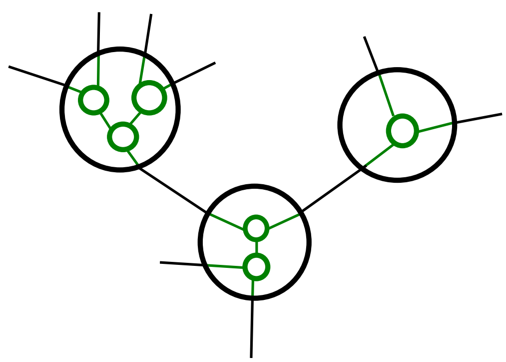
Talbot 2015: Applications of operads

(notes from talks given at the workshop)



Notes taken by Eva Belmont

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Disclaimer

These are notes I took during the 2015 Talbot Workshop. I, not the speakers, bear responsibility for mistakes. If you do find any errors, please report them to:

Eva Belmont
`ekbelmont at gmail.com`

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Finally, thanks to all the participants who submitted corrections and clarifications.

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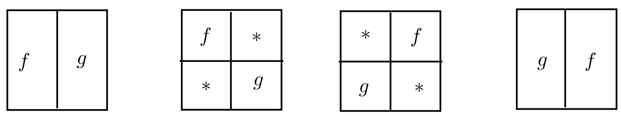
Day 1: Homotopy-theoretic foundations

Talk 1.1: Introduction and general overview (Dev Sinha)

Operads encode operations. The analogy to make is: "groups encode symmetries as operads encode operations." We'll find that there's a lot of data to deal with, but if you keep in mind that operads want to encode a multiplication of some kind, the structure becomes clear. The history of the idea goes back at least to 1898, with Alfred North Whitehead's work on Lie algebras and other algebras. But I want to talk about the history in topology: $\pi_2(X)$ is the group of maps from $I^2 \to X$ sending the boundary to the basepoint of X. This is commutative. Proof by picture:

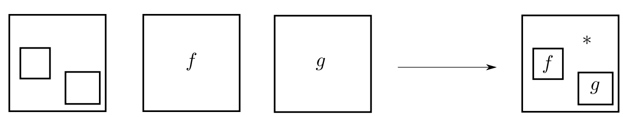


(People originally thought π_2 was uninteresting, because π_1 was noncommutative in general.)

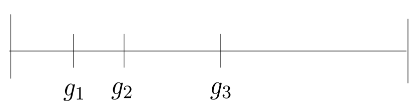
What we can say is that there is a map

$$\text{Rectangles}_2 \times \Omega^2 X \times \Omega^2 X \to \Omega^2 X$$

(here Rectangles_2 means embeddings of 2 rectangles in a rectangle, and Ω^2 is the space of based maps $\text{Maps}_*(S^2, X)$) sending



If G is a group, its classifying space BG is modelled by a line segment on which the elements g_i live and can move around, where if they collide, then you multiply them; if they go off the end, then they vanish; and if $g_i = e$ then it vanishes. An element looks like this:



Formally,

$$BG = \Big(\coprod_n (\Delta^n \times G^{\times n}) \Big) \Big/ \sim$$

The points are $(0 \le t_1 \le t_2 \le \cdots \le 1) \times (g_1, \dots, g_n)$, where the equivalence relation is given by:

$$(t_1 \le \dots . \le t_i = t_{i+1} \le t_{i+2} \le \dots)(g_1, \dots, g_n) \sim (t_1 \le \cdots \le t_i \le t_{i+2} \dots)(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots)$$

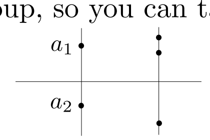
There are two other relations, corresponding to the idea above about points falling off the end or being the identity.

This has the following properties:

- BG classifies principal G-bundles (and thus any bundles are made by "gluing by G").
- If G is discrete, BG is a $K(G, 1)$, and so
- $\Omega BG \simeq G$.

This model of BG is functorial. But only for abelian groups is $G \times G \to G$ a homomorphism.

If A is abelian, then BA itself is a group, where the operation is "take the union" (and multiply where needed). This is a group, so you can talk about $B(BA)$. Elements look like



Equivalently, this is

$$B^{(2)}A = \Big(\coprod_n ((I^2)^{\times n})_{S_n} \Big) \Big/ \sim$$

where now we're thinking of points a_i living in a square. We could replace 2 above by any dimension d. In fact, we can take the space of points in X labelled by A for any X.

Proposition 1.1.1. *Suppose A is discrete, e.g. $\mathbb{Z}/n$ or $\mathbb{Z}$. $B^{(d)}A$ is a $K(A, d)$, the d^{th} Eilenberg-Maclane space for A. That is,*

$$\pi_i(B^{(d)}A) = \begin{cases} A & i = d \\ & i \neq d. \end{cases}$$

These are building blocks for Postnikov towers, in the same way that cells are building blocks for CW complexes. This also means that

$$\widetilde{H}^d(X; A) \cong [X, B^{(d)}A]$$

(note $[-, -]$ means basepoint-preserving maps); that is, it is a representing object for cohomology. (Think of cohomology as a "representation" in the representation-theory sense.)

This generalizes to the fact that homotopy groups of this construction are the homology of X (this is sort of the Dold-Thom theorem).

We could weaken the commutativity requirement if we had a way to define "multiplication along directions."

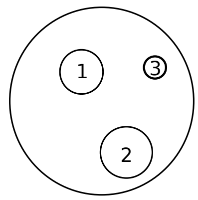
Incomplete definition 1.1.2. A group action is a continuous map $G \times X \to X$, such that some conditions hold. An operad action is a map $\mathcal{O}(n) \otimes (\underbrace{X \otimes \ldots \otimes X}_{n}) \to X$ subject to some conditions (remember than an example of $\otimes$ is product of topological spaces). So this gives you different ways to multiply n things, with some structure, and that structure includes the fact that this is coherent across n in a way that will be explained in a later talk.

Principle: "the smaller the operad, the more stringent the structure." If $\mathcal{O}(n)$ is just a point (if we're working in Top) or k (if we're working in Vect_k) for all n then this says that "there is only one way to multiply." This means that we're dealing with a commutative "algebra."

(Why commutative? The conditions on the operad action map include a Σ_n-equivariance condition, which tells you what happens when you permute the factors X.)

Why should we do this? Here are some main payoffs (which we actually won't talk about in this workshop):

(1) You can interrelate such structures: $\Omega^d X$ is a Disks^d-algebra; that is, there is an action of Disks^d on it. E.g. this



is an element of $\text{Disks}^2(3)$. This tells you that $H_*(\Omega^d X, k)$ is a $H_*(\text{Disks}^d)$-algebra (coefficients are in a field because you need a Künneth formula).

(2) You can make Bar constructions (like $B^{(d)}A$), which in turn allow one to:
   - recognize homotopy invariant algebraic structures. E.g. if Y has a Disks^d action and $\pi_0 Y$ is a group, then you can construct $B_{\text{Disks}^d} Y$ such that $\Omega^d B_{\text{Disks}^d} Y \simeq Y$. This is called a recognition principle.
   - characterize deformations/ (co)homology, in algebraic settings.
   - realize Koszul duality, which says e.g. that differential graded associative algebras are the same as differential graded co-associative coalgebras (Moore), and differential graded commutative algebras are the same as differential graded Lie coalgebras (Quillen). To do this in general, you need bar constructions, and to get bar constructions, you need operads.
     $C_*(G)$ with the G-product is Koszul-dual to $C_*(BG)$ with the natural diagonal map. Similarly, given Y, $C_*(\Omega Y)$ with loop multiplication is dual to $C_*(Y)$ with the natural diagonal.

(3) You can organize complicated algebraic structures, especially some arising in "physical mathematics." (E.g. some crazy rule for different ways n particles or n strings combine.) This is kind of why operads were "reborn" in the 90's.

We will treat some of the more surprising applications of operads. One can set up a homotopy theory for operads themselves. (There are people in the world who like to think about homotopy theories for anything.) This sounds somewhat formal, but it in fact encodes connections to very geometric and number-theoretic topics. Most of the initial ideas are due to Kontsevich.

Part of this idea: a (long) knot $I \hookrightarrow I^d$, i.e. knot in a box with fixed endpoints at the top and bottom of the box, gives rise to a map of configuration spaces $\text{Conf}_n(I) \to \text{Conf}_n(I^d)$. This goes back to Gauss. If my knot is K, and I have a configuration $(t_1, t_2, \dots) \in \text{Conf}_n(I)$, then I can come up with a configuration $(K(t_1), K(t_2), \dots) \in \text{Conf}_n(I^d)$. This extends to the Fulton-MacPherson compactification $\text{Conf}_n[I] \to \text{Conf}_n[I^d]$ which respects operad structures.

