A historical introduction to the K variant problem

ESHT boot camp

April 4, 2016

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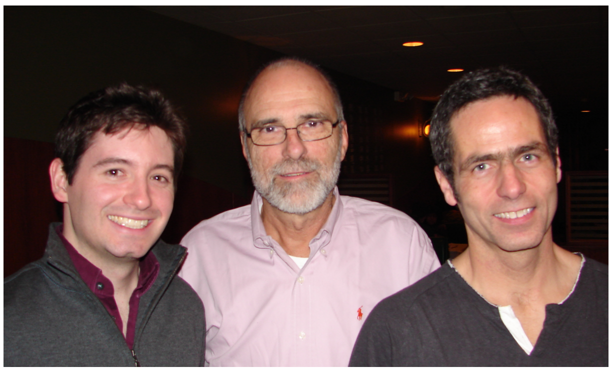

Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010

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1 Background and history

1.1 Classifying exotic spheres

The Kervaire-Milnor classification of exotic spheres

About 50 years ago three papers appeared that revolutionized algebraic and differential topology.

-  John Milnor's *On manifolds homeomorphic to the 7-sphere*, 1956. He constructed the first "exotic spheres", manifolds homeomorphic but not diffeomorphic to the standard S^7. They were certain S^3-bundles over S^4.

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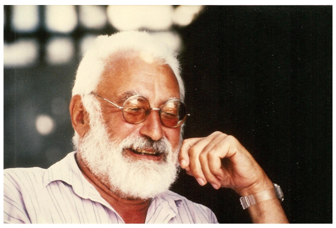
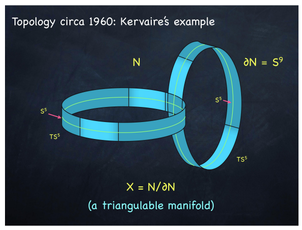
The Kervaire-Milnor classification of exotic spheres (continued)

-  

Michel Kervaire 1927-2007

Michel Kervaire's *A manifold which does not admit any differentiable structure*, 1960. His manifold was 10-dimensional. I will say more about it later.

1.5

The Kervaire-Milnor classification of exotic spheres (continued)

- Kervaire and Milnor's *Groups of homotopy spheres, I*, 1963.

For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

They gave a complete classification of exotic spheres in dimensions ≥ 5, with two caveats:

(i) Their answer was given in terms of the stable homotopy groups of spheres, which remain a mystery to this day.

(ii) There was an ambiguous factor of two in dimensions congruent to 1 mod 4. The solution to that problem is the subject of this talk.

1.6

1.2 Pontryagin's early work on homotopy groups of spheres

Pontryagin's early work on homotopy groups of spheres

Back to the 1930s

Lev Pontryagin 1908-1988

Pontryagin's approach to continuous maps $f : S^{n+k} \to S^k$ was

- Assume f is smooth. We know that any map f can be continuously deformed to a smooth one.
- Pick a regular value $y \in S^k$. Its inverse image will be a smooth n-manifold M in S^{n+k}.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.

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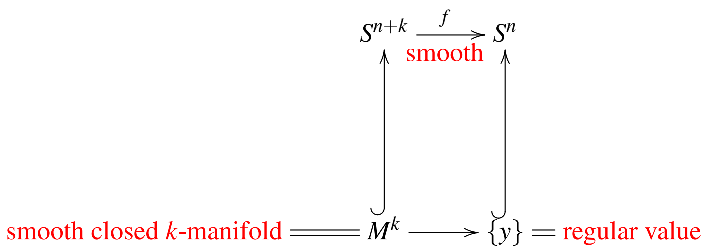
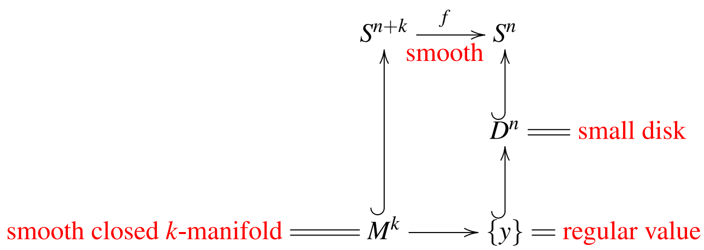
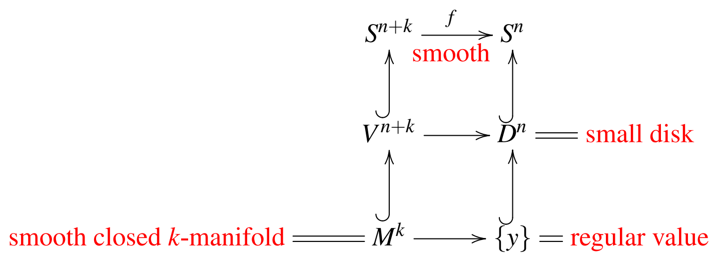
Pontryagin's early work (continued)

