, \mathbb{Z}[v_1^{\pm 1}])$ with the action of C_2 determined by $\gamma(v_1) = -v_1$.

Computing $H^*(C_2, \mathbb{Z}[v_1^{\pm 1}])$ is a great first exercise in group cohomology. The answer is

	-8		-6		-4		-2		0		2		4		6		8
4	$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$
				$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$	
2			$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$		
		$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$				$\mathbb{Z}/2$			
0	$\mathbb{Z}$				$\mathbb{Z}$				$\mathbb{Z}$				$\mathbb{Z}$				$\mathbb{Z}$

Figure 1. The E_2-term of $E_2^{s,t} = H^s(C_2, \pi_t KU) \implies \pi_{t-s} KO$.

Note that $H^0(C_2, \mathbb{Z}[v_1^{\pm 1}]) = (\mathbb{Z}[v_1^{\pm 1}])^{C_2}$, the fixed points, which is why v_1 is not in our picture: it is not fixed! Indeed, the action of C_2 on $\mathbb{Z}\{v_1\}$ is by ± 1. But v_1^2 is the generator for the $\mathbb{Z}$ in degree $(4, 0)$ since $\mathbb{Z}\{v_1^2\}$ is a copy of the trivial C_2–module. In general, the action of C_2 on $\mathbb{Z}\{v_1^i\}$ is by ± 1 if i is odd. We denote this C_2–module by $\mathbb{Z}_-$. It is trivial if i is even, and we denote the trivial C_2–module by $\mathbb{Z}$. Further, $H^*(C_2, \mathbb{Z}) = \mathbb{Z}[x]/(2x)$ where x is in degree $* = 2$. On the other hand, $H^*(C_2, \mathbb{Z}_-) = 0$ if $*$ is even and $\mathbb{Z}/2$ if $*$ is odd.

Since $\eta^4 = 0$ and the spectral sequence is linear over multiplication by η, it follows that $d_2(v_1^2) = \eta^3$. The class detected by v_1^4 must then be a permanent cycle. This forces all the differentials and they are drawn in the picture below.

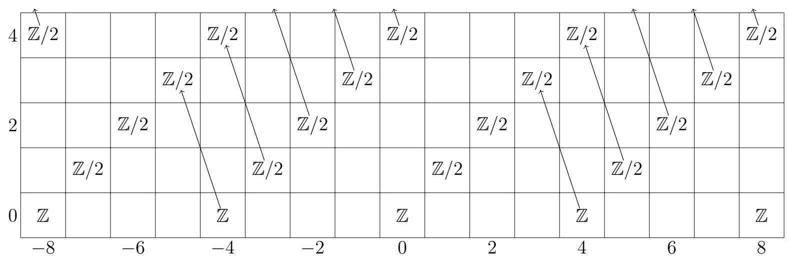


FIGURE 2. The differentials in $E_2^{s,t} = H^s(C_2, \pi_t KU) \implies \pi_{t-s}KO$.

The conclusion is that $\pi_* KO = \mathbb{Z}[b^{\pm 1}, \eta, w]/(2\eta, \eta^3, w^2 = 4b) \cong \underline{\pi}_*^{C_2} K_{\mathbb{R}}(pt)$. The E_∞–term is drawn below.

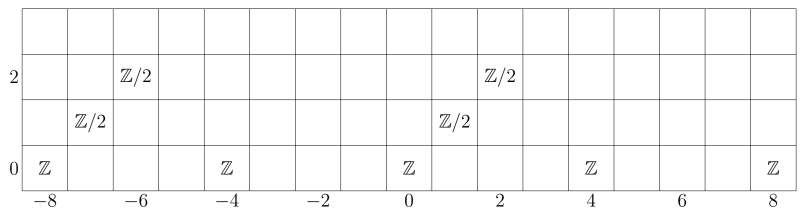


FIGURE 3. The E_∞-term of the spectral sequence $E_2^{s,t} = H^s(C_2, \pi_t KU) \implies \pi_{t-s}KO$.

14.1. Slice spectral sequence. We return to our goal of computing

$$E_2^{s,t} = \underline{\pi}_{t-s}^{C_2} P_t^t K_{\mathbb{R}}(C_2/H) \implies \underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}(C_2/H) = \begin{cases} \pi_{t-s}KO & H = C_2 \\ \pi_{t-s}KU & H = e \end{cases}$$

We need to know the slices. Koen told us that $P_{-1}^{-1} K_{\mathbb{R}} \simeq \Sigma^{-1} H \underline{\pi}_{-1} K_{\mathbb{R}}$. However,

$$\underline{\pi}_{-1} K_{\mathbb{R}}(C_2/H) = \begin{cases} \pi_{-1}KO & H = C_2 \\ \pi_{-1}KU & H = e \end{cases}$$

Since $\pi_{-1}KO = \pi_{-1}KU = 0$, this is the zero Mackey functor. Therefore, $P_{-1}^{-1} K_{\mathbb{R}} \simeq *$.

Here is a result which you can find in Mike's "The equivariant slice filtration: A primer".

Theorem 14.6. *If $\underline{\pi}_0 X$ has all restrictions monomorphisms, then $P_0^0 X = H\underline{\pi}_0 X$.*

You can check that $\underline{\pi}_0 K_{\mathbb{R}} = \underline{\mathbb{Z}}$, the constant Mackey functor, which has all restrictions identities by definition. Therefore, $P_0^0 K_{\mathbb{R}} = H\underline{\mathbb{Z}}$.

Now, recall that we can suspend by regular representations ρ and shift the slices around. However, $\Sigma^\rho K_{\mathbb{R}} \simeq K_{\mathbb{R}}$ by Bott periodicity. So knowing the -1 and 0 slices gives us all of the slices. We get the following result:

Theorem 14.7 (Dugger)**.**

$$P_t^t K_{\mathbb{R}} = \begin{cases} \Sigma^{\frac{t}{2}\rho} H\underline{\mathbb{Z}} & t \text{ even} \\ * & t \text{ odd.} \end{cases}$$

Note that $K_{\mathbb{R}}$ is a *pure isotropic*, a notion that Akhil will define, and that implies the gap theorem for $K_{\mathbb{R}}$.

If we want to compute this spectral sequence, we need to know $\pi_* P_t^t K_{\mathbb{R}}(C_2/H)$. The answer is shown in the next two figures (before the differentials).

Remark 14.8. Here is an idea on how one computes $\pi_* P_{2t}^{2t} K_{\mathbb{R}}(C_2/C_2)$, i.e., how one computes $\pi_*^{C_2}\Sigma^{t\rho} H\underline{Z}$. Note that

$$\pi_*^{C_2}\Sigma^{t\rho} H\underline{Z} = \pi_*^{C_2}\Sigma^{t+t\sigma} H\underline{\mathbb{Z}} \cong \pi_{*-t}^{C_2}(S^{t\sigma} \wedge H\underline{\mathbb{Z}}).$$

Further,

$$\pi_{*-t}^{C_2}(S^{t\sigma} \wedge H\underline{\mathbb{Z}}) \cong \begin{cases} H_{*-t}^{C_2}(S^{t\sigma}; \underline{Z}) & t \geq 0 \\ H_{C_2}^{t-*}(S^{-t\sigma}; \underline{Z}) & t < 0 \end{cases}$$

We focus on the case $t < 0$. For cohomology, we have

$$H_{C_2}^*(S^{-t\sigma}; \underline{Z}) \cong H^*(S^{-t\sigma}/C_2; Z).$$

Here, $H^*(X; Z)$ is non-equivariant cellular cohomology. Further, there is a cofiber sequence

$$S(n\sigma)_+ \to D(n\sigma)_+ \to S^{n\sigma}.$$

Applying $(-)/C_2$, this give a cofiber sequence

$$\mathbb{R}P_+^{n-1} \to S^0 \to S^{n\sigma}/C_2$$

and one can use this to compute $H_{C_2}^*(S^{-t\sigma}; \underline{Z})$ when $t < 0$.

Finally, since we already computed the homotopy groups using the homotopy fixed point spectral sequence, the differentials in the slice spectral sequence are forced and are illustrated below for C_2/e and C_2/C_2. Note that for C_2/e, the spectral sequence collapses.

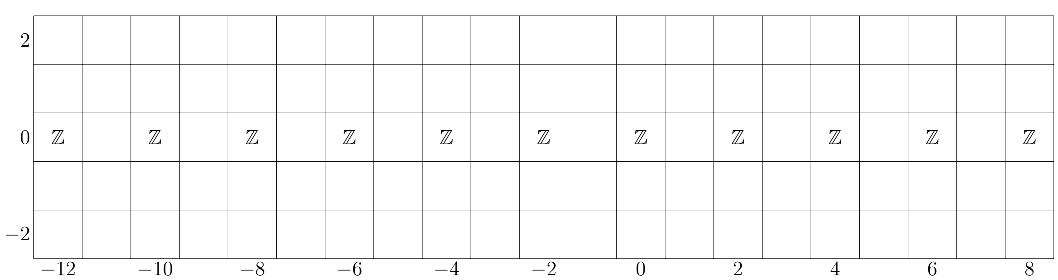


FIGURE 4. $E_2^{s,t} = \underline{\pi}_{t-s}^{C_2} P_t^t K_{\mathbb{R}}(C_2/e) \implies \underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}(C_2/e) \cong \pi_{t-s} KU$

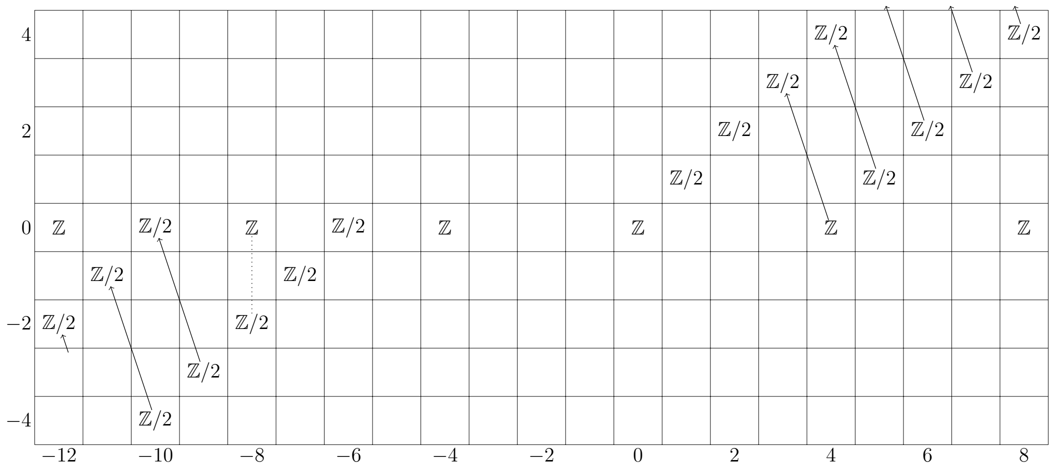


FIGURE 5. $E_2^{s,t} = \underline{\pi}_{t-s}^{C_2} P_t^t K_{\mathbb{R}}(C_2/C_2) \implies \underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}(C_2/C_2) \cong \pi_{t-s}KO$.

