SYLLABUS FOR 2016 TALBOT WORKSHOP EQUIVARIANT STABLE HOMOTPY THEORY AND THE KERVAIRE INVARIANT

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PREREQUISITES

Prerequisites should include elementary category theory (everybody should know what natural transformations, limits and colimits are), chromatic homotopy theory (including formal group laws, the Morava stabilizer group and the statement of the Hopkins-Miller theorem) and model categories.

The standard reference for category theory is [ML98, Chapters I–VII].

For chromatic homotopy theory, see [Rav92, Chapters 1–4]. The Hopkins-Miller theorem is treated in [Rez98, §1-4].

The introduction to model categories can be found in [DS95] is a good start. More thorough treatments are given by [Hov99] and [Hir03], but it is not necessary to know all of the material they cover.

The talks outlined below are intended to cover a lot of ground. In most cases there will not be time for detailed proofs. The speakers should aim at giving enough definitions and background to make clear statements of the appropriate theorems.

The latter talks in the workshop will use tools developed in the earlier ones. Some but perhaps not all of these connections are indicated in the following syllabus. **We recommend that the various speakers be in close contact with each before the workshop in order to be sure that the necessary tools have been developed.**

Online copies of nearly all of the references cited below can be found on Doug's archive at

   http://www.math.rochester.edu/people/faculty/doug/papers.html

One exception is the unfinished manuscript [HHRa], which will be emailed to the participants of this workshop.

TALKS

1. **Historical introduction**, to be given by Hill or Ravenel. The Kervaire invariant problem originated with the work of Kervaire-Milnor [KM63] (and before that [Ker60]) on the classification of differentiable structures on spheres. Their work left an ambiguous factor of two in each dimension congruent to 1 mod 4. A theorem of Browder [Bro69] a few years later related this problem to a question about the Adams spectral sequence. There one has an element known as $\theta_j = h_j^2$ in dimension $2^{j+1} - 2$. The broad impact of Milnor's discovery in [Mil56] of exotic differentiable structures on spheres led to great interest in the question posed by Browder's theorem.

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In the 1970s there were numerous unsuccessful attempts to solve the problem by constructing framed manifilds rperesenting θ_j **for all** $j > 0$. Such manifolds were known to exist for $1 < j < 4$ and later for $j = 5$.

2. **Overview of the proof of the Kervaire invariant theorem**, to be given by Hill or Ravenel. In 2009 we proved in [HHRb] that θ_j does not exist for $j \geq 7$. The status of θ_6 remains open. Our made substantial use of both chromatic and equivariant stable homotopy theory. We constructed a ring spectrum Ω with the following three properties.
   (i) **Detection Theorem.** It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. **This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.**
   (ii) **Periodicity Theorem.** It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
   (iii) **Gap Theorem.** $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.
   Very briefly, the spectrum Ω is the fixed point set of a C_8-equivariant spectrum related to the complex cobordism spectrum MU. The constructions and calculations involved in the proof require a thorough understanding of the equivariant stable homotopy category.

3. **The odd primary Arf invariant theorem.** In the Adams spectral sequence for an odd prime there are elements analogous to h_j^2. As of the mid 70s it was known that the first of these for $p \geq 5$ represents the homotopy element β_1 in the $(2p^2 - 2p - 2)$-stem, while the second one supports a nontrivial differential, which was shown by to Toda [Tod67]. Using the newly discovered apparatus of the chromatic spectral sequence ([MRW77] and [Rav86, Chapter 5]), Ravenel showed in [Rav78] (equivalently [Rav86, §6.4]) that none of the higher cases of these Adams spectral sequence elements represent homotopy elements. In modern language, one has the Morava stabilizer group S_{p-1} acting on the Morava spectrum E_{p-1}. The group is known to have a subgroup of order p, so one has a homotopy fixed point set $E_{p-1}^{hC_p}$. It has properties similar to those of the spectrum Ω used in [HHRb].