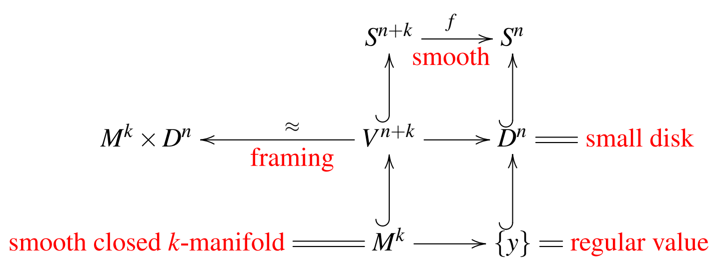
Let f be a smooth map with regular value y.

$$S^{n+k} \xrightarrow[\text{smooth}]{f} S^n$$









A sufficiently small disk D^n around y consists entirely of regular values, so its preimage V^{n+k} is an $(n+k)$-manifold homeomorphic to $M \times D^n$. A local coordinate system around around the point $y \in S^n$ pulls back to one around M called a framing.

There is a way to reverse this procedure. A framed manifold $M^k \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^n$. 1.8

Pontryagin's early work (continued)

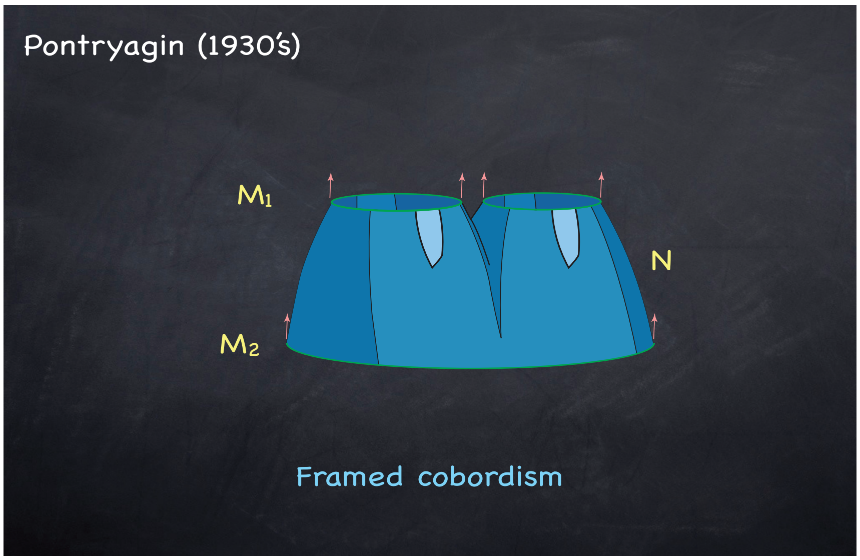
Two maps $f_1, f_2 : S^{n+k} \to S^k$ are homotopic if there is a continuous map $h : S^{n+k} \times [0,1] \to S^k$ (called a homotopy between f_1 and f_2) such that

$$h(x,0) = f_1(x) \qquad \text{and} \qquad h(x,1) = f_2(x).$$

If $y \in S^k$ is a regular value of h, then $h^{-1}(y)$ is a framed $(n+1)$-manifold $N \subset S^{n+k} \times [0,1]$ whose boundary is the disjoint union of $M_1 = f_1^{-1}(y)$ and $M_2 = f_2^{-1}(y)$. This N is called a framed cobordism between M_1 and M_2. When it exists the two closed manifolds are said to be framed cobordant. 1.9

Pontryagin's early work (continued)

Here is an example of a framed cobordism for $n = k = 1$.



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Pontryagin's early work (continued)

Let $\Omega_{n,k}^{fr}$ denote the cobordism group of framed n-manifolds in $\mathbf{R}^{n+k}$, or equivalently in S^{n+k}. Pontryagin's construction leads to a homomorphism

$$\Omega_{n,k}^{fr} \to \pi_{n+k} S^k.$$

Pontyagin's Theorem (1936). *The above homomorphism is an isomorphism in all cases.*

Both groups are known to be independent of k for $k > n$. We denote the resulting stable groups by simply Ω_n^{fr} and π_n^S.