This can all be packaged using Mackey functors. We use the following notation for the relevant C_2–Mackey functors:

Symbol	$\square$	$\overline{\square}$	$\bullet$	$\boxslash$	$\dot{\square}$
Lewis Diagram	$\begin{smallmatrix}\mathbb{Z}\\ 1\downarrow\ \uparrow 2\\ \mathbb{Z}\end{smallmatrix}$	$\begin{smallmatrix}0\\ \downarrow\ \uparrow\\ \mathbb{Z}_-\end{smallmatrix}$	$\begin{smallmatrix}\mathbb{Z}/2\\ \downarrow\ \uparrow\\ 0\end{smallmatrix}$	$\begin{smallmatrix}\mathbb{Z}\\ 2\downarrow\ \uparrow 1\\ \mathbb{Z}\end{smallmatrix}$	$\begin{smallmatrix}\mathbb{Z}/2\\ 0\downarrow\ \uparrow 1\\ \mathbb{Z}_-\end{smallmatrix}$

Then the slice spectral sequence for $K_{\mathbb{R}}$ is described in the following nice picture.

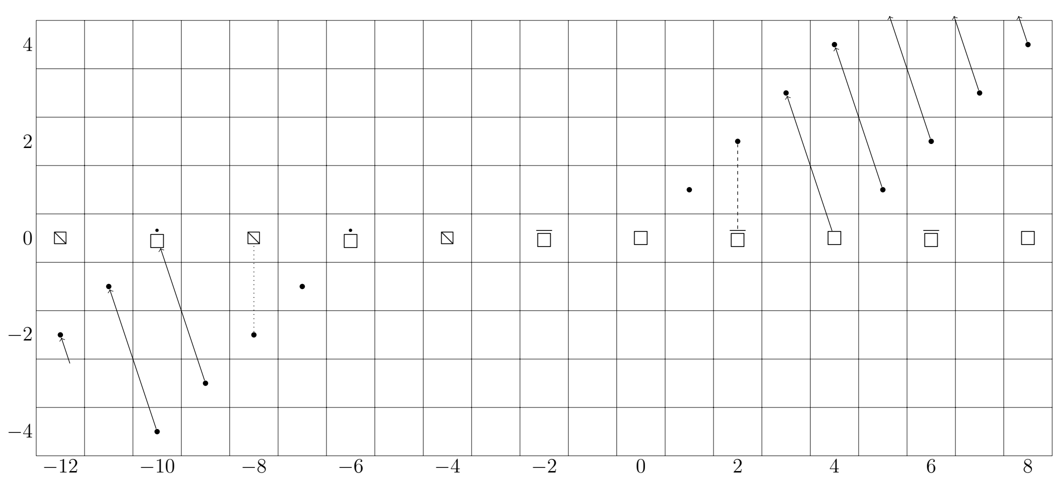


FIGURE 6. The slice spectral sequence $E_2^{s,t} = \underline{\pi}_{t-s}^{C_2} P_t^t K_{\mathbb{R}} \implies \underline{\pi}_{t-s}^{C_2} K_{\mathbb{R}}$. The dashed line in the first quadrant is an exotic transfer and the dotted line in the third quadrant an exotic restriction.

Talk 15: $MU^{((G))}$ and its slice differentials (Eva Belmont)

Let's remember where we were going with the proof of the Kervaire invariant theorem. We had this element h_j^2 in the E_2 page of the Adams spectral sequence, and, for $j \geq 7$, were trying to show that it doesn't survive the ASS.

$$\begin{tikzcd} E_2 = \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) \arrow[r, Rightarrow] & \pi_*^s \widehat{S_2} \arrow[r] & \pi_*\Omega \\ h_j^2 \arrow[r, dotted] & \theta_j \arrow[r] & ? \end{tikzcd}$$

HHR construct a spectrum Ω such that:

- if $\theta_j \neq 0$, then its image in $\pi_*\Omega$ is nonzero;
- $\pi_*\Omega$ is zero in the relevant dimension.

This provides the contradiction showing that there is no such θ_j.

We'll construct Ω in stages:

$$\begin{tikzcd} MU \arrow[r, squiggly] & MU_{\mathbb{R}} \arrow[r, squiggly, "N_{C_2}^{C_8}"] & MU^{((G))} \arrow[r, squiggly, "\text{localize}"] & D^{-1}MU^{((G))} \arrow[r, squiggly, "C_8 \text{ fixed points}"] & \Omega \\ \mathcal{S} & \mathcal{S}^{C_2} & \mathcal{S}^{C_8} & \mathcal{S}^{C_8} & \mathcal{S} \end{tikzcd}$$

This talk will be about $MU_{\mathbb{R}}$ and $MU^{((G))}$; Mingcong will talk about the last two steps.

Convention: In the rest of the talk, everything will be localized at 2, $G := C_8$, and $g = |G| = 8$.

15.1. Construction of $MU_{\mathbb{R}}$ and $MU^{((G))}$. $MU_{\mathbb{R}}$ is a C_2 spectrum. The glib (and slightly incorrect) thing to say is that $MU_{\mathbb{R}}$ is just MU, where you notice that there is a C_2 action on everything that comes from complex conjugation.

But let's try to be a little more correct. In order to specify a G-spectrum, you have to specify a functor out of $\mathscr{J}_G$, i.e. specify what G-space corresponds to each representation V of G. We start by defining spaces corresponding to the representation $\mathbb{C}^n$ with complex conjugation action. Note that $\mathbb{C} \cong 1 \oplus \sigma = \rho_2$, where 1 is the trivial representation, σ is the sign representation, and ρ_2 is the regular representation. Complex conjugation acts on $\mathbb{C}$, hence on $\mathbb{C}$-vector bundles, hence on the classifying space $BU(n)$, hence on the universal bundle $EU(n) \to BU(n)$, and finally on $MU(n) := \mathrm{Thom}(EU(n) \to BU(n))$. So if we let $\mathscr{J}'_G$ denote the full subcategory of $\mathscr{J}_G$ consisting of the representations $\mathbb{C}^n$ with complex conjugation, we can define a functor $\mathscr{J}'_G \to \mathrm{Top}_G$ sending $\mathbb{C}^n \mapsto MU(n)$. To get a functor $\mathscr{J}_G \to \mathrm{Top}_G$, we form the Kan extension

$$\begin{tikzcd} \mathscr{J}'_G \arrow[r] \arrow[d, hook] & \mathrm{Top}_G \\ \mathscr{J}_G \arrow[ur, dotted, "MU_{\mathbb{R}}"'] & \end{tikzcd}$$

(Actually, I just lied a little bit: you're supposed to do a cofibrant replacement on the functor $\mathscr{J}_G' \to \mathrm{Top}_G$ before Kan extending. See HHR §B.12.7.) This has some nice properties:

- $MU_{\mathbb{R}}$ is a commutative ring G-spectrum;
- the underlying (nonequivariant) spectrum of $MU_{\mathbb{R}}$ is just MU;
- $\Phi^{C_2} MU_{\mathbb{R}} \simeq MO$.

The second point is a consequence of the canonical homotopy presentation

$$MU_{\mathbb{R}} \xleftarrow{\simeq} \operatorname{holim} S^{-n\rho_2} \wedge MU(n).$$

15.2. Some elements in $\pi_* MU^{((G))}$.

Theorem 15.1. *There are some elements $r_i \in \pi^u_{2i} MU^{((G))}$ such that:*

- *they "generate", in the sense that $\pi^u_* MU^{((G))} \cong \mathbb{Z}_2[\gamma^j r_i]_{\substack{i\geq 0\\ 0\leq j<4}}$ where $\gamma \in G/C_2$ (so the G/C_2-translates of the r_i's are generators);*
- *they have good fixed-point properties (to be described later).*

It turns out that you can upgrade these non-equivariant homotopy classes to C_2-equivariant homotopy classes:

Theorem 15.2. *We have an isomorphism*

$$\bigoplus_{j\geq 0} \pi^u_{2j} MU^{((G))} \cong \bigoplus_{j\geq 0} \pi^{C_2}_{2\rho_2} i^*_{C_2} MU^{((G))}$$

and moreover

$$\bigoplus_{j\geq 0} \pi^u_{2j} \bigwedge_m MU^{((G))} \cong \bigoplus_{j\geq 0} \pi^{C_2}_{2\rho_2} i^*_{C_2} \bigwedge_m MU^{((G))} \qquad \textit{for any } m \geq 1.$$

Since $i^*_{C_2} MU^{((G))} \cong MU_{\mathbb{R}}^{\wedge 4}$ as C_2-spectra, we now have classes in $\pi^{C_2}_{2\rho_2} i^* MU^{((G))}$; let $\overline{r}_i$ denote the C_2-equivariant homotopy class corresponding to r_i under the above isomorphism.