Theorem 1.1.3 (Arone-Turchin, after Goodwillie-Klein-Weiss)**.**

$$\text{Hom}_{\mathit{Inf.Bimod.}}(FM_1, FM_d) \simeq \overline{\text{Emb}}(I, I^d)$$

for $d > 4$, where FM_d is the Fulton-MacPherson operad.

When $d = 3$, the RHS is a space of knots.

Question: how computable is the LHS? You can get spectral sequences ("Sinha spectral sequence") for the LHS and they collapse; it has the same E_2 as Vassiliev's E_1, but it's not the same spectral sequence. It's combinatorially hard. You can get asymptotics for it.

There are theorems of Dwyer-Hess, Turchin, and Weiss that also give models for deloopings of embedding spaces.

$\mathrm{Hom}(FM_2, FM_2)$ has deep implications for number theory.

Talk 1.2: Homotopy theory of operads and their modules
(Elaine So)

We will write Σ_n for the symmetric group on n letters (also called S_n); set $\Sigma_0 = \Sigma_1$. Let $(V, \otimes, I)$ be a (closed) (symmetric) monoidal category, where I is the unit. For example, think about $(Set, \times, \{*\})$. A nice thing about closed symmetric monoidal category is that you have exponentiation, e.g. for sets $X^Y = \mathrm{Hom}(Y, X)$. In general, $V(X \otimes Y, Z) \cong V(X, Z^Y)$.

When we say "topological operad" we want to work with the categories $(\mathrm{Top}, \times, \{*\})$ or $(sSet, \times, \Delta[0])$, and exponentiation is $X^Y = \mathrm{Map}(Y, X)$.

We also care about algebraic symmetric monoidal categories $(R\text{-mod}, \otimes_R, R)$, $(Ch(R), \otimes[R]_0)$.

Definition 1.2.1. Let G be a (finite) group. Define V^G to be the category whose objects are objects of V with G-action, and the morphisms $V^G(X, Y)$ are G-equivariant maps $X \to Y$ in V.

Definition 1.2.2. Define $\mathbb{N}(V)$ to have objects $X = (X(0), X(1), \dots)$; these are nonsymmetric sequences. Symmetric sequences (a.k.a. collections) are

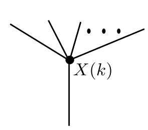
$$\Sigma(V) = \prod_{k \geq 0} V^{\Sigma_k}.$$

Objects: for every k, there's an object $X(k)$ with a Σ_k action for each object. Morphisms: morphisms that respect Σ_k.

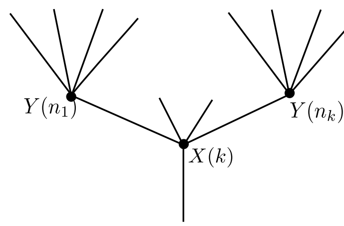
You can define a composition product on symmetric sequences. Given $X, Y \in \Sigma(V)$,

$$(X \circ Y)(n) = \bigsqcup_{k \geq 0} X(k) \otimes_{\Sigma_k} \left(\bigsqcup_{n_1 + \cdots + n_k = n} \bigotimes Y(n_k) \right) \otimes_{\Sigma_{n_1, \dots, n_k}} \bigsqcup_{\Sigma_n} I.$$

If $X(k)$ is a tree with k inputs



then $(X \circ Y)(n)$ is the disjoint union of trees of the form



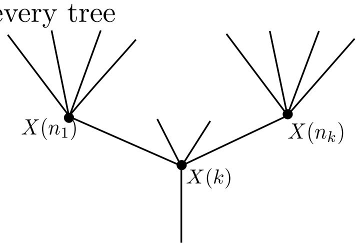
where the disjoint union is over all possible ways to plug Y-things into the k X-inputs so you have a total of n inputs.

(There is a functor from symmetric sequences to endomorphisms, where the "composition" above corresponds to literal composition of endomorphisms.)

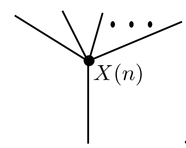
Operads are symmetric sequences where the composition product satisfies certain properties. You want commutativity, i.e. the commutativity of the diagram

$$\begin{array}{ccc} X \circ X & \longrightarrow & X \\ \uparrow & & \uparrow \\ X \circ X \circ X & \longrightarrow & X \circ X \end{array}$$

You want to map every tree



to



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If you have 3 layers, it shouldn't matter in which order you compose them. If I compose the identity in $X(1)$ with anything in $X(k)$, it shouldn't do anything.

name	composition objects	Σ_k-action	composition	objects
I (unit)	$I(1) = I$, empty elsewhere	trivial	trivial	objects
I_* (pointed unit)	$I(0) = I$, empty elsewhere	ditto	ditto	pointed objects
Commutative operad	$C(k) = I$		commutative algebras, commutative monoids	
Associative operad	$Ass(k) = \bigsqcup_{\Sigma_k} I$	Σ_k	Given $\sigma \in Ass(k), \sigma_i \in Ass(n_i)$, get block permutation $\sigma_{\sigma(1)} \otimes \ldots \otimes \sigma_{\sigma(n)}$	associative monoids

There's also a Lie operad, that's terrible to describe, and a Poisson operad.

Given $X \in V$, we have an endomorphism operad End_X, where $\mathrm{End}_X(k) = \mathrm{Hom}(X^{\otimes k}, X)$ which is still in our category because it is the exponential object $X^{X^{\otimes k}} = \mathrm{Hom}(X^{\otimes k}, X)$. Composition is just composition of functions.

Definition 1.2.3. Let $\mathcal{O}$ be an operad on V. An $\mathcal{O}$-algebra structure on $X \in \mathrm{Obj}(V)$ is an operad map $\mathcal{O} \to \mathrm{End}_X$.

Alternate definition 1.2.4. A non-Σ operad is where you forget all instances of Σ_k in the above.

A reduced operad is where $\mathcal{O}(0) = I$.

A non-unital operad is where $\mathcal{O}(0) = \emptyset_V$.

Operads on V act on symmetric collections:

Definition 1.2.5. A *left $\mathcal{O}$-module* is a symmetric sequence X with morphism $\varphi : \mathcal{O} \circ X \to X$. You want the following diagram to commute:

$$\begin{array}{ccccc} \mathcal{O} \circ \mathcal{O} \circ X & \xrightarrow{\mu \circ \mathbb{1}} & \mathcal{O} \circ X & \xleftarrow{\eta \circ \mathbb{1}} & I \circ X \\ \downarrow & & \downarrow \varphi & \swarrow & \\ \mathcal{O} \circ X & \xrightarrow{\varphi} & X & & \end{array}$$

To define a *right $\mathcal{O}$-module*, dualize the above structure. Note that left and right $\mathcal{O}$-modules have really different structures.

A bimodule is a symmetric sequence such that

$$\begin{array}{ccc} \mathcal{O} \circ X \circ \mathcal{O} & \longrightarrow & X \circ \mathcal{O} \\ \downarrow & & \downarrow \\ \mathcal{O} \circ X & \longrightarrow & X \end{array}$$

commutes.

Example 1.2.6. If $\mathcal{O} \xrightarrow{\varphi} \mathcal{O}$ is an operad map, then you get a module structure $\mathcal{O} \circ \mathcal{O} \xrightarrow{\varphi \circ \mathbb{1}} \mathcal{O} \circ \mathcal{O} \xrightarrow{\mu} \mathcal{O}$.

If X is an $\mathcal{O}$-algebra, then $(X, \emptyset, \emptyset, \dots)$ is a left $\mathcal{O}$-module. The only level of $\mathcal{O} \circ X$ is the zeroth level, and that is $\bigsqcup_{k \geq 0} \mathcal{O}(k) \otimes X^{\otimes k} \to X = X(0)$.

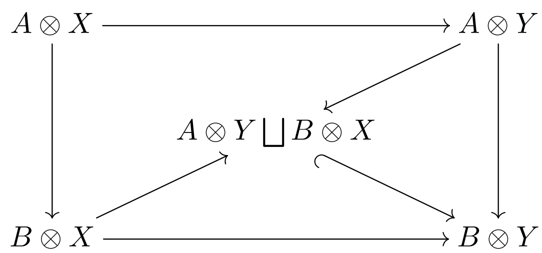
We have adjunctions

$$\begin{array}{ccccc} \mathrm{Operads}(V) & \underset{F}{\overset{U}{\rightleftarrows}} & \Sigma(V) & \underset{F}{\overset{U}{\rightleftarrows}} & \mathbb{N}(V) \\ & & U \uparrow\downarrow F & & \\ & & \mathrm{Bimod}_{\mathcal{O}}(V) & & \end{array}$$

If we have an adjunction $\varphi : V \leftrightarrows W : \psi$, then we also get induced adjunctions on the categories of operads. In particular, there are induced maps Operads(Top) $\overset{C_*}{\to}$ Operads(Ch), and H_* : Operads(Ch) $\to$ Operads(Ch). where U is the forgetful functor, and the F's are left adjoints.