The determination of the stable homotopy groups π_n^S is an ongoing problem in algebraic topology. Experience has shown that unfortunately its connection with framed cobordism is not very helpful for finding homotopy groups. It is not used in the proof of our theorem. 1.11

1.3 Exotic spheres as framed manifolds

Exotic spheres as framed manifolds

Into the 60s again

Following Kervaire-Milnor, let Θ_n denote the group of diffeomorphism classes of exotic n-spheres Σ^n. The group operation here is connected sum.

Each Σ^n admits a framed embedding into some Euclidean space $\mathbf{R}^{n+k}$, but the framing is not unique. Thus we do not have a homomorphism from Θ_n to π_n^S, but we do get a map to a certain quotient. 1.12

Exotic spheres as framed manifolds (continued)

Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on Σ^n. Varying the framing on the standard sphere S^n leads to a homomorphism

Heinz Hopf
1894-1971

$$\pi_n SO(k) \xrightarrow{J} \pi_{n+k} S^k$$

George Whitehead
1918-2004

called the Hopf-Whitehead J-homomorphism. It is well understood by homotopy theorists. 1.13

Exotic spheres as framed manifolds (continued)

Thus we get the Kervaire-Milnor homomorphism

$$\Theta_n \xrightarrow{p} \pi_n^S/\mathrm{Im}\,J.$$

The bulk of their paper is devoted to studying its kernel and cokernel using surgery. The two questions are closely related.

- The map p is onto iff every framed n-manifold is cobordant to a sphere, possibly an exotic one.
- It is one-to-one iff every exotic n-sphere that bounds a framed manifold also bounds an $(n+1)$-dimensional disk and is therefore diffeomorphic to the standard S^n.

They denote the kernel of p by bP_{n+1}, the group of exotic n-spheres bounding parallelizable $(n+1)$-manifolds. 1.14

Exotic spheres as framed manifolds (continued)

Hence we have an exact sequence

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \xrightarrow{p} \pi_n^S/\mathrm{Im}\,J.$$

Kervaire-Milnor Theorem (1963).

- *The homomorphism p above is onto except possibly when $n = 4m+2$ for $m \in \mathbf{Z}$, and then the cokernel has order at most 2.*
- *Its kernel bP_{n+1} is trivial when n is even.*
- *bP_{4m} is a certain cyclic group. Its order is related to the numerator of the mth Bernoulli number.*
- *The order of bP_{4m+2} is at most 2.*
- *bP_{4m+2} is trivial iff the cokernel of p in dimension $4m+2$ is nontrivial.*

We now know the value of bP_{4m+2} in every case except $m = 31$. 1.15

Exotic spheres as framed manifolds (continued)

In other words have a 4-term exact sequence

$$0 \longrightarrow \Theta_{4m+2} \xrightarrow{p} \pi^S_{4m+2}/\mathrm{Im}\,J \longrightarrow \mathbf{Z}/2 \longrightarrow bP_{4m+2} \longrightarrow 0$$

The early work of Pontryagin implies that $bP_2 = 0$ and $bP_6 = 0$.

In 1960 Kervaire showed that $bP_{10} = \mathbf{Z}/2$.

To say more about this we need to define the Kervaire invariant of a framed manifold. 1.16

2 The Arf-Kervaire invariant

The Arf invariant of a quadratic form in characteristic 2

Back to the 1940s

Cahit Arf 1910-1997

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i\colon 1 \le i \le n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

1.17

The Arf invariant of a quadratic form in characteristic 2 (continued)

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

1.18

The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of λ is a map $q: \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x,y)$$

Its Arf invariant is

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q. 1.19

Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997

1.20

The Kervaire invariant of a framed $(4m+2)$-manifold


Into the 60s
a third time

Let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$. Let $H = H_{2m+1}(M;\mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an embedding $i_x : S^{2m+1} \hookrightarrow M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