Just as $MU^{((G))}$ was obtained from $MU_{\mathbb{R}}$ from taking the norm, we can produce G-equivariant homotopy classes $N\overline{r}_i \in \pi^G_{i\rho_G} MU^{((G))}$ as well by taking the norm:

$$\begin{aligned}
[S^{i\rho_2}, i^*_{C_2} MU^{((G))}] &\xrightarrow{N^G_{C_2}} [N^G_{C_2} S^{i\rho_2}, N^G_{C_2} i^*_{C_2} MU^{((G)})] \\
&\cong [S^{\mathrm{Ind}^G_{C_2} i\rho_2}, N^G_{C_2} i^*_{C_2} MU^{((G))}] \\
&\cong [S^{i\rho_G}, N^G_{C_2} i^*_{C_2} MU^{((G))}] \\
&\xrightarrow{\text{counit}} [S^{i\rho_G}, MU^{((G))}]
\end{aligned}$$

where the last map is the counit of the adjunction between restriction $i^*_{C_2}$ and norm that happens in Comm^G. This is sometimes called the *internal norm*.

Recall that $\pi_* \Phi^G MU^{((G))} \cong \pi_* MO = \mathbb{Z}/2[h_i \,:\, i \neq 2^k - 1]$. It turns out that $\Phi^G N\overline{r}_i = h_i$ (and $\Phi^G N\overline{r}_i$ when $i = 2^k - 1$ for some k). This is what I meant earlier when I said the r_i had "good fixed point properties".

15.3. Slice spectral sequence review and slice theorem preview. Recall the slice tower for a G-spectrum X looks like

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P^1X \arrow[d] & P^1_1X \arrow[l, hook', "\text{fiber}"'] \\
X \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & P^0X \arrow[d] & P^0_0X \arrow[l, hook', "\text{fiber}"'] \\
 & \vdots &
\end{tikzcd}$$

where $P^n_nX \to P^nX \to P^{n-1}X$ are fiber sequences and P^nX is made out of slice cells of dimension $\leq n$.

Before, we talked about the spectral sequence obtained by taking integer-graded homotopy of this tower; now, we're going to take $RO(G)$-graded homotopy of the slice tower. The result is a spectral sequence

$$E_2^{s,V} = \pi^G_{V-s} P^{\dim V}_{\dim V} X \implies \pi^G_{V-s} X,$$

where the differentials have the form $d_r : E_2^{s,V} \to E_2^{s+r,V+(r-1)}$. Fortunately, we will only be considering the piece of this with $V = * - 2^k\sigma$ where $* \in \mathbb{Z}$ and k is fixed; the resulting thing is now doubly-integer-graded, and the differentials stay in this piece.

We want to apply this to $X = MU^{((G))}$.

In general, computing P^n_nX is hard. But,

$$\bigvee_n P^n_n\Big(\bigvee \text{slice cells}\Big) \simeq H\underline{\mathbb{Z}} \wedge (\text{those slice cells}).$$

$MU^{((G))}$ isn't a wedge of slice cells, but the slice theorem, which we'll hear more about in Akhil's talk, says that it's the next best thing:

Theorem 15.3 (Slice theorem)**.** *There is a map*

$$\bigvee \textit{slice cells} \to MU^{((G))}$$

that is an equivalence after applying $P^n_n(-)$. In particular, this wedge of slice cells can be written $S^0[G \cdot \overline{r}_i]_{i \geq 0}$ (notation explained below). So, there is an equivalence

$$H\underline{\mathbb{Z}} \wedge S^0[G \cdot \overline{r}_i] \xrightarrow{\sim} \bigvee_{n \geq 0} P^n_n MU^{((G))}.$$

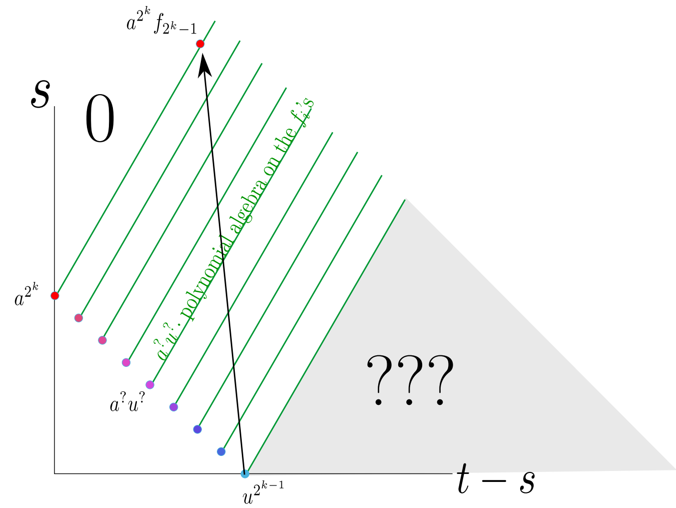
About the notation: first let $S^0[S^V] := \bigvee_{n \geq 0} (S^V)^{\wedge n}$ be the free associative algebra on S^V (the notation should remind you of a polynomial algebra). If x is the class of the canonical bottom map $S^V \to S^0[S^V]$, then we also write $S^0[x] := S^0[S^V]$. That's a polynomial ring in one variable. Our wedge of slice cells $S^0[G \cdot \overline{r}_i] := S^0[\gamma^j \cdot \overline{r}_i]_{\substack{i \geq 0 \\ 0 \leq j < 4}}$ should be thought of as a polynomial ring in infinitely many variables, one for each G/C_2-translate of $\overline{r}_i$. (Here γ is again the generator of G/C_2.)

15.4. Elements in E_2. Now we are ready to describe elements in the E_2 page of the slice spectral sequence for $MU^{((G))}$. Recall we're restricting to V of the form $i+j\sigma$ for $i,j\in\mathbb{Z}$ (we'll restrict to $*-2^k\sigma$ for fixed k later). So we're looking for maps $S^{*+*\sigma}\to\bigvee P^n_n MU^{((G))}$, and by the slice theorem it's clear that we should be looking for maps $S^{*+*\sigma}\to H\underline{\mathbb{Z}}\wedge S^0[G\cdot\overline{r}_i]$.

There are three classes of elements we're going to talk about; the first two are gotten by mapping into $H\underline{\mathbb{Z}}$ and the third is gotten by mapping into $S^0[G\cdot\overline{r}_i]$.

- a_V is the Hurewicz image of $S^0\to S^V$. (This is the Euler class of V.) In this talk, we'll only care about $\boxed{a}:=a_\sigma\in\pi_{-\sigma}H\underline{\mathbb{Z}}$, which is in $E_2^{1,1-\sigma}=\pi_{-\sigma}P^0_0MU^{((G))}$. It is a permanent cycle.
- For any oriented representation[12] V, I claim that the restriction map
$$H^G_V(S^V;\underline{\mathbb{Z}})\to H^u_{|V|}(S^V;\mathbb{Z})$$
is an isomorphism. In particular, $2\times$ any representation is an orientable representation. So define $\boxed{u}:=u_{2\sigma}\in H^G_{2\sigma}(S^{2\sigma};\underline{\mathbb{Z}})$ to be the class corresponding to the orientation class in $H^u_2(S^{2\sigma};\mathbb{Z})$. This is in $E_2^{0,2-2\sigma}=\pi_{2-2\sigma}P^0_0MU^{((G))}$.
- Recall $N\overline{r}_i$ is a homotopy class given by a map $S^{i\rho_G}\to S^0[\gamma^j\overline{r}_i]\to MU^{((G))}$. To get it into the right degree, we consider $\boxed{f_i}:S^i\xrightarrow{a_{\overline{\rho}_G}}S^{i\rho_G}\to S^0[G\cdot\overline{r}_i]\to H\underline{\mathbb{Z}}\wedge S^0[G\cdot\overline{r}_i]$. This is in $E_2^{(g-1)i,gi}=\pi_iP^{gi}_{gi}MU^{((G))}$ and it is a permanent cycle (this is just a suspension of $N\overline{r}_i$, which was an actual G-equivariant homotopy class).

Here's a picture of where these elements sit in the E_2 term. Soon I'll discuss some properties of this picture.



The vanishing line comes from the vanishing line on the slice spectral sequence that Koen talked about, but it's a little different because we've got the piece of the spectral sequence where $V=*-2^k\sigma$ for $*\in\mathbb{Z}$ (Koen only discussed the case where both indices are integers).

[12] An orthogonal representation $G\to O(N)$ is oriented if it factors through $SO(n)$; or equivalently, if there is an element in $\bigwedge^{|V|}V$ fixed by G; or equivalently, if there is an element (an orientation class) in $H^G(S^V;\underline{\mathbb{Z}})$.

15.5. A differential in the slice spectral sequence for $MU^{((G))}$.

Theorem 15.4. *There is a differential $d_{1+(2^k-1)g}u^{2^{k-1}} = a^{2^k}f_{2^k-1}$, and all previous differentials on $u^{2^{k-1}}$ are zero.*

Proof. Note: this proof is not quite the same as the one in the HHR paper; I learned this from Po Hu's MSRI talk.

Here are the main steps of the proof:

(1) Show we've described everything in the colored region.

(2) $a^{-1}\pi_*X \cong \pi_*\Phi^G X$. In particular, $a^{-1}\pi_*MU^{((G))} \cong \pi_*\Phi^G MU^{((G))} \cong \pi_*MO \cong \mathbb{Z}/2[h_i : i \neq 2^j - 1 \text{ for some } j]$.

(3) $a^? f_{2^k-1}$ needs to be hit by a differential (for some value of ?).

(4) It has to be $a^{2^k} f_{2^k-1}$.

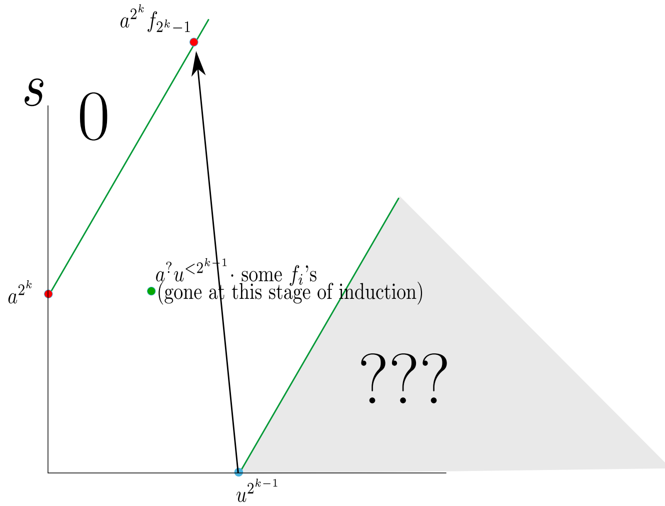
(5) And it has to be killed by the advertised differential.