Definition 1.2.7. A *monoidal model category* is a model category such that

(1) we have a unit axiom (not necessary if the unit is cofibrant);
(2) if $A \hookrightarrow B$ and $X \hookrightarrow Y$ are cofibrations then we have



where the marked cofibration is a trivial cofibration if $A \hookrightarrow B$ and $X \hookrightarrow Y$ are.

Theorem 1.2.8 (Transferral theorem)**.** *Suppose D has a model structure. Given an adjunction $L : D \leftrightarrows E : R$ (where E is sufficiently nice), then there is a transferral model structure on E:*

- *f is a weak equivalence iff Rf is a weak equivalence on D.*
- *f is a fibration iff Rf is.*

This works if L preserves small objects and $R(col(po(L(J))))$ is an equivalence.

A path object is a factorization of $X \to X \times X$ into a weak equivalence followed by a fibration (well, it's the middle object in such a factorization).

In sets, recall that $I = *$, and the diagonal map can be rewritten $X^I \to X^{I \cup I}$, which is true in general. If I can find a J and a factorization $I \overset{\sim}{\leftarrow} J \hookleftarrow I \cup I$ of the right map, then then by model categorical nonsense, you get a functorial path object X^J.

Definition 1.2.9. H is a segment object of V if $I \cup I \to H \to I$ is the right map, and there exists $v : H \otimes H \to H$ with $i_0 \otimes X \overset{v}{\mapsto} X$ and $i_1 \otimes X \overset{v}{\mapsto} i_1$. (Here I'm assuming i_0 is the image of the first I in $I \cup I \to X$.)

It is an interval object if $i_0 \sqcup i_i$ is a cofibration. Say H is coassociative if there's an associative comultiplication $H \to H \otimes H$ that plays nicely with the v, and similarly for cocommutative.

There's really only one good example: in simplicial sets, let $H = \Delta[1]$ and $I = \Delta[0]$. Then we have $\Delta[0] \cup \Delta[0] \overset{\delta_0 \cup \delta_1}{\to} \Delta[1] \overset{\sim}{\to} \Delta[0]$ where the first map sends the first $*$ to the first endpoint of $\Delta[1]$ and the second $*$ to the second endpoint.

The multiplication $v : \Delta[1] \times \Delta[1] \to \Delta[1]$ sends $(0,1), (1,1), (1,0)$ in $\Delta[1] \times \Delta[1]$ to $1 \in \Delta[1]$, and the $(0,0) \mapsto \Delta[1]$. There's a comultiplication on H where you include your interval into the diagonal of the square $\Delta[1] \times \Delta[1]$.

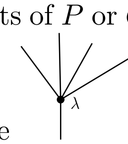
In Top, $H = [0,1]$, and $I = \{*\}$; this is cocommutative.

Theorem 1.2.10 (Reese-Moerdijk)**.** *Let V be a monoidal category where I is cofibrant, V has functorial fibrant replacement, V has cocommutative interval objects. Then there is an adjunction $F : \Sigma(V) \leftrightarrows Operads(V) : U$. This works for* Top *and sSet.*

If V/I has functorial fibrant replacements, and V has a coassociative interval object, then you get an adjunction out of $Operads^{red}$. This works when $V = Ch(R)$.

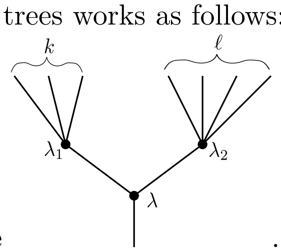
Talk 1.3: Boardman-Vogt tensor product of operads and their bimodules (Amelia Tebbe)

Let P and Q be operads; we want to construct $P \otimes Q$. Remember that elements of P or Q are equivalence classes of labelled planar trees. For example, $\lambda \in P(4)$ looks like



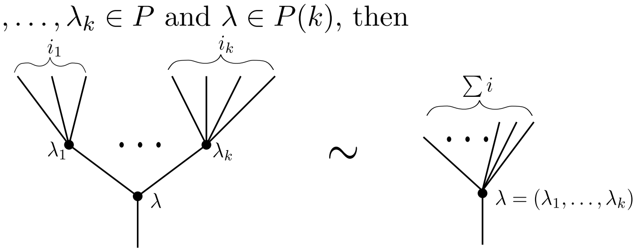
, and $\lambda \in P(0)$ is just a single vertex (with a tail representing output). This is called a "stump".

Grafting trees works as follows: if $\lambda_1 \in P(k)$, $\lambda_2 \in Q(\ell)$, and $\lambda \in P(4)$ then the grafted tree looks like

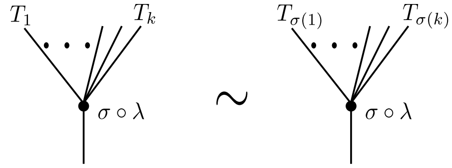


. Then $P \otimes Q$ is a quotient of all possible graftings of elements from P and Q with some equivalence relations as follows:

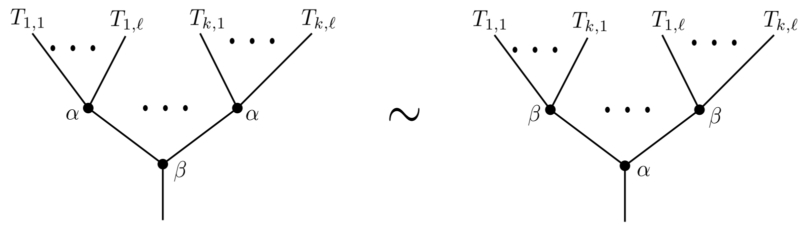
- $\mathbb{1}_P \sim \mathbb{1}_Q$
- (collapsing) if $\lambda_1, \ldots, \lambda_k \in P$ and $\lambda \in P(k)$, then



- (Σ-action) let $T_1, \ldots, T_k$ be trees, where $\sigma \in \Sigma_k$; then



- (interchange) if $\alpha \in P(k)$, $\beta \in Q(\ell)$, trees T_{ij} for $1 \le i \le k$, $1 \le j \le \ell$, then

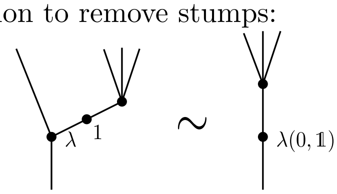


Let X be a P-algebra and a Q-algebra. Then we have a commutative diagram

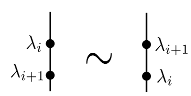
$$\begin{array}{ccccc} (X^k)^\ell & \longrightarrow & (X^\ell)^k & \xrightarrow{\beta^k} & X^k \\ \downarrow{\scriptstyle \alpha^\ell} & & & & \downarrow{\scriptstyle \alpha} \\ X^\ell & & \xrightarrow{\quad\beta\quad} & & X \end{array}$$

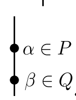
(This is usually done for simplicial sets or topological spaces.) If $\beta \in Q(0)$ and $\alpha \in P(0)$ I could do this diagram with zero's everywhere, and get that α and β have to be identified. So no matter how many 0-objects you have to start off with, $P \otimes Q$ can only have one zero-object.

You can use the collapsing relation to remove stumps:



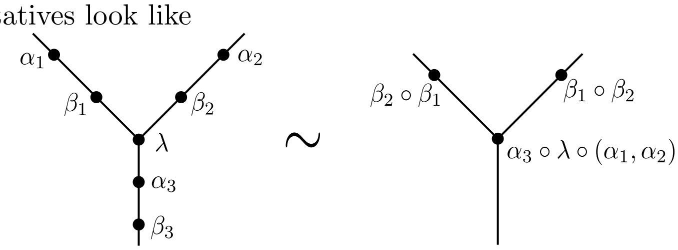
To get a feel for this, let's look at it in low arities.

Arity 1: If $\lambda_i \in P$, $\lambda_{i+1} \in Q$, then 

by the interchange relation. So every element in arity 1 can be expressed as 

. So $P \otimes Q(1) = P(1) \times Q(1)$.

Arity 2: Representatives look like



I claim that $(P \otimes Q)(2)$ is the pushout of the diagram

$$\begin{array}{ccc} P(2) \times Q(2) & \longrightarrow & P(2) \times Q(1) \times Q(1) \\ \downarrow & & \downarrow \\ P(1) \times P(1) \times Q(2) & \longrightarrow & ? \end{array}$$

where these maps have the following effect on elements

$$\begin{array}{ccc} (\alpha, \beta) & \longrightarrow & (\alpha, \beta(\mathbb{1}, 0), \beta(0, \mathbb{1})) \\ \downarrow & & \\ (\alpha(\mathbb{1}, 0), \alpha(0, \mathbb{1}), \beta) & & \end{array}$$

In general, everything can be expressed as a colimit, but things get really complicated, fast.