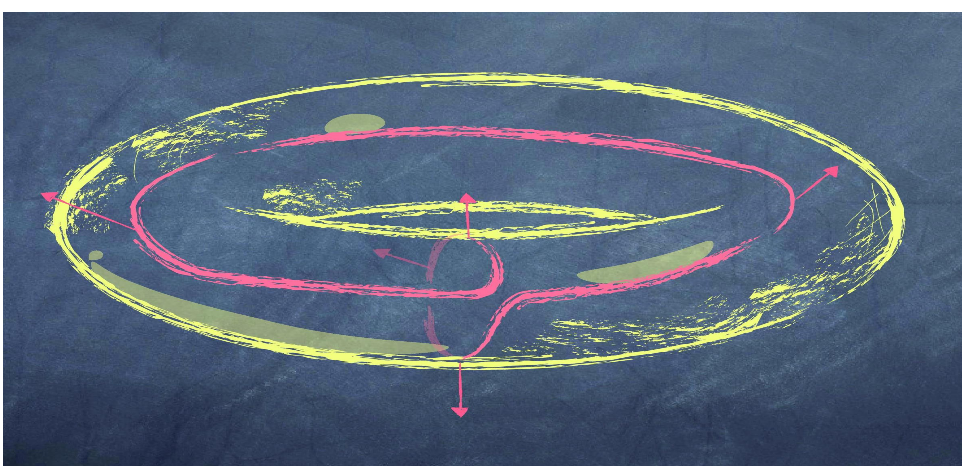
Here is a simple example. Let $M = T^2$, the torus, be embedded in S^3 with a framing. We define the quadratic refinement

$$q : H_1(T^2;\mathbf{Z}/2) \to \mathbf{Z}/2$$

as follows. An element $x \in H_1(T^2;\mathbf{Z}/2)$ can be represented by a closed curve, with a neighborhood V which is an embedded cylinder. We define $q(x)$ to be the number of its full twists modulo 2. 1.21

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

For $M = T^2 \subset S^3$ and $x \in H_1(T^2;\mathbf{Z}/2)$, $q(x)$ is the number of full twists in a cylinder V neighboring a curve representing x. This function is not additive!



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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

Again, let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$, and let $H = H_{2m+1}(M;\mathbf{Z})$. Each $x \in H$ is represented by an embedding $S^{2m+1} \hookrightarrow M$. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

Kervaire defined a quadratic refinement q on its mod 2 reduction $\overline{H}$ in terms of each sphere's normal bundle. The Kervaire invariant $\Phi(M)$ is defined to be the Arf invariant of q.

Recall the Kervaire-Milnor 4-term exact sequence

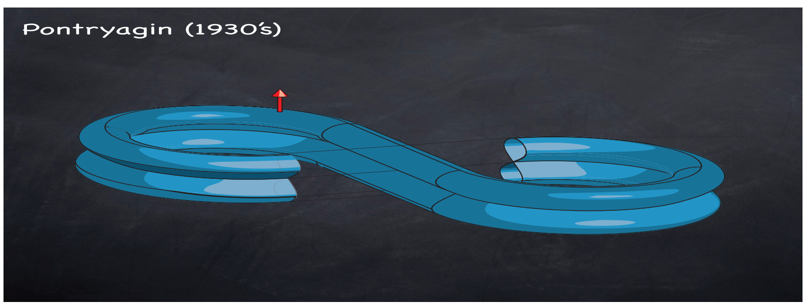
$$0 \longrightarrow \Theta_{4m+2} \xrightarrow{p} \pi^S_{4m+2}/\mathrm{Im}\,J \longrightarrow \mathbf{Z}/2 \longrightarrow bP_{4m+2} \longrightarrow 0$$

Kervaire-Milnor Theorem (1963). *$bP_{4m+2} = 0$ iff there is a smooth framed $(4m+2)$-manifold M with $\Phi(M)$ nontrivial.* 1.23

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

What can we say about $\Phi(M)$?

For $m = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant.

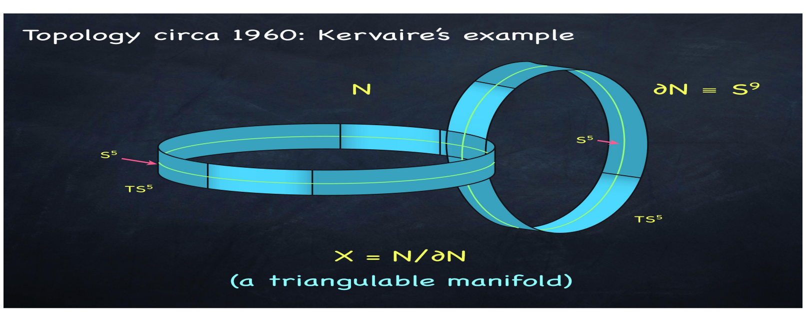


Pontryagin used it in 1950 (after some false starts in the 30s) to show $\pi_{k+2}(S^k) = \mathbf{Z}/2$ for all $k \geq 2$. There are similar framings of $S^3 \times S^3$ and $S^7 \times S^7$. This means that bP_2, bP_6 and bP_{14} are each trivial. 1.24