I'm not going to talk about (1) because it's not the interesting part; this corresponds to most of §9.1 in the paper before the proof of 9.9. The point is to show that induced slice cells don't contribute in this region, and then explicitly analyze the non-induced slice cells.

(2) is Lemma 15.5. (3) follows from this.

(4) I'm not going to talk about this either – you assume some $a^n f_{2^k-1}$ were hit by a longer differential and get a contradiction because the degree of the source of the differential doesn't make sense.

(5) Use induction on k. The claim is that everything in the known region except the two green lines in the diagram is zero on our stage of the induction – this is because all of the elements in the middle (see hypothetical green dot in the picture) are a product of some permanent cycles (a's and f_i's) and u raised to a power $< 2^{k-1}$. Even though we've only shown that u^{2^i} for $i < k-1$ support differentials, we've also shown that $d_r(u^{2^i}) = 0$ for $r <$ the differential that is actually nonzero. So e.g. if we have $u^6 = u^2u^4$, and if $d_j(u^2) \neq 0$, we have $d(u^2u^4) = d(u^2)u^4 \neq 0$. So our hypothetical green dot (and everything else in the middle) is gone at this stage of the induction. We've already claimed that $a^{2^k} f_{2^k-1}$ has to get hit, and it has to get hit at this stage because there are no more opportunities afterwards.



□

Lemma 15.5. $\pi_* \Phi^G X \simeq a^{-1} \pi_* X$

Proof. It has been shown in previous talks that $a^{-1}\pi_*^G X = \operatorname{colim}(\pi_*^G X \xrightarrow{a} \pi_*^G X \xrightarrow{a} \dots)$ is just smashing X with S^σ a bunch of times. On the other hand, we've also shown that $\pi_*(\widetilde{E}\mathcal{P} \wedge X)^G = \pi_*^G((\lim_{n\to\infty} S^{n\sigma}) \wedge X)$. (I guess I'm ignoring fibrant replacement. . . the claim is that it doesn't matter.) □

For the record, I'll actually tell you some facts about degree:

$$\begin{aligned}
a_\sigma &\in E_2^{1,1-\sigma} = \pi_{-\sigma} P_0^0 MU^{((G))} \\
f_i &\in E_2^{(g-1)i,gi} = \pi_i P_{gi}^{gi} MU^{((G))} \\
u = u_{2\sigma} &\in E_2^{0,2-2\sigma} = \pi_{2-2\sigma} P_0^0 MU^{((G))}
\end{aligned}$$

where $g = |G| = 7$.

Talk 16: The reduction, gap, and slice theorems (Akhil Mathew)

Consider $MU^{((G_8))} = N_{C_2}^{C_8}(MU_\mathbb{R})$ from last time. There's going to be a class $D \in \pi_{19\rho_8} MU^{((C_8))}$ that Mingcong will talk about. Then they define $\Omega_\mathbb{O} = (MU^{((C_8))})[D^{-1}]$. Then we're going to construct a non-equivariant spectrum $\Omega := \Omega_\mathbb{O}^{C_8}$ and that is $\cong \Omega_\mathbb{O}^{hC_8}$ by the homotopy fixed point theorem (the $\mathbb{O}$ stands for octonions).

One of the key theorems is

Theorem 16.1 (Gap theorem). $\pi_i \Omega = 0$ *for* $-4 < i < 0$

Ω is 256-periodic, so this is going to imply that the Kervaire classes go to 0 if they existed.

We have

$$\Omega_{\mathbb{O}} \simeq \operatorname{hocolim}(MU^{((C_8))} \xrightarrow{D} \Sigma^{-19\rho_8} MU^{((C_8))} \to \dots) \tag{16.1}$$

HHR proved that the gap is visible at each stage. In fact, $\pi_i^{C_8}(\Sigma^{n\rho_8} MU^{((C_8))}) = 0$ for $-4 < i < 0$ and all n. They proved this by showing that this gap is true for a large class of G-spectra. This is proved using the slice spectral sequence.

We cared about a non-equivariant spectrum Ω, but the important thing is to pass through a C_8-spectrum and consider its non-homotopy-fixed points.

I'm going to review a definition from Koen's talk. Throughout, G will be a cyclic 2-group.

Definition 16.2. A slice in $\mathcal{S}^G$ is a *pure cellular* spectrum if it is a wedge of $H\mathbb{Z} \wedge \widehat{S}(m, K)$ where $\widehat{S}(m, K) = G_+ \wedge_K S^{m\rho_K}$ for $K \subset G$.

Definition 16.3. A G-spectrum X is *pure and isotropic* if all the slices $P_n^n X$ are pure cellular, i.e. wedges of $H\mathbb{Z} \wedge \widehat{S}(m, K)$ where $m|K| = n$ and where $K \neq 1$.

In particular, these are all even slices, so this is saying that all the odd-dimensional slices vanish.

There are two theorems that will imply the gap theorem.

Theorem 16.4 (Slice Theorem)**.** *$MU^{((G))} = N_{C_2}^G(MU_{\mathbb{R}})$ is pure and isotropic.*

Theorem 16.5 ((General) gap theorem)**.** *If $X \in \mathcal{S}^G$ is pure and isotropic and $G \neq 1$, then $\pi_i^G X = 0$ for $-4 < i < 0$.*

Remark 16.6. If X is pure and isotropic, so is $S^{m\rho_G} \wedge X$; smashing with a regular representation sphere just shifts the slice filtration.

So if we know that $MU^{((C_8))}$ is pure and isotropic, then so are all the other terms in the hocolim in (16.1). (Note that the homotopy colimit of pure and isotropic things needn't be pure and isotropic, but it works in this case, because these maps are inclusion of a wedge summand.)

Example 16.7. As Agnes explained, $K\mathbb{R} \in \mathcal{S}^{C_2}$ has slices of the form $H\mathbb{Z} \wedge S^{n\rho_2}$. We can see the gap in the homotopy: $\pi_i^{C_2} K\mathbb{R} \cong \pi_i KO = 0$ if $-4 < i < 0$.

You can try to run this story with p-groups, and still get a gap $-4 < i < 0$.

If X is pure and isotropic, we know that $X \simeq \varprojlim P^n X$ and the fiber of $P^n X \to P^{n-1} X$ looks like a wedge of summands of the form $H\mathbb{Z} \wedge \widehat{S}(m, K)$ where $m|K| = n$ and $K \neq 1$. By using this tower, it suffices to show that you have the gaps on each of these fibers, because then you can use a LES. It is enough to show that $\pi_i^G(H\mathbb{Z} \wedge (G_+ \wedge_K S^{m\rho_K})) = 0$ in the range. Induction and coinduction are the same for G-spectra; by the adjunction this is $\cong \pi_i^K(H\mathbb{Z} \wedge S^{m\rho_K})$. It suffices to prove:

Lemma 16.8 (Cell Lemma). *If $G \neq 1$ is a cyclic 2-group, then $\pi_i^G H\mathbb{Z} \wedge S^{k\rho_G} = 0$ for $-4 < i < 0$.*

Aside from one edge case, G can be any finite group. It suffices to make a calculation of Bredon homology.

Proof. We only describe the proof for $i = -1$ and -2, because those are all that is needed for the Kervaire result.

This is $[S^i, H\mathbb{Z} \wedge S^{k\rho_G}] \simeq [\Sigma^i S^{-k\rho_G}, H\mathbb{Z}]$. We need a general observation about Bredon homology with coefficients in the constant Mackey functor: if X is a G-CW complex, $[\Sigma^{-j} X, H\mathbb{Z}]_{\mathcal{S}^G} \simeq \widetilde{H}^j(X/G; \mathbb{Z})$. (Why? Check on orbits, and then use the cell decomposition. Warning: this doesn't work in homology.)

Consider maps $[\Sigma^i S^{-k\rho_G}; H\mathbb{Z}]_G$. This vanishes for $k \geq 0$: it is $\cong [S^0, H\mathbb{Z} \wedge S^{-i} \wedge S^{k\rho_G}]$. If $k \geq 0$, the RHS is 0-connected (also remember we're considering $-4 < i < 0$), which means there are no maps from S^0 into it. So it suffices to consider the case when $k < 0$.

So take $k = -\ell$ ofr $\ell > 0$. We're considering $[\Sigma^i S^{\ell\rho_G}; H\mathbb{Z}]_G \cong \widetilde{H}^{-i}(S^{\ell\rho_G}/G; \mathbb{Z})$. I need to show that this is zero for $i = -1, -2$ (for the general gap theorem, you also need -3, but for the part needed for the Kervaire theorem you only need these i's). If $\ell \geq 2$, $S^{\ell\rho_G}$ is a 2-fold suspension $\Sigma^2 S^{(\ell\rho_G - 2)}$. We're using the fact that if you have a connected space and take it's orbits, it's still connected, and also that orbits commute with suspension. So you're taking $\widetilde{H}^{-i}$ of the 2-fold suspension of a connected space $S^{\ell\rho_G - 2}/G$. So $i = -1$ is OK.