4. **Review of category theory** including adjoint functors, limits and colimits, ends and coends, and Kan extensions, enriched category theory, including symmetric monoidal categories and, enriched functors and the Day convolution. See [HHRa, §2] (which gives many additional references), [ML98] and [Rie14]. The speaker should consult with the person giving the definition of G-spectra to be sure all the necessary tools are introduced here.

5. and 6. **Introduction to equivariant homotopy theory** including the Bredon theorem, Mackey functors, G-CW complexes, representation spheres and orthogonal representation theory for cyclic 2-groups, $RO(G)$-graded homotopy groups and the isotropy separation sequence. This material is covered

in [GM95], [HHRb, §2-3], and [HHRa, §5], which gives many additional references. Also consult [HHRc, §2, §3, and §6].

7. and 8. **Model categories** including cofibrantly generated ones, Bousfield localization and maybe enriched model categories. Model category structures for the category of pointed G-spaces (Mandell-May [MM02, III.1]) and for classically defined spectra (Bousfield-Friedlander [BF78]). Explain why Bousfield localization (of either spaces or spectra) and the passage from "spectra" to "Ω-spectra" are both forms of fibrant replacement. State the Kan Recognition and Transfer Theorems of [HHRa, §3.8]. See [HHRa, §3] and the references therein. Consult with the person giving the lecture on the positive complete model structure.

9. **The Mandell-May** [MM02] **definition of G-spectra** including Yoneda spectra (aka S^{-V}), the tautological presentation and smash product via the Day convolution. This is explained in [HHRa, §6] and the references therein. It relies heavily on categorical concepts developed earlier in the book and hopefully covered in lecture 4. You should consult with the person doing that lecture. For an expository talk on this topic, see Doug's lecture at the Lehigh conference of May, 2015. There is a link to it at at `http://www.math.rochester.edu/people/faculty/doug/AKtalks.html`. Also see [HHRb, §A.2]. Consult with the person giving the lecture on the positive complete model structure.

The definition of spectra, meaning the foundation of stable homotopy theory, has changed several times since they were first introduced over fifty years ago. Remarkably, no new definition has invalidated any calculations based on previous definitions. This validates the adage "computation precedes theory." It seems that our intuition about spectra has been right all along, despite the shifting sands upon which the edifice rests.

The first definition with a strictly associative and commutative smash product was that of [EKMM97]. Prior to its appearance in the mid '90s, there were many technical headaches associated with the smash product. The definition given in [MM02] a few years later, along with similar definitions discussed in [MMSS01], is far simpler. Amazingly, the wonderful properties of the smash product (in all cases except [EKMM97]) follow formally using categorical tools (the Day convolution) available since the early 70's. A very helpful discussion of this development can be found in [Sch07, pp. 2-3].

10. **The homotopy of G-spectra** including long exact sequences of homotopy groups for fiber and cofiber sequences. See [HHRa, §7], [HHRb, §B.1-3] and [MM02]. The results here can be stated before defining a model category structure on $\mathcal{S}^G$, the category of G-spectar and equivaraint maps defined in the provious talk. One only needs to know what a weak equivalance is, and this is dictated by our long standing intuition about stable homotopy theory. This leads to the notion of a homotopical category, first introduced by Kan et al in [DHKS04], the "blue beast." Be sure to include a discussion of h-cofibrations and flat maps. Consult with the person giving the lecture

on the positive complete model structure, for which these concepts are needed.

11. **The positive complete model category structure and why we need it.** See [HHRa, §8] and [HHRb, §B.4]. You will need material from the previous 4 lectures, so you should consult with the people giving them. This structure differs from the positive stable model category structure given by [MM02]. It plays more nicely with change of group functors and with wedges and smash products indexed by G-sets. See [HHRb, Remark 2.23].