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

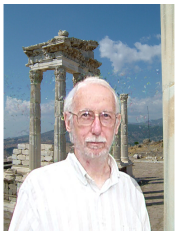
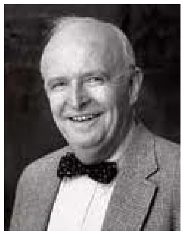
Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbf{Z}/2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.



This construction generalizes to higher m, but Kervaire's proof that the boundary is exotic does not. 1.25

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

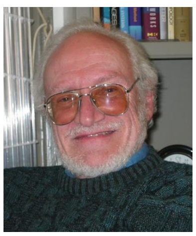
 

Ed Brown — Frank Peterson 1930-2000

Brown-Peterson (1966) showed that it vanishes for all positive even m. This means $bP_{8\ell+2} = \mathbf{Z}/2$ for $\ell > 0$. 1.26

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

- 

  Bill Browder

  Browder (1969) showed that the Kervaire invariant of a smooth framed $(4m+2)$-manifold can be nontrivial (and hence $bP_{4m+2} = 0$) only if $m = 2^{j-1} - 1$ for some $j > 0$. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence. The corresponding element in $\pi_{n+2^{j+1}-2}(S^n)$ for large n is θ_j, the subject of our theorem. This is the stable homotopy theoretic formulation of the problem.
- θ_j is known to exist for $1 \le j \le 5$, i.e., in dimensions 2, 6, 14, 30 and 62. In other words, bP_2, bP_6, bP_{14}, bP_{30} and bP_{62} are all trivial.

1.27

And then . . . the problem went viral!

A wildly popular dance craze

Drawing by Carolyn Snaith 1981
London, Ontario

1.28

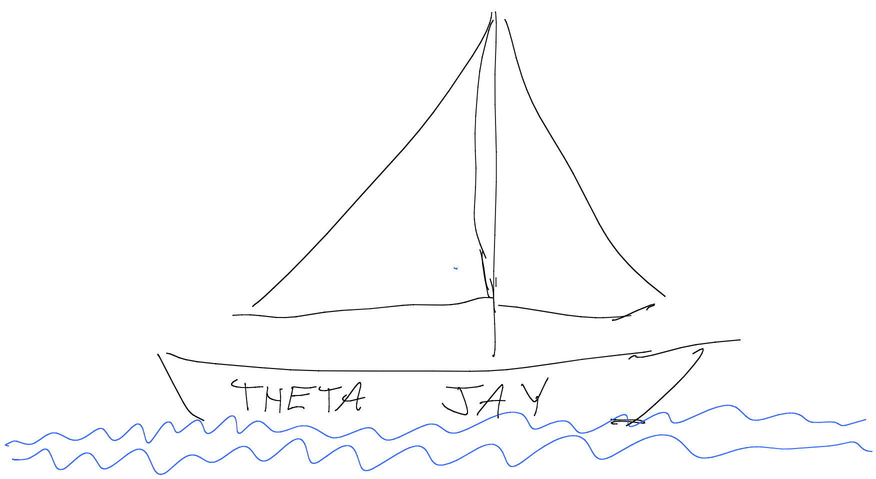
Speculations about θ_j after Browder's theorem

In the decade following Browder's theorem, many topologists tried without success to construct framed manifolds with nontrivial Kervaire invariant in all such dimensions, i.e., to show that $bP_{2^{j+1}-2} = 0$ for all $j > 0$.

Mark Mahowald

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. He derived numerous consequences about homotopy groups of spheres. 1.29 The possible nonexistence of the θ_j for large j was known as the Doomsday Hypothesis.

Mark Mahowald's sailboat



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3 The main theorem

Our main result

Our main theorem can be stated in three different but equivalent ways:

- Manifold formulation: It says that the Kervaire invariant $\Phi(M^{4m+2})$ of a smooth $2m$-connected framed $(4m+2)$-manifold must vanish (and $bP_{4m+2} = \mathbf{Z}/2$) for all but 5 or 6 values of m.
- Stable homotopy theoretic formulation: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.
- Unstable homotopy theoretic formulation: It says something about the EHP sequence, which has to do with unstable homotopy groups of spheres.