There's a forgetful functor $U : \text{Operads}_0 \to \text{Monoids}$ (here $_0$ means reduced) sending $P \mapsto P(1)$. This has a right adjoint which we call R, and $RM(k) = M^k$. This does something special to the associative operad:

$$Ass \otimes B \cong (Ass \times RU(B))/\sim$$

where the relation identifies $(\sigma, b_1, \dots, b_k) \sim (\tau, b_1, \dots, b_k)$ if for each (i, j) such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $\tau^{-1}(i) > \tau^{-1}(j)$, there exists $c \in \beta(2)$ such that $(b_i, b_j) = (c \circ (\mathbb{1}), c \circ (0, \mathbb{1}))$.

We want to extend this to a tensor product of bimodules. Start with $M \times N$, where M has a left P-action and a right Q-action, and N has a left R-action and a right T-action. We want to end up with $M \widetilde{\otimes} N$ which has a left $P \otimes R$ action and a right $Q \otimes T$ action. That is, we want:

$$\begin{array}{ccc} \text{Bimod} \times \text{Bimod} & \dashrightarrow & \text{Bimod} \\ {\scriptstyle \text{free}\times\text{free}} \uparrow \downarrow {\scriptstyle \pi\times\pi} & & {\scriptstyle \text{free}} \uparrow \downarrow {\scriptstyle \pi} \\ \text{Op}^2 \times \text{Op}^2 & \longrightarrow & \text{Op}^2 \end{array}$$

(free means it's the thing with one generator in arity 1). On objects, this does

$$\begin{array}{ccc} (P \circ Q) \times (R \circ T) & & (P \otimes R) \circ (Q \otimes T) \\ \uparrow & & \uparrow \\ (P, Q) \times (R, T) & \longrightarrow & (P \otimes R, Q \otimes T) \end{array}$$

so we want $(P \circ Q) \widetilde{\otimes} (R \circ T) = (P \otimes R) \circ (Q \otimes T)$.

Any bimodule is a coequalizer of free modules. So it suffices to define

$$F_{P,Q}(X) \widetilde{\otimes} F_{R,T}(Y) = F_{P \otimes R, Q \otimes T}(-)$$

where X and Y are symmetric sequences. (Here $F_{P,Q}(X)$ means the free left-P right-Q bimodule.) This is hard!

But we can do it in a special case, where the symmetric sequences are concentrated in arity 1. Then the operads are simplicial monoids and the bimodules are simplicial bisets. We can just use cartesian products: $F_{P,Q}(X) = P \times X \times Q$.

In this case, we can just define

$$F_{P,Q}(X) \widetilde{\otimes} F_{R,T}(Y) = F_{P \otimes R, Q \otimes T}(X, Y) \overset{\substack{\text{by arity 1} \\ \text{computation}}}{=} F_{P \times R, Q \times T}(X \times Y)$$

because we have cartesian products and cartesian products do all we need. But in a general case, we don't have $\times$; we have $\circ$.

But in the general case, replace $X \times Y$ in the middle term by $X \square Y$. This is called the matrix monoidal product.

Suppose M is an R, T-bimodule. We get an adjunction

$$\mathrm{Bimod}_{P,Q} \underset{\mathrm{Map}_{R,T}(M, \gamma_*(-))}{\overset{-\widetilde{\otimes} M}{\rightleftarrows}} \mathrm{Bimod}_{P \otimes R, Q \otimes T}$$

Here $\gamma_n(X)(k) = i^*(X(kn))$ where the Σ_{kn} action is induced by the forgetful map $\Sigma_{kn} \to \Sigma_k$.

Question: is there a colored version of this?

Answer: Yes. And the tensor product gives a closed monoidal structure.

Talk 1.4: Little disks and little cubes operads (Alex Yarosh)

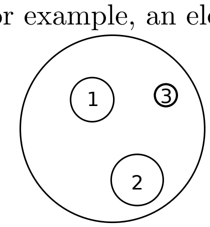
The word "operad" showed up when trying to figure out when a space has the homotopy type of a loop space. What properties do loop spaces have that are special? We have concatenation of loops, which is homotopy-associative with higher coherences. We have double loop spaces which are not only homotopy-associative, they end up being homotopy-commutative (by Eckmann-Hilton) as well, with higher coherences. This is too much to think about; hence, operads.

Associative operads just have the discrete symmetric group in each level. Now we want to encode homotopies; instead of discrete points, let's take blobs around those points. We also need a composition – a way to put these blobs inside each other.

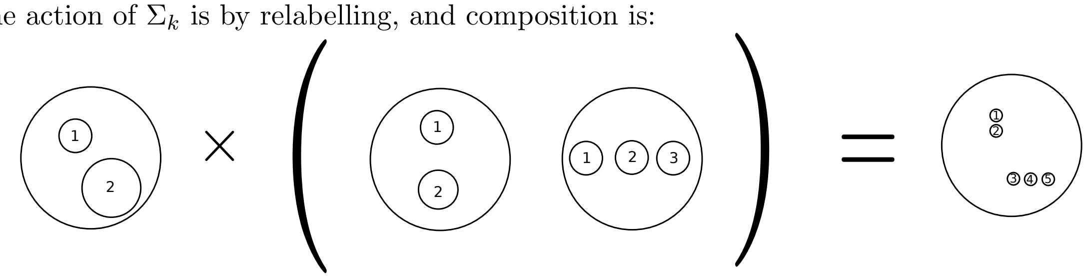
Definition 1.4.1. The little n-disks operad D_n is defined by

$$D_n(k) = k\text{-tuple of standard embeddings } D^n \hookrightarrow D^n \text{ with disjoint images}$$

(where $D^n = \{x \in \mathbb{R}^n : |x| < 1\}$). For example, an element of $D_2(3)$ looks like:



The action of Σ_k is by relabelling, and composition is:



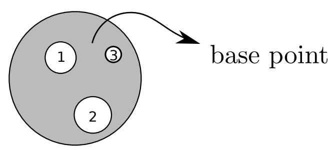
Consider the equatorial embedding $D^n \hookrightarrow D^{n+1}$ (e.g. including D^1 as the equator in D^2, and thickening the chosen subintervals of D^1). We can define $D_\infty(k) = \operatorname{colim}_n D_n(k)$. This inherits an operad structure.

It is obvious that $D_n(k) \simeq F(\mathbb{R}^n, k)$ where the latter is the configuration space of k points in $\mathbb{R}^n$ (such that no two points are in the same place). The homotopy just shrinks the little disks to a point. We have an equivalence of spaces $D_1 \simeq Ass$ given as follows: given a tuple of intervals $(c_1, \dots, c_k) \in D_1(k)$, there's an ordering $c_{i_1}(0) < \cdots < c_{i_n}(0)$. The image of Σ_n is the permutation $(i_1, i_2, \dots, i_k)$.

This gives a map of operads $D_1 \to Ass$, but there is no inverse map of operads.

Similarly, there is a map of operads $D_\infty \to Comm$ that induces an equivalence on the space level. (Again, no operad map the other way.)

We saw earlier that D_n acts on loop spaces. That is, we want a map $D_n(k) \times (\Omega^n)^k \to \Omega^n X$. Recall that $\Omega^n X = [(D^n, \partial D^n) \to (X, *)]$. Given an element of $D_2(3)$ and $(\alpha_1, \alpha_2, \alpha_3) \in (\Omega^2 X)^3$, the resulting element of $\Omega^3 X$ can be represented by:



Theorem 1.4.2 (Recognition principle (Boardman-Vogt, May))**.** *A group-like space X (i.e. $\pi_0 X$ is a group) has the homotopy type of a loop space iff it is a D_n-algebra.*

The little cubes operads is the same, but replaces disks with cubes. This is nice because it's easy to suspend. Little disks are nice because of the action of the orthogonal group. There's an operad that has both these properties: Steiner's operad $S_n(k)$ is k-tuples of paths of embeddings. (This is important when you want to find the action of the linear isometries operad on the little cubes operad.)

Definition 1.4.3. Say that $\mathcal{O}$ is an E_n-operad if $\mathcal{O}$ is Σ_n-free and there is a chain of weak equivalences $\mathcal{O} \xleftarrow{\sim} \dots \xrightarrow{\sim} C_n$ (where C_n is the little cubes operad).

It's not obvious that the little n-disks operad is actually an E_n-operad: they're obviously equivalent as spaces, but there isn't an obvious operadic map. All the ingredients are in later talks. The idea is that you build the Boardman-Vogt construction WC_n, which comes with a canonical equivalence $WC_n \xrightarrow{\sim} C_n$, and get an equivalence $WC_n \to FM_n$ (the Fulton-MacPherson operad – a compactification of configuration spaces). The idea is that

$$D_n \xleftarrow{\sim} WD_n \xrightarrow{\sim} FM_n \xleftarrow{\sim} WC_n \xrightarrow{\sim} C_n.$$

Proposition 1.4.4. *An operad $\mathcal{O}$ is an E_1-operad iff:*

(1) each path component is contractible, and
(2) Σ_k acts freely and transitively on $\pi_0 \mathcal{O}(k)$.