One odd feature of ths model category structure is the fact that **the sphere spectrum**, which we denote in [HHRa] by S^{-0}, **is not cofibrant**. This has to do with the bad behavior of the nth symmetric power functor Sym^n, which fails to preserve weak equivalences. For example applying it to the weak equivalence $S^{-1} \wedge S^1 \to S^{-0}$ does not yield a weak equivalence. In order to get a model category stucture on the category of commutative G-equivariant ring spectra, we need a model structure for which Sym^n preserves weak equivalences between cofibrant objects.

12. **The norm functor, multiplicative properties and their relation to geometric fixed points.** These two constructions are the computational mainsprings of the proof of the Kervaire invariant theorem. See [HHRb, §2.5] for an introduction to the geometric fixed point functor. For a G-spectrum X, the geometic fixed point spectrum $\Phi^G(X)$ is defined to be the G-fixed point set of the fibrant replacement of the smash product of X with a certain G-space $E\mathcal{P}_+$. This functor is shown to preserve weak equivalances, smash products and filtered colimits, and we have a convenient description of it when X is a suspension spectrum, spelled out in [HHRb, Prop. 2.45]. There is a similar functor denoted in [HHRb] by Φ^G_M (monoidal geometric fixed points) defined in [MM02] that is more explicit in that its definition does not involve fibrant replacement. The relation between Φ^G and Φ^G_M is studied in [HHRb, B.10].

Meanwhile for a subgroup $H \subseteq G$ we have the norm functor N^G_H from H-spectra to G-spectra. It is introduced in [HHRb, §2.2.3]. Roughly speaking, for an H-spectrum X, the G-spectrum $N^G_H X$ is the smash power $X^{\wedge |G/H|}$ where the action of G permutes the smash coordinates with each factor being invariant under H. The functor N^G_H converts commutative H-ring spectra to commutative G-ring spectra and is the left adjoint of the forgetful functor from the latter to the former.

At this point a serious technical issue arises. The spectra underlying commutative rings are almost never cofibrant since the sphere spectrum is not cofibrant. This means that there is no guarantee that the norm of a commutative ring has the correct homotopy type. The fact that it does is one of the main results of [HHRb, Appendix B]. (This paragraph appears nearly verbatim just before [HHRb, Prop. 2.30].)

Finally one has the fact that the functors $\Phi^G N^G_H$ and Φ^H are weakly equivalent. This is discussed in [HHRb, §2.5.4].

13. **Slice filtration and slice spectral sequence.** Classically one can attach cells to a space or spectrum X to kill off all of its homotopy groups above dimension n. We denote the resulting space or spectrum by $P^n X$, the nth Postnikov section of X. The fiber of the map $X \to P^n X$ is $P_{n-1} X$, the n-connected cover of X. The map $X \to P^n X$ factors through $P^{n+1} X$, and the fiber $P_n^n X$ of the map $P^{n+1} X \to P^n X$ is an Eilenberg-Mac Lane space or spectrum capturing $\pi_n X$. This construction can be done functorially and described in terms of Dror localization as in [Far96]; see also [HHRa, §10]. For each n the subcategory of $(n-1)$-connected spaces or spectra is the smallest subcategory containing S^n that is closed under certain operations. Formally this collection of subcategories determines all the functors in sight.

   The slice filtration is a similar construction in the category $\mathcal{S}^G$ of G-spectra. The point of departure is a collection of subcategories generalizing those of $(n-1)$-connected spectra for all n. The nth subcategory is the smallest one that is closed under similar operations and contains a certain collection of finite G-CW spectra, each of which is underlain by a finite wedge of spheres of dimension $\geq n$. This leads to functors P^n, P_n and P_n^n as before.

   In favorable cases $P_n^n X$, the nth slice of X, is a G-spectrum related to the integer Eilenberg-Mac Lane spectrum $H\underline{\mathbf{Z}}$, but its equivariant homotopy groups are **not** concentrated in a single dimension. This means we get an interesting spectral sequence associated with the filtration of X by its "connective" covers. This is the **slice spectral sequence**, the main computational tool of [HHRb].