There were several unsuccessful attempts in the 1970s to prove the opposite of what we have proved, namely that $bP_{2^{j+1}-2} = 0$ for all $j > 0$. 1.31

Our main result

Here is the stable homotopy theoretic formulation.

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ for large n do not exist for $j \geq 7$.*

The θ_j in the theorem is the name given to a hypothetical map between spheres represented by a framed manifold with nontrivial Kervaire invariant. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are 2 less than a power of 2.

Corollary. *The Kervaire-Milnor group $bP_{2^{j+1}-2}$ is nontrivial for $j \geq 7$.*

It is known to be trivial for $1 \leq j \leq 5$. The case $j = 6$, i.e., bP_{126}, is still open. 1.32

Questions raised by our theorem

Adams spectral sequence formulation. We now know that the h_j^2 for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

Unstable homotopy theoretic formulation. In 1967 Mahowald published an elaborate conjecture about the role of the θ_j (assuming that they all exist) in the unstable homotopy groups of spheres. Since they do not exist, a substitute for his conjecture is needed. We have no idea what it should be.

Our method of proof offers a new tool, the slice spectral sequence, for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future. I will illustrate it at the end of the talk. 1.33

4 Our strategy

4.1 Ingredients of the proof

Ingredients of the proof

Our proof has several ingredients.

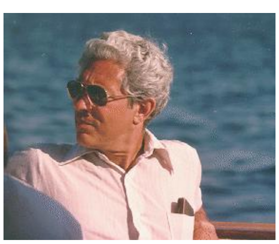
- We use methods of stable homotopy theory, which means we use spectra instead of topological spaces. The modern definition of spectra, due to Mandell-May, will be given in a talk later this week. It makes use of enriched category theory, which is also the subject of a later talk.

  For the sphere spectrum S^{-0} (new notation), $\pi_n(S^{-0})$ (previously denoted by π_n^S) is the usual homotopy group $\pi_{n+k}(S^k)$ for $k > n+1$. The hypothetical θ_j is an element of this group for $n = 2^{j+1} - 2$. 1.34

Ingredients of the proof (continued)

More ingredients of our proof:

- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbf{Z}$, but by $RO(G)$, the real representation ring of G. Our calculations make use of this richer structure.


Peter May


John Greenlees


Gaunce Lewis
1949-2006

1.35

Ingredients of the proof (continued)

More ingredients of our proof:

- We use complex cobordism theory. It is used in the proof in two ways:
  (i) We construct a C_2-equivariant spectrum $MU_{\mathbf{R}}$. Roughly speaking it is the complex Thom spectrum MU equipped with complex conjugation.
  (ii) We use related formal group law methods to prove the detection theorem, to be stated in the next slide. 1.36

4.2 The spectrum Ω

The spectrum Ω

We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence, which I will illustrate at the end of the talk. 1.37

The spectrum Ω (continued)

Here again are the properties of Ω

(i) Detection Theorem. If θ_j exists, it has nontrivial image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$.

(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 1.38

4.3 How we construct Ω

How we construct Ω

Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The geometric fixed point set of this action is the set of real points, known to topologists as MO, the unoriented cobordism spectrum. The geometric fixed point functor Φ^G is an important tool in equivariant stable homotopy theory. 1.39

How we construct Ω (continued)

To get a C_8-spectrum, we use the following general construction for getting from a space or spectrum X acted on by a group H to one acted on by a larger group G containing H as a subgroup. Let

$$Y = \mathrm{Map}_H(G, X),$$

the space (or spectrum) of H-equivariant maps from G to X. Here the action of H on G is by left multiplication, and the resulting object has an action of G by left multiplication. As a space, $Y = X^{|G/H|}$, the $|G/H|$-fold Cartesian power of X. A general element of G permutes these factors, each of which is invariant under the action of the subgroup H. 1.40

How we construct Ω (continued)

There is a similar construction in the category of spectra called the norm functor, denoted by N_H^G. It uses smash products rather than Cartesian products.