An operad $\mathcal{O}$ is an E_∞ operad iff:

(1) each $\mathcal{O}(k)$ is contractible, and
(2) Σ acts freely on $\mathcal{O}(k)$.

Any E_2-operad has a universal covering space of the braid operad.

Theorem 1.4.5 (Recognition principle for E_2)**.** *An operad $\mathcal{O}$ is E_2 iff*

(1) Universal covering spaces $\widetilde{\mathcal{O}(k)}$ are contractible on each level, and
(2) the braid group B_k acts freely and transitively on $\widetilde{\mathcal{O}(k)}$.

What is the braid group? All the possible braids of k strands, attached at the bottom and top.

There is no recognition principle for higher E_k (yet!). Also not clear what the applications would be. (The recognition principle for E_2 is useful for computations involving stable homotopy groups of spheres.)

Definition 1.4.6. A cellular decomposition of a space X w.r.t. a poset A is a collection $\{C_\alpha\}_{\alpha\in A}$, where C_n are closed contractible subspaces of X, such that

(1) $C_\alpha \subset C_\beta$ if $\alpha \leq \beta$
(2) $X = \operatorname{colim}_A C_\alpha$
(3) inclusions $C_\alpha \hookrightarrow X$ are cofibrations.

Note that I'm *not* implying that these are disks.

Note that $X \simeq |NA|$.

How do you decompose the little cubes operad? You probably want the n^{th} level to be n-cubes, etc.

Suppose I have three labelled little cubes. What's the essential information?

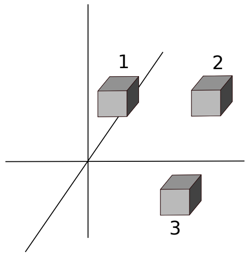
- labels
- the cubes are disjoint, i.e. separated by hyperplanes
- where the cubes are w.r.t. the separating hyperplane

I want to abstract this definition...

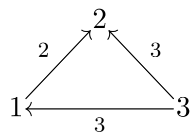
The complete graph operad is given by

$$K_n(p) = \{\mu, \sigma \in K(p) \,:\, \mu_{ij} \leq n \ \forall i \neq j\} \text{ where } K(p) = \mathbb{N}^{\binom{p}{2}} \times \Sigma_p.$$

Elements (μ_{ij}, σ) for $i < j$ can be represented as a graph: start with a complete graph on p vertices, where the ij edge is labeled by μ_{ij}, and with orientation given by σ. E.g. the graph associated to



is



because cubes 1 and 3 are separated by hyperplane 3 (the one perpendicular to the z axis), and the direction encodes which side 1 and 3 are on.

Define a partial order on $K(p)$ as follows: $(\mu, \sigma) \leq (\nu, \tau)$ if either $\mu_{ij} < \nu_{ij}$ for all $i < j$, or whenever $\mu_{ij} = \nu_{ij}$ we have $\sigma|_{ij} \leq \tau|_{ij}$.

σ defines an acylic orientation $(\sigma^n(1) \to \sigma^n(2) \to \dots)$.

An operad $\mathcal{O}$ is a cellular E_n operad if

(1) $\mathcal{O}(2)$ has a cellular decomposition w.r.t. $K_n(2)$
(2) "Cells"
$$\mathcal{O}(k)^{(\mu,\sigma)} = \{x \in \mathcal{O}(k) \,:\, \gamma_{ij}(x) \in \mathcal{O}^{(\mu,\sigma)}(2)\}$$
where γ are the operad structure maps form a cellular decomposition of $\mathcal{O}(k)$ w.r.t. $K_n(k)$. (I.e. the cellular decomposition for $k > 2$ comes from the cellular decomposition for $k = 2$.)
(3) Operad multiplication is compatible with complete graph operad multiplication.

Theorem 1.4.7.

(1) The little n-cubes operad is cellular;
(2) any cellular E_n operad $\mathcal{O}$ is $\mathcal{O} \xrightarrow{\sim} |NK_n|$;
(3) all cellular E_n-operads are equivalent.

Fulton-MacPherson is cellular (in a sense?).

This is good because any cellular operad is the geometric realization of a simplicial operad.

Talk 1.5: Model categories and derived mapping spaces (Kyle Gray)

Goals:

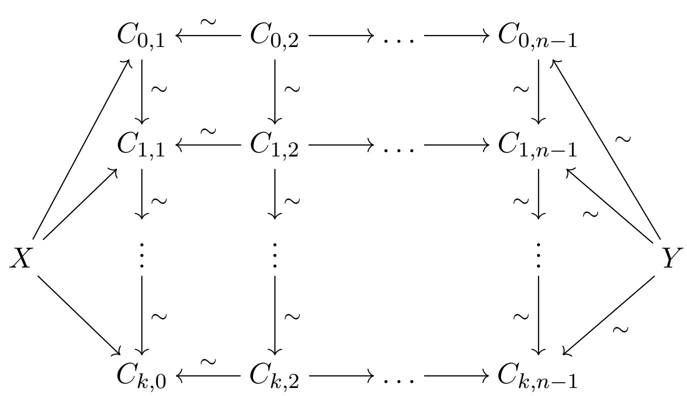
(1) Define simplicial localization via hammocks
(2) Define derived mapping spaces
(3) Discuss derived adjunction theorem
(4) Application to operads

Simplicial localizations. Our setup is as follows: let C be a model category with weak equivalences W. This gives rise to a homotopy category $Ho(C)$, where we formally invert the weak equivalences. Sometimes this is denoted $C[W^{-1}]$. Problem: in passing from the original category to the homotopy category, we lose "higher homotopical information" about the original model category. One solution to this problem, proposed through a series of papers in 1980 by Dwyer and Kan, is to study a richer object called the simplicial localization.

The following definition makes sense in a weaker setting than model categories.

Definition 1.5.1. Let C be a categories with weak equivalences W (any subcategory which contains all the objects). The *hammock localization* of C w.r.t. W, denoted $L^H C$, is the simplicial category (i.e. enriched over simplicial sets) defined as follows:

- $L^H C$ has the same objects as C
- for objects $X, Y \in C$ the simplicial set $L^H C(X, Y)$ has as k-simplices the "reduced hammocks" of height $k+1$



  where arrows with $\sim$ are arrows in W. Note that n is arbitrary. k shows up as the number of rows.

The i^{th} face map is given by omitting the i^{th} row (and the map is given by composition), and the i^{th} degeneracy map is given by repeating the i^{th} row. Composition is just concatenation of hammocks.

Proposition 1.5.2. *For any $X, Y \in C$, there is a bijection of sets*

$$L^H C(X, Y) = C[W^{-1}](X, Y)$$

(when things are small enough for this to make sense).

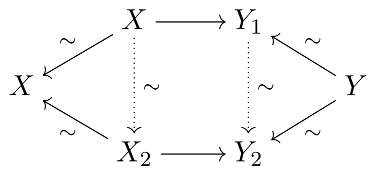
Definition 1.5.3. Let C be a model category, and $X, Y \in C$. The *derived mapping space* from X to Y is the following simplicial set

$$\mathrm{Map}^h_C(X, Y) = L^H C(X, Y)$$

This gives a bifunctor $\mathrm{Map}^h(-,-) : C^{op} \times C \to sSet$ which converts weak equivalences in either variable to weak homotopy equivalences.

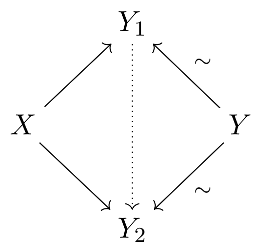
Here are some other descriptions. (In the following, $\sim$ will mean "weakly homotopy equivalent to.")

- If C is a model category, then $\mathrm{Map}^h(X,Y)$ is $\sim$ the nerve of the category whose objects are diagrams $X \xleftarrow{\sim} X_1 \to Y_1 \xleftarrow{\sim} Y$ and whose morphisms are



  Advantage: we don't have arbitrary length anymore.

- (Dugger) Moreover, if X is cofibrant this is $\sim$ nerve of the less complicated category whose morphisms are

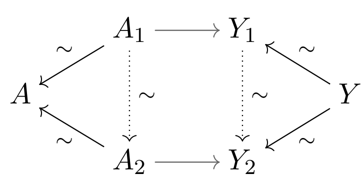


- If C is a simplicial model category, then $\mathrm{Map}^h(X,Y) \sim \mathrm{Map}_C(X^c, Y^f)$ (cofibrant and fibrant replacements, respectively).