   The first reference is [HHRb, §4]. Also consult [Hil].

14. **Dugger's computation for real K-theory.** The slice spectral sequence for Atiyah's real K-theory spectrum $K_{\mathbf{R}}$ was worked out by Dugger in is thesis [Dug05]. The group here is C_2, which acts on the complex K-theory spectrum by conjugation. It is described in more modern language as a spectral sequence of Mackey functors in [HHRc, §8]. Also see Doug's talk at the Northwestern University conference of March, 2013, for which there is a link at
   `http://www.math.rochester.edu/people/faculty/doug/AKtalks.html`

15. **The construction of $MU_{\mathbf{R}}$ and its slice differentials.** This C_2-spectrum is the star of our show. It is obtained from the complex cobordism spectrum MU by defining a group action in terms of complex conjugation. See [HHRb, §5 and B.12]. It is a commutative ring in the category $\mathcal{S}^{C_2}$. Applying the norm functor $N_{C_2}^{C_{2^n}}$ (which we abbreviate by $N_2^{2^n}$) gives a commutative ring in the category $\mathcal{S}^{C_{2^n}}$. One can show that $\Phi^{C_2} MU_{\mathbf{R}} = MO$, the unoriented cobordism spectrum. By comparing the long known homotopy groups of MO with those of MU, one finds some nontrivial differentials in the slice spectral sequence for $N_2^{2^n} MU_{\mathbf{R}}$. These are spelled out in [HHRb, Thm. 9.9]. They determine the slice spectral sequence completely for the case $n = 1$. For $n > 1$ (in particular for $n = 3$, the case relevant to the spectrum Ω) they give enough information for the computations we need.

16. **The Slice, Reduction and Gap Theorems.** The slice theorem enables us to identify the slices for spectra of interest, which are related to $MU_{\mathbf{R}}$.

Formally his problem can be reduced to identifying the 0-slice, which is the subject of the reduction theorem. Once this has been done, the gap theorem is an easy corollarly See [HHRb, §6-8]. For an expository talk on the gap theorem, see Doug's lecture at the Princeton Conference of March, 2015. There is a link to it at at
`http://www.math.rochester.edu/people/faculty/doug/AKtalks.html`.

17. **The Periodicity Theorem.** The spectrum Ω is constrcuted as follows. One starts with the C_8-spectrum $N_2^8 MU_{\mathbf{R}}$ described above. For reasons which become apparent in the proof, one inverts a certain element $D \in \pi_{19\rho_8}$ (where ρ_8 denotes the regular representation of C_8) by forming a mapping telescope, which we denote by $\widetilde{\Omega}$. Then Ω itself is the fixed point spectrum $\widetilde{\Omega}^{C_8}$. The element D is chosen so that inverting it will convert a certain element into a permanent cycle in the slice spectral sequence even though it supports a nontrivial differential before inverting D. This permanent cycle enables us to construct the desired equivalence $\Sigma^{256}\Omega \to \Omega$. The construction can be adapted to $N_2^n MU_{\mathbf{R}}$ for any n. The dimension of the resulting periodicity is $2^{n+1+2^{n-1}}$. For $n = 1$, this value is 8, and it is related to classical Bott periodicity. See [HHRb, §9].

This lecture and the next one could be cut if we run out of time.

18. **The Detection Theorem.** This guarantees that if θ_j exists we will see it in $\pi_*\Omega$. It is the reason for choosing $n = 3$. This theorem fail for smaller n. It holds for larger n but would yield a weaker result about the nonexistence of θ_j. For example the peridocity dmnsion for $n = 4$ is 2^{13}, so we would get a theorem saying that θ_j does not exist for $j \geq 12$.

The proof involves some 2-adic number theory and a study of the classical Adams-Novikov 2-line. See [HHRb, §11].

19. **Future directions:** the 3-primary problem, the C_4 computation, the prospects for θ_6 and questions related to the slice filtration. To be given by Hill or Ravenel.

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