It suffices to consider $\ell = 1$, $i = -2$. Write $\rho_G = 1 \oplus \widetilde{\rho}_G$, so $H^2(S^{\rho_G}/G; \mathbb{Z}) = 0$ iff $H^1(S^{\widetilde{\rho}_G}/G; \mathbb{Z}) = 0$. There is a cofiber sequence $S(\widetilde{\rho}_G)_+ \to S^0 \to S^{\widetilde{\rho}_G}$ called the Euler sequence. Quotienting by the action of G, you have $S(\widetilde{\rho}_G)_+/G \to S^0 \to S^{\widetilde{\rho}_G}/G$. So $S^{\widetilde{\rho}_G}/G = \Sigma(S(\widetilde{\rho}_G)/G)$ which is the suspension of a connected space and hence it has no H^1. (To do $i = -3$, you have to push the calculation one step further.) $\square$

We're trying to understand the slice tower of $MU^{((G))}$. We're going to build $P^n MU^{((G))}$ by "algebraic methods"; then you have to prove that the thing you've produced is the slice tower. We start with a toy case.

Example 16.9. Let R be a ring spectrum with $\pi_* R = \mathbb{Z}[x]$ for $|x| = 2$. R/x is defined to be the cofiber of $\Sigma^2 R \xrightarrow{x} R$, and in our case that is $\simeq H\mathbb{Z}$. Similarly, $R/x^n \simeq \mathrm{cofib}(\Sigma^{2n} R \xrightarrow{x^n} R) \simeq \tau_{\leq 2n-2} R$. I claim that the sequence $\{R/x^n\}$ is the Postnikov tower.

Example 16.10. Let R be an E_∞ ring spectrum, where $\pi_* R = \mathbb{Z}[x, y]$ where $|x| = 2$, $|y| = 4$. Again we're going to try to construct the Postnikov tower. Now it's not sufficient to mod out by powers of x and y alone; you need to mod out by powers of the augmentation ideal. Define $A_1 = S^0[S^2] = \bigvee_{i \geq 0} S^{2i}$. There's a natural map $A_1 \to R$ of associative rings that sends $S^2 \mapsto x$. Let $A_2 = S^0[S^4] = \bigvee_{j\geq 0} S^{4j}$; there's a map $A_2 \to R$ sending $S^4 \mapsto y$. Get a map $A_1 \wedge A_2 \to R \wedge R \to R$, where $A_1 \wedge A_2 \simeq \bigvee_{i,j\geq 0} S^{2i+4j}$. Its homotopy looks like the homotopy groups of spheres where you've added an extra class in degree 2 and in degree 4. Here we use that R is commutative. If R is associative, you have a map $R \wedge R \to R$ but it's only a map of rings if R is commutative.

I'm going to define an ideal (sub-wedge) in this ring spectrum of classes that have degree $\geq 2n$: define $I_{2n} = \bigvee_{\substack{i,j \\ 2i+4j\geq 2n}} S^{2i+4j} \subset A$. This is an (A, A)-bimodule. So I can form a tower $\{A/I_{2n}\}$, and its associated graded $I_{2n}/I_{2n+2} \simeq \bigvee_{\substack{i,j \\ 2i+4j=2n}}$., an (A, A)-bimodule whose action is coming from $A \to S^0$ (the classes S^2 and S^4 aren't doing anything because they push you into higher filtration).

How to construct the Postnikov tower of R? Using the augmentation ideal, form $R \wedge_A S^0$, and the claim is that that is $\simeq H\mathbb{Z}$. More generally, $R \wedge_A (A/I_{2n}) \simeq \tau_{\leq 2n-2}R$.

(These are monoid rings, but not twisted yet.)

HHR try to do this equivariantly.

Theorem 16.11. $\pi_*^u MU^{((G))} = \mathbb{Z}[G \cdot r_1, G \cdot r_2, \dots]$ *where* $G \cdot r_i = \{r_i, \gamma r_i, \dots, \gamma^{|G|/2-1} r_i\}$ *where* $\gamma \in G$ *is a generator, and* $\gamma^{|G|/2} r_i = (-1)^i r_i$.

Each r_i refines $S^{i\rho_2} \xrightarrow{\overline{r}_i} i^*_{C_2} MU^{((G))}$.

For each i, define $S^0[S^{i\rho_2}] \simeq \bigvee_{j\geq 0} S^{ij\rho_2}$ to be the free associative algebra, and define $S^0[G \cdot \overline{r}_i] = N_{C_2}^G(S^0[S^{i\rho_2}])$. Norming from a polynomial ring on one generator gets you a polynomial ring on a bunch of generators that are permuted by the group.

The observation is that for each i, I get a map $S^0[G \cdot \overline{r}_i] \to MU^{((G))}$ which does what you think it does on π_*^u. Why? $\overline{r}_i$ is a map $S^0[S^{i\rho_2}] \to i^*_{C_2} MU^{((G))}$, and so you can form $S^0[G \cdot \overline{r}_i] \simeq N_{C_2}^G(S^0[S^{i\rho_2}]) \to N_{C_2}^G(i^*_{C_2} MU^{((G))}) \to MU^{((G))}$; we're using the fact that $MU^{((G))}$ is commutative and the last map is the counit map.

Form $S^0[G \cdot \overline{r}_i] = \bigwedge_i S^0[G \cdot \overline{r}_i] \to MU^{((G))}$; this is $\simeq A$ and is called a "refinement of homotopy".

Theorem 16.12 (Reduction theorem)**.** $MU^{((G))} \wedge_A S^0 \simeq H\mathbb{Z}$.

This is due to HHR in general, but for C_2 it is due to Hu-Kriz. There's an analogue of this is motivic homotopy theory, where it is due to Hopkins-Morel, separately published by Hoyois.

Now more about the slice theorem. We had

$$A = S^0[G \cdot \overline{r}_i]_{i\geq 0} = \bigvee_{\substack{f: J \to \mathbb{Z}_{\geq 0} \\ \text{finitely supported}}} S^{\rho_f}$$

where

- $J = \coprod_{i>0} G/C_2$
- $K_f = \mathrm{stab}(f)$
- ρ_f is the multiple of the regular representation of K_f of dimension $2\sum_{j\in J} if(j)$

Let $I_{2n} \subset A$ be all representation spheres of dimension $\geq 2n$. This is a sub-bimodule of A. You don't need to use G-spectra; this is happening at the level of ordinary spectra. The I_{2n} will form a descending filtration of A, and I_{2n}/I_{2n-2} is a wedge of isotropic slice cells of dimension $2n$ acted on through the augmentation.

I claim that $MU^{((G))} \wedge_A (A/I_{2n}) \simeq P^{2n-2} MU^{((G))}$. The proof is similar to the non-equivariant case. The associated graded is $MU^{((G))} \wedge_A (I_{2n}/I_{2n+2})$. Because this is acted on by the augmentation ideal, this is the same as $(MU^{((G))} \wedge_A S^0) \wedge I_{2n}/I_{2n+2}$. This is the same as $H\mathbb{Z}\wedge$ (pure and isotropic slice cells). So each of the terms in the associated graded is made out of slice cells. There's a recognition theorem that tells you this is actually the slice tower.

For the reduction theorem, you're trying to prove that $MU^{((G))} \wedge_A S^0 \simeq H\mathbb{Z}$. For $G = 1$, this is due to Quillen and Milnor. In general, this is proved along with the slice theorem. Prove the reduction theorem for C_{2^n}, use that to prove the slice theorem for C_{2^n}, use that to prove the reduction theorem for $C_{2^{n+1}}$, use that to prove the slice theorem for $C_{2^{n+1}}$, etc.

Suppose we know it for proper subgroups. Use the isotropy separation sequence: to show $X \to Y$ is an equivalence, it suffices to show the top and bottom maps in

$$\begin{tikzcd} E\mathcal{P}_+ \wedge X \arrow[r] \arrow[d] & E\mathcal{P}_+ \wedge Y \arrow[d] \\ X \arrow[r] \arrow[d] & Y \arrow[d] \\ X \wedge \widetilde{E}\mathcal{P} \arrow[r] & \widetilde{E}\mathcal{P} \wedge Y \end{tikzcd}$$

are equivalences. Using the reduction and slice theorems for the smaller groups, you can show the top equivalence. The hard part is to show that $\Phi^G(MU^{((G))} \wedge_A S^0) \simeq \Phi^G H\mathbb{Z}$. Identify each side explicitly: the LHS is $MO \wedge_{S^0[h_i]} S^0$ and the RHS is $\mathbb{Z}/2[b]$ for a class with $|b| = 2$. Then you have to show that the map you get between these is not the zero map.

Talk 17: The periodicity theorem and the homotopy fixed point theorem (Mingcong Zeng)

17.1. Periodicity theorem. The idea is to find an element $D \in \pi^{C_8}_{\ell\rho_8} MU^{((C_8))}$ such that $(D^{-1}MU^{((C_8))})^{hC_8}$ has periodicity. Start with important elements $\overline{\partial}_k := N^{C_8}_{C_2}\overline{r}_{2^k-1} \in \pi^{C_8}_{(2^k-1)\rho_8} MU^{((C_8))}$. This has the property that $\Phi^G(\overline{\partial}_k) = h_{2^k-1} = 0$ in $\pi_* MO = \mathbb{Z}/2[h_i : i \neq 2^k-1]$. Eva "defined" $f_{2^k-1} = a_{\overline{\rho}_8}^{2^k-1}\overline{\partial}_k$ where a_V is the map $S^0 \hookrightarrow S^V$. We have $|f_i| = i$ in homotopy.

Let $G = C_8$ and $g = |G| = 7$.

Corollary 17.1. *In the slice spectral sequence for $MU^{((G))}$, $\overline{\partial}_k u_{2\sigma}^{2^k}$ are permanent cycles.*

Proof. By brutal computation, $d_{1+(2^{k+1}-1)g}(\overline{\partial}_k u_{2\sigma}^{2^k}) = \overline{\partial}_k a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$. By definition, $f_{2^{k+1}-1}\overline{\partial}_k = a^{2^{k+1}-1}$ By magic, $d_{1+(2^k-1)g}(u_{2\sigma}^{2^{k-1}} a_\sigma^{2^k} a_{\overline{\rho}_8}^{2^k} \overline{\partial}_{k+1}) = \overline{\partial}_k a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$. The target got killed earlier, so $\overline{\partial}_k u_{2\sigma}^{2^k}$ survives $d_{1+(2^{k+1}-1)g}$. □

For the periodicity theorem, you want to find $D \in \pi_{\ell\rho_8}^{C_8} MU^{((C_8))}$ such that $\overline{\partial}_1 \mid D$ (this is a technical requirement that I'll explain soon).