Theorem 1.5.4 (The Derived Adjunction Theorem)**.** *Suppose we have model categories C and D and a Quillen pair $\lambda : C \leftrightarrows D : \rho$. For all objects $A \in C$, $Y \in D$, there is a natural weak homotopy equivalence*

$$\mathrm{Map}^h_C(A, \rho(Y^f)) \sim \mathrm{Map}^h_D(\lambda(A^c), Y).$$

Comments on the proof. We can assume without loss of generality that A is cofibrant and Y is fibrant; this is because the derived mapping space takes equivalences in either variable to weak homotopy equivalences. The strategy is as follows. Let Z be the category described by $A \xleftarrow{\sim} A_1 \to Y_1 \xleftarrow{\sim} Y$ where the gray map represents a map $A_1 \to \rho(Y_1)$ or equivalently $\lambda(A_i) \to Y_i$. The morphisms are "natural transformations"



The goal is to show $Z \sim \mathrm{Map}^h_D(\lambda(A), Y)$.

Consider the functor $F : Z^c \to (\mathscr{M}(C) \downarrow A)$ ($\mathscr{M}$ is the moduli category whose morphisms are just the weak equivalences, and $\downarrow$ denotes the overcategory over A). This "picks off" $A_1 \xrightarrow{\sim} A$. One can show that for all objects $U \in (\mathscr{M}(C^c) \downarrow A)$,

$$(F \downarrow U) \sim \mathrm{Map}^h_D(\lambda(A), Y).$$

Use Quillen's theorem B, which in this setting allows us to conclude that $\mathrm{Map}^h(\lambda(A), Y)$ is weakly equivalent to the homotopy fiber of F. Since $(\mathscr{M}(C^c) \downarrow A)$ has a terminal object, it is contractible, which gives that $Z \sim \mathrm{Map}^h_D(\lambda(A, Y))$. $\square$

Corollary 1.5.5. *Let C and D be model categories. Let $\lambda : C \leftrightarrows D : \rho$ be a Quillen pair such that λ and ρ preserve all weak equivalences. For all $A \in C$ and $Y \in D$,*

$$\mathrm{Map}^h_C(A, \rho(Y)) \sim \mathrm{Map}^h_D(\lambda(A), Y).$$

If in addition, each component $\eta_A : A \to \rho\lambda(A)$ of the unit of the adjunction is a weak equivalence, then

$$\mathrm{Map}^h_C(A, A') \sim \mathrm{Map}^h_D(\lambda(A), \lambda(A'))$$

for all $A, A' \in C$.

If in addition, each component $\varepsilon_Y : \lambda\rho(Y) \to Y$ of the counit of the adjunction is a weak equivalence, then

$$\mathrm{Map}^h_D(Y, Y') \sim \mathrm{Map}^h_C(\rho(Y), \rho(Y')).$$

An application to operads. The point is that derived mapping spaces of operads are invariant under change of the underlying category $sSet \leftrightarrow \mathrm{Top}$.

If I start with the Quillen equivalence $|-| : sSet \leftrightarrows \mathrm{Top} : S$ (geometric realization, singular subfunctor), we saw earlier that such an adjunction will pass to an adjunction of operads on the respective categories. Elaine wrote up a way to transfer the model structure on simplicial sets and Top to the categories of operads. It turns out that this satisfies all hypotheses of Corollary 1.5.5.

Proposition 1.5.6. *If P, P' are operads on spaces, and Q, Q' are operads on simplicial sets, then*

$$\mathrm{Map}^h_{Op(\mathrm{Top})}(P, P') \sim \mathrm{Map}^h_{Op(sSet)}(SP, SP')$$
$$\mathrm{Map}^h_{Op(sSet)}(Q, Q') \sim \mathrm{Map}^h_{Op(\mathrm{Top})}(|Q|, |Q'|)$$

Point: you might not have a *space* of maps between two objects, but the derived mapping space gives you one.

Day 2: Configuration spaces and knot theory

Talk 2.1: (Co)homology and compactifications of configuration spaces (Felicia Tabing)

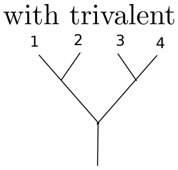
Definition 2.1.1. The configuration space of n points in X is

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) : x_i \neq x_j, i \neq j\}$$

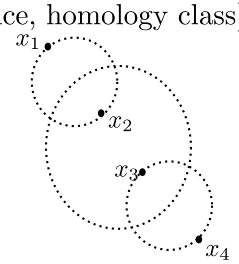
topologized as a subspace of the product X^n.

Let's start by talking about $\mathrm{Conf}_2(\mathbb{R}^d)$. I claim it deformation retracts onto $P_{12} = \{(x_1, x_2) : x_1 = -x_2, |x_i| = 1\}$, which is $\simeq S^{d-1}$. This retract sends a configuration (y_1, y_2) to $\frac{y_1 - y_2}{\|y_1 - y_2\|}$ (i.e. direction from the midpoint to y_1).

We want to know $H_*(\mathrm{Conf}_n(X))$, especially when $X = \mathbb{R}^d$. We use trees to define homology classes. Let T be a rooted planar tree with trivalent vertices (except for the root). Here,



corresponds to the submanifold (hence, homology class) depicted below:



This diagram means the set of configurations $(x_1, \dots, x_4)$ such that x_1 and x_2 are diametrically opposed on the top circle, x_3 and x_4 are diametrically opposed on the bottom circle, and these two circles are allowed to be in orbit around the big circle. (This is all in $\mathbb{R}^2$; for $\mathbb{R}^d$, we would have copies of S^{d-1} instead of circles.)

You can talk about trees with less than n leaves; in that case, the unspecified coordinates are free to go anywhere.

Suppose T is a tree, and let H denote its set of internal vertices. Then let

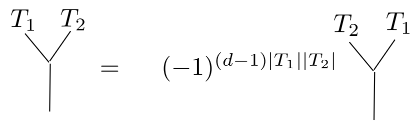
$$P_T : (S^{d-1})^H \to \mathrm{Conf}_n(\mathbb{R}^d)$$

be the map sending $(u_{v_1}, \dots, u_{v_{|H|}}) \mapsto (x_1, \dots, x_n)$ where $x_i = \sum_{v_j} \pm \varepsilon^{h_i} u_{v_j}$. Here ε is some fixed constant $< \frac{1}{3}$.

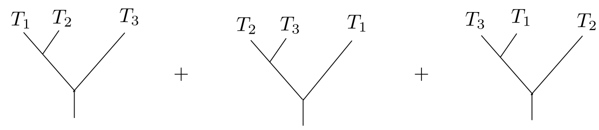
You can extend this to forests (a bunch of trees whose roots are all on a line), denoted P_F. These induce classes in $H_*(\mathrm{Conf}_n(\mathbb{R}^d))$.

Proposition 2.1.2. *The classes given by forests in $H_*(\mathrm{Conf}_n(\mathbb{R}^d))$ satisfy*

(1) (anti-symmetry)



(2) (Jacobi identity)



$= 0.$

We can describe $H^*(\mathrm{Conf}_n(\mathbb{R}^d))$ using graphs.

Definition 2.1.3. Let $\alpha_{ij} : \mathrm{Conf}_n(\mathbb{R}^d) \to S^{d-1}$ be the map sending $(x_1, \dots, x_n) \mapsto \frac{x_i - x_j}{|x_i - x_j|}$. Then let $\alpha_{ij}^* \in H^{d-1}(\mathrm{Conf}_n(\mathbb{R}^n))$ denote the pullback of the dual to the fundamental class of S^{d-1}.

Definition 2.1.4. Let $\Gamma(n)$ be the free module generated by graphs with labelled vertices $\{1, \dots, n\}$ with oriented and ordered edges.

We get a map $\Gamma(n) \to H(\mathrm{Conf}_n(\mathbb{R}^d))$ sending $(i \to j)$ to α_{ij}^*.

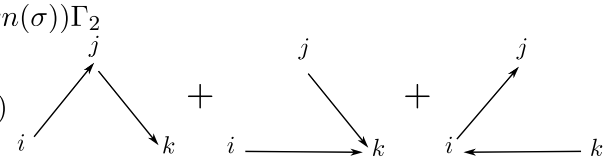
Example 2.1.5. This map sends $4 \to 2 \;\; 1 \to 3$ to $a_{42}a_{13}$. (Why are they multiplied in that order? Remember that the edges of the graph are ordered.)

The graph $1 \overset{e_1}{\to} 2 \overset{e_3}{\to} 3 \overset{e_2}{\to} 4$ corresponds to $a_{12}a_{34}a_{23}$.