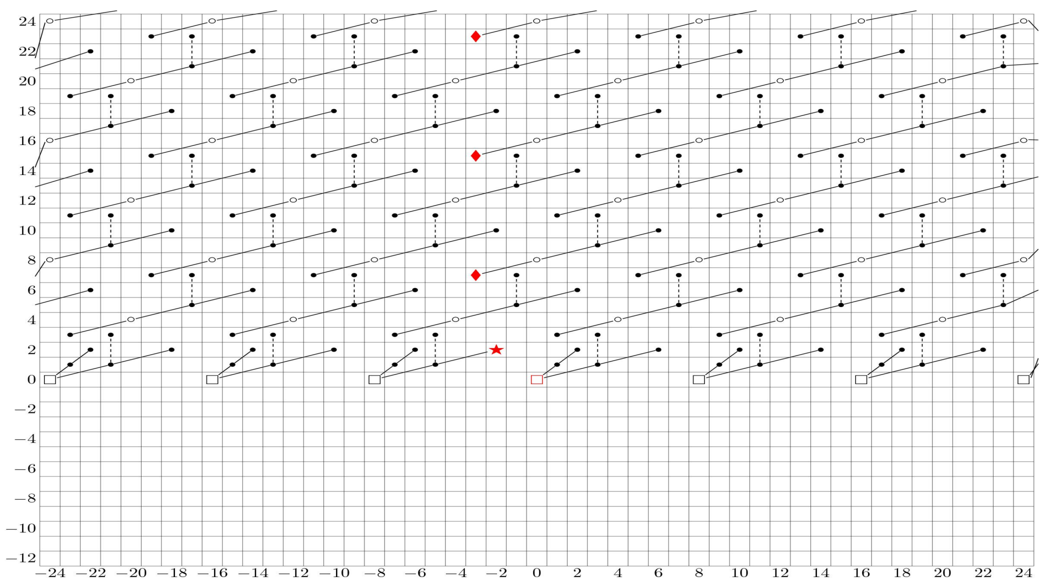
In particular we get a C_8 commutative ring spectrum

$$MU_{\mathbf{R}}^{((C_8))} = N_{C_2}^{C_8} MU_{\mathbf{R}}.$$

We can make it periodic by inverting a certain element $D \in \pi_{19\rho_8} MU_{\mathbf{R}}^{(4)}$. Our spectrum Ω is its C_8 fixed point set. 1.41

4.4 The slice spectral sequence

A homotopy fixed point spectral sequence



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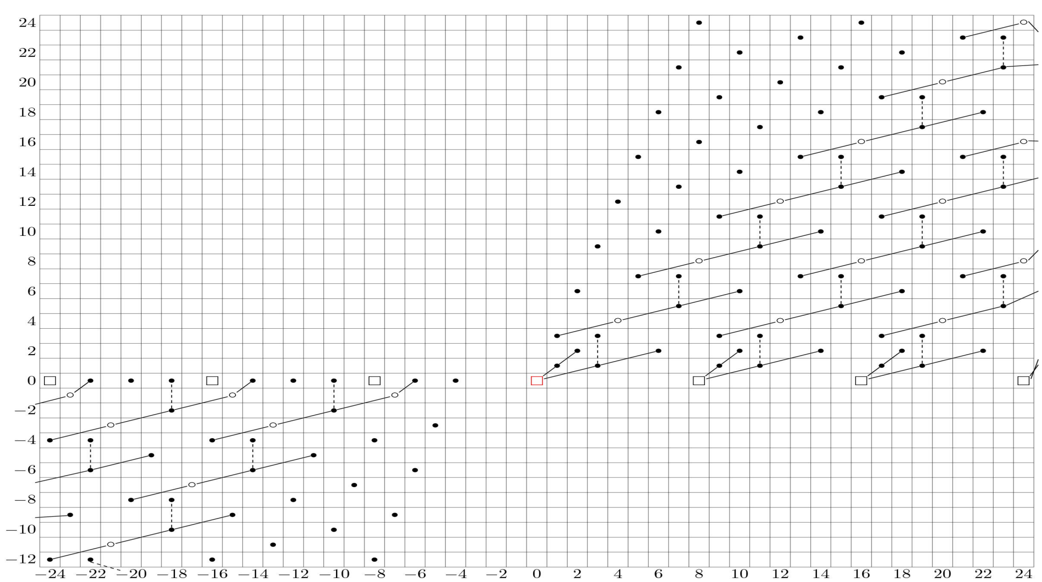
The corresponding slice spectral sequence



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