You hope that $u_{2\rho_G}^k$ becomes a permanent cycle after inverting D (for D yet to be announced). Recall that $u_{2\rho_G} \in \pi_{2g-2\rho_G}^{C_8} H\underline{\mathbb{Z}}$. Use $\overline{\partial}_1$ to pull this back into integer degree, i.e. consider $u_{2\rho_G}^k \overline{\partial}_1^{2k} \in \pi_{2gk}^{C_8} D^{-1} MU^{((C_8))}$. Realize that $i_0^*(u_{2\rho_G}^k) = 1$. So $i_0^*(\overline{\partial}_1^{2k})$ is invertible. This gives a map $D^{-1}MU^{((C_8))} \to \Sigma^{2gk} D^{-1} MU^{((C_8))}$ which gives a weak equivalence when you forget the G-action. Lemma 17.2 below shows that it is an equivalence after taking $(-)^{hG}$, and that gives periodicity.

Lemma 17.2. *If a G-map $f : X \to Y$ is a weak equivalence when you forget the G-action, then f^{hG} is a weak equivalence.*

Our task is to find this D.

The identity map $MU^{((G))} \to MU^{((G))}$, by adjunction, gives a map $MU^{((H))} \to i_H^* MU^{((G))}$. Someone claimed that this was an injection. There are elements $\overline{r}_i^H \in \pi_{i\rho_2}^H MU^{((H))}$ and we use the same notation to denote their image in $MU^{((G))}$. They have horrible formulas.

Now I can state the main theorem. Let $G = C_8$, $H \subset G$, $g = |G|$, and $h = |H|$.

Theorem 17.3. *Let $D \in \pi_{\ell\rho_8}^{C_8} MU^{((G))}$ be a class such that:*

(1) For any nontrivial $H \subset G$, $N_H^G i_H^* D$ divides a power of D.

(2) Define $\overline{\partial}_i^H := N_{C_2}^H \overline{r}_{2^i-1}^H$. Then $\overline{\partial}_{g/h}^H \mid i_H^* D$.

Assuming we have such a D, then $u_{2\rho_G}^{2^{g/2}}$ is a permanent cycle in $\pi_\star^{C_8} D^{-1} MU^{((C_8))}$.

Proof. $\overline{\partial}_{g/h}^H u_{2\sigma_H}^{2^{g/h}}$ are permanent cycles. By (2), $u_{2\sigma_H}^{2^{g/2}}$ is a permanent cycle in $D^{-1}MU^{((C_8))}$. Now things get very easy. We have a nice formula $u_{2\rho_G} = u_{2\sigma_G}^{g/2} \cdot N_H^G u_{2\rho_H}$ if $[G : H] = 2$. This is somewhere very early in HHR and it is a geometric proof; I'd like to just assume this. Now we can do the same thing for H and the index-2 subgroup of H.

We're trying to show that, for some k, the $h \cdot k/2$-power of $u_{2\rho_G} = \prod_{0 \neq H \subset G} N_H^G (u_{2\sigma_H}^{h/2})$ is a permanent cycle for nontrivial $H \subset G$. The smallest k we can pick is $2^{g/2}$ when $|H| = 2$. $u_{2\sigma_H}^{2^{g/2}}$ is a permanent cycle. As H grows larger, this grows larger, but $u_{2\sigma_H}^{2^{g/h}}$ gets smaller by a power of 2. So $u_{2\rho_G}^{2^{g/2}}$ is a permanent cycle. □

We've never used condition (1); that is used to prove that $D^{-1}MU^{((C_8))}$ is a commutative algebra, so the norm works.

Why does g/h happen? We can replace all g/h's by 1, and we will get a k much smaller and the periodicity will work. But you need it for the detection theorem. Take

$$\Delta = (u_{2\rho_G}(\overline{\partial}_1^G)^2)^{2^{g/2}}.$$

This is something you can pull back to integer degree, so it's a permanent cycle. D is really flexible if you don't need detection. Set

$$D = (N_{C_2}^{C_8}\overline{\partial}_4^{C_2})(N_{C_2}^{C_8}\overline{\partial}_2^{C_4})\overline{\partial}_1^{C_8}$$

This element will give you the magic number $19 = 15 + 3 + 1$ (from the three factors above). Each of those numbers are 1 less than a power of 2.

17.2. Homotopy fixed point theorem. Why do we need this? The periodicity and detection theorems are about homotopy fixed points. But the gap and slice spectral sequence computations are in actual fixed points. To make the whole proof work, we need to show that they are equal.

How to do this? Brutally define:

Definition 17.4. A G-spectrum X is cofree if $X \to X^{EG_+}$ is a weak equivalence.

(Here X^{EG_+} means the function spectrum. This is adjoint to $EG_+ \wedge X \to X$, which is an equivalence if X is free, so that's why this is called cofree.)

Lemma 17.5. *For a G-spectrum X, TFAE:*

(1) For all nontrivial $H \subset G$, $\Phi^H X$ is contractible.

(2) The map $EG_+ \wedge X \to X$ is a weak equivalence.

(3) The spectrum $\widetilde{E}G \wedge X$ is contractible.

(Don't get $\widetilde{E}G$ confused with $\widetilde{E}\mathcal{P}$!)

Proof. (2) $\iff$ (3) because of the cofiber sequence $EG_+ \to S^0 \to \widetilde{E}G$.

(2) $\implies$ (1) Geometric fixed points were defined as $\Phi^H X := (\widetilde{E}\mathcal{P}_H \wedge X)^H_f$. Use the hypothesis to write $\widetilde{E}\mathcal{P} \wedge EG_+ \wedge X \cong \widetilde{E}\mathcal{P}_H \wedge X$, but $\widetilde{E}\mathcal{P}_H \wedge EG_+$ is contractible because EG_+ has trivial geometric fixed points.

(1) $\implies$ (2) I claim it suffices to check $H = \{e\}$. But $\Phi^{\{e\}}(\widetilde{E}G \wedge X) \simeq *$ since $\widetilde{E}$ is contractible as a space. $\square$

Corollary 17.6. *Let R be an equivariant homotopy ring spectrum. If R satisfies the equivalent conditions in Lemma 17.5 then all modules over R are cofree.*

The proof is pretty simple.

Proof of the homotopy fixed points theorem. M is a retract of $R\wedge M$, so $\Phi^H(M)$ is a retract of $\Phi^H(R\wedge M)$. So if R satisfies Lemma 17.5, then so do all modules over it. In particular, this is true for M and M^{EG_+}. We want to show that $M \to M^{EG_+}$ is a weak equivalence. Plug this into a cofiber sequence

$$\begin{tikzcd} EG_+ \wedge M \arrow[r] & M \arrow[r] \arrow[d] & \widetilde{E}G \wedge M \\ & M^{EG_+} & \end{tikzcd}$$

Smash the map with a cofiber sequence:

$$\begin{tikzcd} EG_+ \wedge M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \underbrace{\widetilde{E}G \wedge M}_{\simeq} * \\ EG_+ \wedge M^{EG_+} \arrow[r] & M^{EG_+} \arrow[r] & \underbrace{\widetilde{E}G \wedge M^{EG_+}}_{\simeq *} \end{tikzcd}$$

$\widetilde{E}_G \wedge M$ vanishes so $\widetilde{E}G \wedge M^{EG_+}$ vanishes. $M \to M^{EG_+}$ is an underlying weak equivalence. This gives the left hand vertical map is an equivalence. EG_+ is built of free G-cells; so when smashing with it all you need is the underlying homotopy type. The space-level analogue is $G \times X \cong G \times X_{\substack{\text{trivial}\\ \text{action}}}$. So the middle map is an equivalence as well. $\square$

Finally, I need to show that $D^{-1}MU^{((C_8))}$ satisfies the condition. This comes from the construction. Geometric fixed points commutes with smash products and homotopy colimits. So we only need to look at $\Phi^H(D)^{-1}\Phi^H(MU^{((G))})$. For example, if $H = C_8$, then you're involving $\Phi^{C_8}(\cdots \cdot \overline{\partial}_1^{C_8}) = 0$. In the beginning, $\Phi^G(\overline{\partial}_i^G) = 0$ is sort of the defining property (remember how $N\overline{r}_i$ related to the missing element in $\pi_* MO$).

You want E_n's to have pure and isotropic slices; then you could use the SSS instead of the homotopy fixed points.

Talk 18: The detection theorem (Zhouli Xu)

Theorem 18.1 (The detection theorem)**.** *If $\theta_j \in \pi_{2^{j+1}-2}S^0$ is an element with Kervaire invariant 1 and $j > 2$, then the Hurewicz image in $\pi_{2^{j+1}-2}\Omega$ is nonzero.*

Recall we had a C_8-spectrum $\Omega_{\mathbb{O}}$ and $\Omega = \Omega_{\mathbb{O}}^{hC_8}$. We have maps of spectral sequences

$$\begin{tikzcd}
\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*) \arrow[rr, "f"] \arrow[dr, Rightarrow] \arrow[dd, two heads] & & H^2(C_8, \pi^H_{2^{j+1}}\Omega_{\mathbb{O}}) \arrow[dr, Rightarrow] & \\
& \pi_{2^{j+2}-2}S^0 \arrow[rr] & & \pi_{2_{j+1}-2}\Omega \\
\mathrm{Ext}^{2,2^{i+1}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

Theorem 18.2 (The algebraic detection theorem)**.** *If $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*)$ is any element that maps to h_j^2, then the image of x in $H^2(C_8; \pi_{j+2}\Omega_{\mathbb{O}})$ is nonzero.*

The next thing is to show that the algebraic detection theorem implies the topological detection theorem.