Cohomology classes coming from graphs satisfy relations

(1) $\Gamma_2 = (-1)^{d-1}(sgn(\sigma))\Gamma_2$

(2) (Arnold relation)



$= 0.$

Now we define the Poisson operad. Let $\mathrm{Pois}^d(n)$ be generated by n forests moduli antisymmetry and the Jacobi relation, and let $Siop^d(n) = \Gamma(n)/$Arnold relation, arrow-reversing relation.

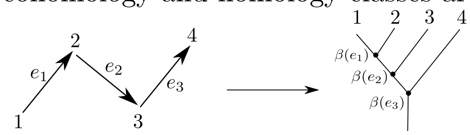
Let $\Gamma \in \Gamma(n)$ and $T \in T(n)$ (a tree with n leaves). We can define a pairing between graphs and trees. First, define a map

$$\beta_{P,T} : \{\text{edges from } \Gamma\} \to \{\text{internal vertices of } T\}$$

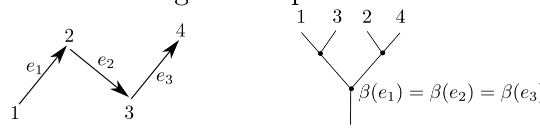
sending an edge $i \to j$ to the highest shared vertex in the path in T from leaf i to j. Define the configuration pairing

$$\langle \Gamma, T \rangle = \begin{cases} \pm 1 & \text{if } \beta_{\Gamma,T} \text{ is a bijection} \\ 0 & \text{otherwise.} \end{cases}$$

For example, the following cohomology and homology classes are dual to each other



(i.e. they pair to ± 1) but the following classes pair to zero:



Theorem 2.1.6.

$$Pois^d(n) \cong H_*(\mathrm{Conf}_n(\mathbb{R}^d))$$
$$Siop^d(n) \cong H^*(\mathrm{Conf}_n(\mathbb{R}^d))$$

Furthermore, the pairing between homology and cohomology corresponds to the configuration pairing.

It follows from the Arnold relation and other things that any graph with a cycle goes to zero.

Compactification of $\mathrm{Conf}_n(M)$. Assume M is a submanifold of $\mathbb{R}^d$.

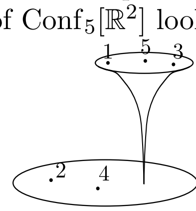
Definition 2.1.7. Given $i, j \in \{1, \dots, n\}$ where $i \neq j$ let $\alpha_{ij} : \mathrm{Conf}_n(\mathbb{R}^d) \to S^{d-1}$ be the map sending $\underline{x} \mapsto \frac{x_i - x_j}{|x_i - x_j|}$. This encodes the direction between 2 points.

Let $I = [0, \infty]$ and consider i, j, k be three distinct indices in $\{1, \dots, n\}$. Let $\rho_{ijk} : \mathrm{Conf}_n(\mathbb{R}^d) \to I$ be the map sending $\underline{x} \mapsto \frac{|x_i - x_j|}{|x_i - x_k|}$. This encodes "relative distance."

Given $A_n[M] = M^n \times (S^{d-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$, define

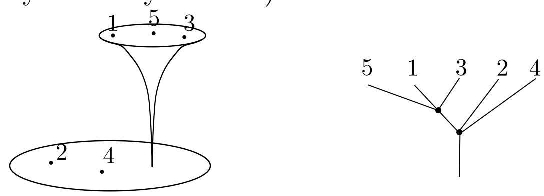
$$\alpha_n : \mathrm{Conf}_n(M) \to A_n[M]$$

where $\alpha_n = i \times \alpha_{ij} \times \rho_{ijk}$. Define $\mathrm{Conf}_n[M]$ be the closure of the image of α_n in $A_n[M]$. This is called the canonical compactification or completion. If M is compact, then this recovers $\mathrm{Conf}_n[M]$. For example, an element of $\mathrm{Conf}_5[\mathbb{R}^2]$ looks like



where the points 1, 3, and 5 are "infinitesimally close", but you still remember their relative distance and direction.

Let T be a tree with no bivalent internal vertices. Let $C_T(M)$ be the subspace of all $X = ((X_i), (u_{ij}), (\rho_{ijk})) \in \mathrm{Conf}_n[M]$ where $\rho_{ijk} = 0$ when the paths from i and j to the root meet each other before either meets the path from k to the root. (I.e. i and j are nestled up in some subtree, and k is only distantly related.) Here



1, 3, and 5 all belong to the same subtree, so they are infinitesimally close in the element of the compactification.

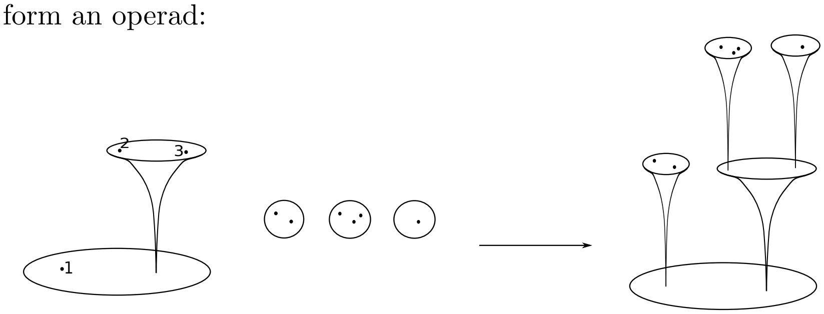
When T is trivalent, P_T is homologous to the $C_T(M)$ stratum. (i.e. let ε go to zero, which makes sense in the compactification).

There's an alternate definition that says that this is an operadic completion of the configuration spaces.

Properties: $\mathrm{Conf}_n[M]$ is (/ has):

- compact
- stratified
- a manifold with corners
- smooth structure
- independent of embedding
- functorial
- the inclusion $\mathrm{Conf}_n(M) \to \mathrm{Conf}_n[M]$ is a homotopy equivalence (the former is the interior of the latter)

$\{\mathrm{Conf}_n[\mathbb{R}^d]\}$ form an operad:



There's also an intrinsic definition in terms of blowups that was given first.

$$\partial C_{1\;2\;3}[\mathbb{R}^d] \cong C_{1\;2\;3} \cup C_{2\;3\;1} \cup C_{3\;1\;2}$$

Talk 2.2: Formality of the E_n operad (Umut Varolgunes)

We will use the Fulton-MacPherson operad as a model for E_n, and denote it $\{FM_n(d)\}_{n\geq 1}$ (the confusing notation is to be consistent with the configuration space notation). Let $C_n(\mathbb{R}^d) = \mathrm{Conf}_n(\mathbb{R}^d)/$ (dilation, translation); recall we defined $FM_n(d)$ as the closure of $C_n(\mathbb{R}^d)$ inside $\prod_{i\neq j} S^{d-1} \times \prod_{i\neq j\neq k}[0,\infty]$ under $(x_1,\dots,x_n)\mapsto \left(\frac{x_i-x_j}{|x_i-x_j|}, \frac{|x_i-x_j|}{|x_i-x_k}\right)$. The funnel diagrams are nice, but remember we have equations too!

$FM_n(d)$ is a manifold with corners; its interior is canonically identified with $C_n(\mathbb{R}^d)$.

For a manifold, the de Rham forms is a DGA; say the manifold is *formal* if you can find a zigzag (or map) $\Omega_{dR}(M) \xrightarrow{\sim} \dots \xleftarrow{\sim} H^*_{dR}(M)$ that is an quasi-isomorphism of algebras. (This condition is similar to saying that all Massey products in $\Omega_{dR}(M)$ vanish.)

By *formality of an operad*, I will mean formality of the duals $\mathcal{O}_n$ in the sense of rational homotopy theory (think of Ω_{dR}, not $A_{PL}(\mathcal{O}(n))$) where the maps in the zigzag respect the operad structure. We have maps $\theta_{ij} : FM_n(d) \to S^{d-1}$ extending the map $C_n(\mathbb{R}^d) \to S^{d-1}$ sending $(x_1,\dots,x_n)\mapsto \frac{x_i-x_j}{|x_i-x_j|}$. (These are the same maps that have previously been called π_{ij} and α_{ij}.) Fix a volume form $\mathrm{vol}_{S^{d-1}}$. For all $d\geq 2$,

$$H^*(F_n(d)) = \mathbb{C}[\{[w_{ij}]_{i\neq j}\}]\Big/[\omega_{ij}] = -[\omega_{ji}], [\omega_{ij}]^2 = 0, [\omega_{ij}][\omega_{jk}] \pm [\omega_{ik}][\omega_{kj}] \pm [\omega_{ik}][\omega_{ij}] = 0.$$

The last relation is called the Arnold relation. For $d = 2$, the Arnold relation holds on the chain level (i.e. you don't need the $[-]$'s).

Exercise 2.2.1. For $d > 2$, $\omega_{ij}\omega_{jk} + \cdots = d\beta$.

You're adding a β for each ijk; then you need to keep adding more stuff. The point of this talk is to show how to add this stuff in a way that is systematic, i.e. in a way that respects the operad structure.