First, we have to show that h_j^2, x, and $f(x)$ are non-boundaries. In the Adams grading, differentials d_r have grading $(-1, r)$. These are in Adams filtration 2, so the only possible differentials come from filtration 0 or 1. But there are classical facts that $\mathrm{Ext}_A^{0,2^{j+1}-1} = 0$ and $\mathrm{Ext}^{0,2^{j+1}-1}_{MU_*MU} = 0$. So these elements can't be killed by a d_2. For $f(x)$, the possible d_2 is $H^0(C_8; \pi_{2^{i+1}-1}\Omega_{\mathbb{O}}) \to H^2(C_8; \pi_{2^{j+1}}\Omega_{\mathbb{O}})$. But $\pi_{\mathrm{odd}}\Omega_{\mathbb{O}} = 0$ and that implies the LHS is 0.

Since the ASS and ANSS converge to the same thing, the preimage of h_j^2 has to be in filtration ≤ 2. If h_j^2 survives, there's at least one preimage of h_j^2, say x, that survives in ANSS. x is a p-cycle in the homotopy fixed point spectral sequence (HFPSS). Therefore, $f(x)$ is a p-cycle in the HFPSS. That's the topological detection theorem.

I haven't told you yet why there's a map of spectral sequences; I'll do that later.

There's a map $\pi_*^u\Omega_{\mathbb{O}} \to A_*$ where A_* is a C_8-spectrum such that $A = \mathbb{Z}_2[\zeta]$ where ζ is an 8^{th} root of unity, and $A_* = A[u]$ for $|u| = 2$. If $\gamma \in C_8$ is a generator, then $\gamma \cdot a = a$ for all $a \in A$ and $\gamma \cdot u = \zeta u$. A is a DVR, and its maximal ideal is generated by $1 - \varphi = \pi$. Note that $2 = (1-\zeta)^4 \cdot \mathrm{unit}$.

$$\begin{tikzcd}
\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*) \arrow[rrr, bend left, shift left, end anchor=north] \arrow[dr, bend right, end anchor=north west] & & & \\
& \mathrm{Ext}^{1,2^{j+1}}_{MU_*MU}(MU_*, MU_*) \arrow[ul, two heads] \arrow[r] \arrow[d, two heads] & H^1(C_8; \pi_{2^{j+1}}\Omega_{\mathbb{O}}/2) \arrow[r] \arrow[d] & H^2(C_8; \pi_{2^{j+1}}\Omega_{\mathbb{O}}) \arrow[d] \\
& \underbrace{\mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)}_{\mathbb{Z}/2} \arrow[r, equal] & \underbrace{H^1(C_8; A_{2^{j+1}}/(\pi))}_{\mathbb{Z}/2} \arrow[r, hook] & H^2(C_8; A_{2^{i+1}})
\end{tikzcd} \tag{18.1}$$

$H^2(C_8; A_{2^{j+1}})$ has a trivial action, so it's much easier to compute.

The SES $0 \to A_* \xrightarrow{\pi} A_* \to A_*/(\pi) \to 0$ gives a boundary map $H^1(C_8; A_{2j+1}/(\pi)) \hookrightarrow H^2(C_8; A_{2j+1})$ and I claim it's an injection. Also, this H^1 is $\cong \mathbb{Z}/2$.

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "\pi"] & A_* \arrow[r] & A_*/(\pi) \arrow[r] & 0 \\
0 \arrow[r] & \pi_*\Omega_{\mathbb{O}} \arrow[r, "2"] \arrow[u] & \pi_*\Omega_{\mathbb{O}} \arrow[r] \arrow[u] & \pi_*\Omega_{\mathbb{O}}/2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The goal is to show that all the boxes in (18.1) commute, and to show all the claimed surjectivity, injectivity (but that's homological algebra and I probably won't say a lot about it).

Recall that a formal group law over R is a power series $F(x, y) \in R[[x, y]]$ such that $F(x, 0) = F(0, x) = x$, $F(x, y) = F(y, x)$, and $F(x, F(y, z)) = F(F(x, y), z)$. Write $x +_F y = F(x, y)$. Then $[2](x) := x +_F x$ is another power series in one variable and inductively define $[n](x) = x +_F [n-1](x)$. There is also a power series $i(x)$ such that $x +_F i(x) = 0$; then $[-1](x) = i(x)$.

Let F, G be formal group laws over R. A morphism $f : F \to G$ is a power series $f(x) \in R[[x]]$ such that $f(F(x, y)) = G(f(x), f(y))$. Then $f(0) = 0$. Say that f is an *isomorphism* if f is invertible, or equivalently if $f'(0)$ is a unit. Say that f is a *strict isomorphism* if $f'(0) = 1$.

Example 18.3. $[n](x)$ is an endomorphism of F. For example, $[3](x) = x +_F x +_F x$. So there is a map $\mathbb{Z} \to \mathrm{End}(F)$ sending $n \mapsto [n](x)$. This is a ring map. Sometimes we might want to extend this to maps out of the ring over which this is defined.

Let E be a complex oriented commutative ring spectrum. This gives a formal group law F over π_*E that classifies the tensor product of line bundles. Write $F(x, y) = x+y+\sum_{i,j} a_{ij}x^iy^j$. Give this a grading where $|x| = |y| = -2$ and $|a_{ij}| = 2(i + j - 1)$. (Note that, by the unit axiom, any formal group law $F(x, y)$ is $\equiv x + y \pmod{(x, y)^2}$.)

Theorem 18.4 (Lazard-Quillen)**.** *MU_* admits the universal formal group law. As maps of graded rings, there is a 1-1 correspondence*

$$\{MU_* \to R_*\} \longleftrightarrow \{\textit{homogeneous f.g.l. } F \textit{ over } R\}$$

Every map $MU_*MU = \pi_*(MU \wedge MU) \to R_*$ gives the data of two formal group laws over R, one from inclusion of the left MU factor and the other from the right; these are isomorphic. There is a 1-1 correspondence

$$\{MU_*MU \to R_*\} \longleftrightarrow \left\{ \begin{tikzcd} F \arrow[rr, "\cong"] \arrow[dr] & & G \\ & R_* \arrow[ur] & \end{tikzcd} \right\}$$

(MU_*, MU_*MU) is a Hopf algebroid.

Let M_{FG} be the category whose objects are pairs (R, F) where F is a formal group law over R; the morphisms $(f, \psi) : (R_1, F_1) \to (R_2, F_2)$ consist of a map $f : R_1 \to R_2$ and an isomorphism

$$\begin{tikzcd} F_2 \arrow[rr, "\cong", "\psi"'] \arrow[dr] & & f^*F_1 \arrow[dl] \\ & R_2 & \end{tikzcd}$$

There is a similar category M_{FG}^h of homogeneous formal group laws.

18.1. Group actions on formal group laws.

Definition 18.5. Say that G acts on a formal group law (R, F) if there is a monoid morphism $G \to M_{FG}((R, F) \to (R, F))$. Similarly, a *strict action* is an action over (R_*, F^h) where R_* denotes a graded ring, and F^h denotes a homogeneous formal group law.

Example 18.6. Any formal group law (R_*, F) has a C_2-action by conjugation. This replaces x and y by $[-1](x)$ and $[-1](y)$.

Example 18.7. Let E be a complex oriented commutative ring spectrum. This gives rise to a formal group law, and if E is G-equivariant, the formal group law inherits a G-action in this sense.

Fact 18.8. *$MU_{\mathbb{R}}$ admits the universal C_2-conjugation formal group law.*

Proposition 18.9. *$MU_{\mathbb{R}}^{((G))}$ admits the universal G-equivariant formal group law that extends the C_2-conjugation action.*

Now I'm going to justify the map from the ANSS to the HFPSS. Conceptually, if G acts on a formal group law (R, F), I can think of it as a one-object category including into M_{FG}. The Hopf algebroid (MU_*, MU_*MU) co-represents the category M_{FG}; I will construct another Hopf algebroid that fits in the diagram:

$$\begin{tikzcd} (R_*, F^h) \arrow[r, hook] \arrow[d, dotted] & M_{FG} \arrow[d, dotted] \\ (R_*C(G, R_*)) & M_{FG} \arrow[l] \end{tikzcd}$$

$C(G, R_*)$ is the ring of functions from G to R_*. This map of Hopf algebroids gives rise to a map on cohomology

$$\mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*) \to H^*(G, R_*).$$

Example 18.10. If I have a G-equivariant complex oriented commutative ring spectrum E, then we said there was a G-action on the associated FGL (π_*E, F). The aforementioned map corresponds to the map from the ANSS to the HFPSS.

18.2. Lubin-Tate formal A-modules. For any formal group law (R, F) there is a map $\mathbb{Z} \to \mathrm{End}(F)$ taking $n \mapsto [n](x)$. Sometimes we might want to extend this to a map out of R; in our case this helps give things a C_8 action.

Theorem 18.11 (Lubin-Tate)**.** *If R is a DVR with maximal ideal π, and given $f(x)$ that satisfies $f(x) \equiv \pi x \pmod{x^2}$, $f(x) \equiv x^2 \pmod{\pi}$ (e.g. $f(x) = \pi x + x^2$) then there is a formal group law F_f and a map $R \to \mathrm{End}(F_f)$ sending $\pi \mapsto [\pi](x) = f(x)$.*

In our case, $A = \mathbb{Z}_2[\zeta]$, $A_* = A[u]$ for $|u| = 2$, then the Lubin-Tate theorem gives a formal group law F_f and a map $A_* \to \mathrm{End}(F_f)$ sending $2 \mapsto [2](x)$, $\pi \mapsto [\pi](x) = f(x)$, and $\zeta \mapsto [\zeta](x)$. The whole point is that ζ is an 8^{th} root of unity, and $[\zeta](x)$ is an automorphism of F_f. So C_8 acts on F_f as a formal group law over A_*.

The universal property guarantees a map $\pi_* MU^{((C_8))} \to A_*$. We know that $\Omega_{\mathbb{O}} = D^{-1}MU^{((C_8))}$. To have a map $\pi_*^u \Omega_{\mathbb{O}} \to A_*$, we need to check that D maps to a unit in A_*.

There are two valuations that agree with each other. First, we know that π generates the maximal ideal so $v(1-\zeta) = 1$, hence $v(2) = 4$. Second, we have a map $A \to \mathrm{End}(F_f)$ sending $a \mapsto [a](x)$, and $[a](x) \equiv ?x^d + \dots \pmod{(\pi)}$. The second valuation is $v(a) = \log_2 d$ and $v(1-\zeta) = v(\pi) = 1$. These are the same(?).

Let's do some computations. We know that $v(1+\zeta) = 1$ so $\frac{1+\zeta}{1-\zeta}$ and $\frac{1-\zeta}{1+\zeta}$ are units. So $v(1-\zeta^2) = 2$, $v(1+\zeta^2) = 2$, and $v(1-\zeta^4) = 4$.

$[1-\zeta](x) \equiv x^2 + \dots \pmod{(\pi)}$. So

$$\begin{aligned} [\zeta](x) &= x +_{F_f} [\zeta - 1] \\ &\equiv x +_{F_f} (x^2 + \dots) \pmod{(\pi)} \\ &\equiv x + x^2 + \dots \pmod{(\pi)} \end{aligned}$$

Similarly,

$$\begin{aligned} [\zeta^2](x) &\equiv x + x^4 + \dots \pmod{(\pi)} \\ [\zeta^4](x) &\equiv x + x^{16} + \dots \pmod{(\pi)} \\ [2](x) &\equiv x^{2^4} + \dots \pmod{(\pi)} \end{aligned}$$

This shows that this has height 4. Recall $\pi = 1 - \zeta$ and $2 = \pi^4 \cdot \text{unit}$.

Talk 19: Further directions (Doug Ravenel)

19.1. Summary. The main theorem says that θ_j does not exist for $j \geq 7$. The program is to construct a ring spectrum $S^{-0} \to \Omega$ satisfying three properties:

(1) Detection theorem: if θ_j exists, we see it in $\pi_* \Omega$

(2) Periodicity theorem: $\Sigma^{256}\Omega \simeq \Omega$

(3) Gap theorem: $\pi_{-2}\Omega = 0$

This implies that $\pi_{254}\Omega = 0$ and that's where the image of θ_7 lives, a contradiction. Constructing Ω was very difficult. To do this right, we had to learn about model categories and equivariant homotopy theory, and construct the norm and the slice spectral sequence.

19.2. θ_6**.** The case of θ_6 is open. One approach is that you might be able to squeeze a little bit more out of the periodicity theorem, going from 256 to 128; then the same argument would show that θ_6 does not exist. Failing this, if you could study Ω or something like it and look at $\pi_{126}\Omega$, then you could try to reach similar conclusions. These methods could only lead to a non-existence result.

At the other end of the problem, if you're really gutsy you could look at the Adams spectral sequence in the 126-stem and prove that the thing actually exists.

Renee: Atiyah thinks it exists, but this is just a heuristic. The Hopf invariant manifolds have something to do with division algebras; the reason there are only three of them is that they correspond to division algebras. The idea is that they have something to do with special Lie algebras. You can write down Freudenthal's magic square:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \times A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

It takes two division algebras and outputs a Lie algebra.

The idea is that Kervaire manifolds tell you about going along the first row and down the last column.

j	0	1	2	3	4	5	6
2^{j+1}	$\mathbb{R}P^2$	$\mathbb{C}P^2$	$\mathbb{H}\mathrm{P}^2$	$\mathbb{O}\mathrm{P}^2$			

There are three special manifolds in the remaining dimensions, of which the symmetry groups are exactly E_6, E_7, and E_8, respectively. I know that F_4 is the symmetry group of $\mathbb{O}\mathrm{P}^2$. The manifolds are called the Rosenfeld projective planes, or the bi-octonian plane, the quad-octonian plane, and the octo-octonian planes, and are denoted $(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2$, $(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$, and $(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$. The manifolds in the table aren't actually Kervaire manifolds; they're in dimensions 2 higher than the Kervaire manifolds. The heuristic is actually due to Bökstedt.

The method got a manifold for θ_4. Last Mike heard, θ_5 hasn't been constructed yet.

19.3. Odd-primary case.

Theorem 19.1 (Ravenel, 1978)**.** *For $p \geq 5$, θ_j does not exist for $j \geq 2$.*

What *is* θ_j? It corresponds to an element that lives in $\mathrm{Ext}_A^{2,2p^j(p-1)}(\mathbb{F}_p, \mathbb{F}_p)$; it's sometimes denoted by b_j. Now b_1 is also known as β_1; it's the first one. Toda proved that the next one, θ_2, also known as $\beta_{\varphi/p}$ supports a differential that hits $\alpha_1\theta_1^p$. Then the argument in Ravenel's paper leverages this, but for various technical reasons it doesn't work at the prime 3.

At $p = 3$, θ_1 exists in the 10-stem, θ_2 does not exist (Toda's theorem), but θ_3 does exist in the 106-stem "by something of an accident". Ravenel thought about this a lot, thought about using all the machinery discussed in this workshop, but does not know how to construct $MU_{\mathbb{R}}$. Look at Ravenel's slides from Banff four years ago discussing this.

If you did have $MU_{\mathbb{R}}$, the $\Omega_{\mathbb{O}}$ you're looking for would be $D^{-1}N_{C_3}^{C_9}MU_{\mathbb{R}}$; if everything worked, you'd have a 972-dimensional periodicity theorem. The dimension of θ_5 is 970. (Even in fantasy land, you're planning on leaving out θ_4.) θ_4 would still be unsettled. Instead of working at height 4, you would be working at height 6; instead of C_8, it would be C_9.

19.4. Differential on h_j^2. θ_j must support a nontrivial differential in the ASS and ANSS. We have no idea what differential it is and what it hits.

19.5. EHP sequence. In the 60's, Mahowald assumed for simplicity that the θ_j's all exist; then he had a good idea about how they fit into the EHP spectral sequence.

The following is all in the green book, §1.5.

It was proved in the 50's by James that there are 2-local fibrations $S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$. This leads to a LES

$$\cdots \to \pi_{n+k}S^n \xrightarrow{E} \pi_{n+k+1}S^{n+1} \xrightarrow{H} \mathfrak{p}\pi_{n+k+1}S^{2n+1} \xrightarrow{P} \pi_{n+k-1}S^n \to \dots$$

For historical charm:

- E stands for Einhängng (the German word for suspension)
- H stands for Hopf
- P stands for Whitehead product

There's a way to combine all these LES's to get a spectral sequence.

If $k = n$, $\pi_{n+k+1}S^{2n+1} = \mathbb{Z}$. Call the generator x_k. It might pull back to $\pi_{n+k+1}S^{n+1}$, or it might not. If $k < n$ then the group is zero, so the previous map is onto; this is part of the Freudenthal suspension theorem. If you have something in the k-stem, you can ask what the smallest value of n is so that it pulls back to $\pi_{n+k}S^n$.

For $k = n$, we have $\pi_{2n+1}S^{2n+1} = \mathbb{Z}$ generated by x_k. Let $\varphi(j) = \begin{cases} 2j & j \equiv 1, 2 \pmod 4 \\ 2j+1 & j \equiv 0 \\ 2j+2 & j \equiv 3. \end{cases}$

j	$\varphi(j-1)$
0	0
1	1
2	3
3	7
4	8
5	9
6	11
7	15
8	16

These are the dimensions where the orthogonal group has nontrivial homotopy.

Let $\overline{\alpha}_j$ denote the generator of the image of J in dimension $\varphi(j)-1$. If you have an element in a stable stem, you pull back as far as you can; the Hopf invariant (related to the H map) is an obstruction to pulling it back further.

$$\begin{aligned}
H(\theta_1) &= \overline{\alpha}_1 = \eta \in \pi_1 \\
H(\theta_2) &= \overline{\alpha}_2 = \nu \in \pi_3 \\
H(\theta_3) &= \overline{\alpha}_3 = \sigma \in \pi_7 \\
H(\theta_4) &= \overline{\alpha}_4 \in \pi_8 \\
H(\theta_5) &= \overline{\alpha}_5 \in \pi_9 \\
H(\theta_6) &= \overline{\alpha}_6 \in \pi_{11} \\
H(\theta_7) &= \overline{\alpha}_7 \in \pi_{15}
\end{aligned}$$

$$\begin{aligned}
\theta_4 &= x_{22}\overline{\alpha}_4 \\
\theta_5 &= x_{53}\overline{\alpha}_5 \\
\theta_6 &= x_{115}\overline{\alpha}_6 \\
\theta_7 &= x_{241}\overline{\alpha}_7
\end{aligned}$$

Mahowald has some theorems about the $\overline{\alpha}_j$'s etc. But we know that this doesn't happen. So what happens? Some of the $\overline{\alpha}_j$'s job was to correspond to θ_j. See Mahowald's EHP sequence paper. Some of this is even in his metastable homotopy theory paper ("Chairman Mahowald's little red book"). There are a lot of partial results.

The statements about θ_4 and θ_5 are true.

19.6. Concluding thoughts. I'm going to tell you a story about a dream I had [some time in the 70's] when I was attending a conference in Aarhus, Denmark in August. The latitude is very high, so if you go there in the summer, the sun comes up very early, which is a problem if you're trying to sleep. There were two people in this dream; one of them was

Haynes Miller, who at the time was a young mathematician like me. The other was Michael Barrett, who was British, and at the time he was totally obsessed with the Kervaire invariant problem. He tried over and over to construct the θ_j's. He is the person who invented the term "Doomsday hypothesis" for the possibility that only a finite number of θ_j's existed.

In this dream, the Kervaire invariant was a painting in an art gallery in Berkeley. It was abstract, yellow and black. Michael Barratt persuaded Haynes and I to break into the museum one night and steal the painting. Following Michael's instructions we stole several other paintings as a decoy so they wouldn't know which one we were really after. But the police weren't fooled. They didn't know it had anything to do with the Kervaire invariant. They hired a psychic to come up with a description of a person who would want to steal it. She really got a good description of Michael Barratt's personality. So the police tracked us down. The last thing I remember is that the police had closed in on us and I was trying to hide under some floorboards; there was a knothole in the floorboards, and the policeman was shining his flashlight into my eyes, and that was the sun. $\square$