We computed cohomology using only the Arnold forms. We want to push this strategy as far as possible, using only formal properties and the structure of the Fulton-MacPherson operad, but without writing down any more forms explicitly.

$FM_2(\mathbb{R}^d)$ is naturally identified with S^{d-1}. With respect to this identification, we have maps

$$\theta_{ij} : FM_n(\mathbb{R}^d) \to FM_2(\mathbb{R}^d)$$

which remembers ij. We've been pulling back forms. Now we want to try to push forward forms.

The idea is to use all of $FM_{n+m}(d) \to FM_n(d)$ and push forward; this is the strategy to make the β's in the exercise.

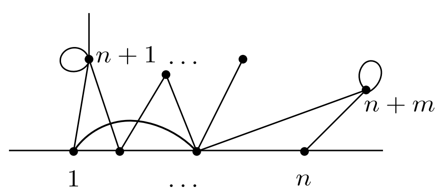
Interlude 2.2.2 (Push-forward in de Rham theory)**.** Let X and Y be manifolds, $X \to Y$ the locally trivial fiber bundle with compact oriented fibers of dimension ℓ with differential k-form on X. Idea: take a form on X; given a (small enough) chain on Y, define the value of the pushforward on that chain by integrating the original form on the preimage. If the manifolds are closed, it's the same as doing Poincaré dual, pushforward, and Poincaré dual again.

Problem: the maps $FM_{n+m} \to FM_m$ are not submersions, so de Rham theory doesn't work. But we can still make sense of the pushforward if we enlarge the class of "forms" that we consider: replace smooth forms by semi-algebraic forms.

Exercise 2.2.3. The Fulton-MacPherson operad on $\mathbb{R}$ is a bunch of copies of the Stasheff polytope. Think about $FM_4(\mathbb{R}) \to FM_3(\mathbb{R})$ (4! copies of the 2-Stasheff polytope maps to 3! copies of the 1-Stasheff polytope, i.e. the interval).

Forms generated by the ω_{ij}'s are called *Arnold forms*. Define *generalized Arnold forms* to be pushforwards of Arnold forms under the maps $FM_{n+m} \to FM_m$.

Generalized Arnold forms can be represented by diagrams, where there are a bunch of external vertices labelled $1, \dots, n$ (represented by being on the horizontal line), and a bunch of internal vertices labelled $n+1, \dots, n+m$. It's a graph, so there are some edges. There's also more rather boring data (ordering of edges etc), which should be obvious from the construction below.



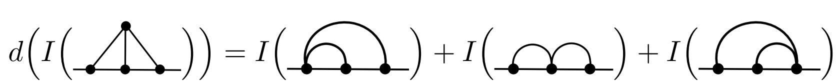
Given a graph, I can come up with a form, namely $\bigwedge_{\substack{\text{edges} \\ i \to j}} \omega_{ij} \in \Omega^*(FM_{n+m}(d))$. Push this forward along the map that forgets the internal vertices to get a form $\pi_*(\bigwedge \omega_{ij}) \in \Omega_*(FM_n(d))$. Call this $I(\Gamma)$ (where Γ was the graph).

Proposition 2.2.4. *Generalized Arnold forms form a subalgebra. More precisely,*

$$d(I(\Gamma)) = \sum I(\Gamma')$$

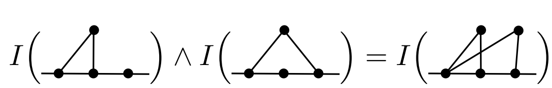
where the sum is over edges of Γ connecting internal and external vertices, and Γ' is the diagram obtained by collapsing that edge.

Here's an example of the formula for the differential


(2.2.1)

The formula for the wedge product is $I(\Gamma) \wedge I(\Gamma') = I(\Gamma'')$ where Γ'' is the diagram obtained by stacking Γ and Γ' along the external vertices.

For example,



Using these rules, define a CDGA $\widetilde{D}_n$ of diagrams with fixed n and varying m.

Question: what is the "G"? Where in this whole construction does the dimension d come up? The map I should also preserve grading, but that's the only place where the dimension comes in. (This is an exercise.)

Proposition 2.2.5. *$I(\Gamma) = 0$, where Γ contains a loop, double edge, or internal vertex of valency ≤ 2.*

Define D_n to be $\widetilde{D}_n$ modulo the ideal of relations generated by such diagrams.

Proposition 2.2.6. *$\{D_n\}_{n=1}^{\infty}$ has the structure of a cooperad of CDGA's.*

Now we want to show formality of $\{FM_n(d)\}_{n=1}^{\infty}$. We will come up with a zigzag

$$H^*(FM_n(d)) \leftarrow D_n \xrightarrow{I} \Omega^*(FM_n(d)).$$

We already talked about the second map. Let us describe the first map. If you have a diagram with only external vertices and one edge, say between i and j, send that to $[\omega_{ij}]$. Extend this to diagrams with only external vertices as a map of algebras. Send all the diagrams with internal vertices to 0.

Remark 2.2.7.

- $\Omega^*(-)$ is not monoidal. Given two spaces X, Y, you don't have a natural map
$$\Omega^*(X \times Y) \to \Omega^*(X) \otimes \Omega^*(Y)$$
but you do have a map
$$\Omega^*(X) \otimes \Omega^*(Y) \to \Omega^*(X \times Y).$$
This is a quasi-isomorphism, but the inverse is not canonical. You should be able to work with homotopy classes of maps instead of maps. Instead, they make do with the second map and get some kind of zigzag.
- If $d = 2$, then D_n is not a CDGA model; it might have negatively graded things.
- For deformation quantization, all we need is stable formality (homology instead of cohomology, chains instead of cochains, and don't care about the product structure). This is good because chains are monoidal.
- The map $FM_n(\mathbb{R}^d) \to FM_n(\mathbb{R}^D)$ induced by $\mathbb{R}^d \hookrightarrow \mathbb{R}^D$ is formal when $D \geq 2d+1$. This property is called relative formality.

We promised something about β: look at (2.2.1).

Talk 2.3: Universal Vassiliev invariants via integration on configuration spaces (Daniel Alvarez-Gavela)

Plan for the talk:

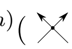
- Vassiliev's construction
- The Birman-Lin axioms
- Chord diagrams
- The linking number
- The first Bott-Taubes integral
- The Universal Bott-Taubes integral
- The Universal Kontsevitch integral

Vassiliev's construction. We're interested in knots, i.e. embeddings $S^1 \to \mathbb{R}^3$. Consider $\mathcal{K} = \mathrm{Emb}(S^1, \mathbb{R}^3) \hookrightarrow C^\infty(S^1, \mathbb{R}^3) \sim *$. After replacing this inclusion by a finite dimensional approximation and applying Alexander duality, morally speaking the study of $H^*(\mathcal{K})$ reduces to the study of $H_*(\mathcal{K}')$ where $\mathcal{K}' = C^\infty(S^1, \mathbb{R}^3) - \mathrm{Emb}(S^1, \mathbb{R}^3)$. The space $\mathcal{K}'$ of singular knots is naturally filtered

$$\mathcal{K}^{(1)} \subset \mathcal{K}^{(2)} \subset \dots$$

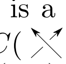
where you can think of $\mathcal{K}^{(m)}$ as the space of m-singular knots, i.e. knots with m transverse self-intersections. One can obtain homology classes on $\mathcal{K}'$ by running a spectral sequence on a simplicial resolution of this filtration. This is Vassiliev's construction. By the process described above, one obtains cohomology classes on $\mathcal{K}$. Degree zero cohomology classes are knot invariants. The knot invariants obtained in this way are called the Vassiliev knot invariants.

The Birman-Lin axioms. We give an axiomatic description of the knot invariants described in the previous section. Let $u : \mathcal{K} \to \mathbb{R}$ be a knot invariant (i.e $u \in H^0(\mathcal{K}; \mathbb{R})$). Define $u^{(m)} : \mathcal{K}^{(m)} \to \mathbb{R}$ recursively via:

$u^{(m)}($  $) = u^{(m-1)}($  $) - u^{(m-1)}($  $)$

Recall $\mathcal{K}^{(m)}$ is the space of immersions $S^1 \to \mathbb{R}^3$ that fail to be injective at exactly $2m$ points in the source S^1, and such that the image of those $2m$ points consists of m distinct images in $\mathbb{R}^3$ at which the immersion has a transverse self-intersection (the two tangent vectors to the knot at the self-intersection are linearly independent). Given $u = u^{(0)} : \mathcal{K} \to \mathbb{R}$, the above recursive formula defines $u^{(1)} : \mathcal{K}^{(1)} \to \mathbb{R}$, an isotopy invariant on singular knots with one unique self-intersection. The we use $u^{(1)}$ to define $u^{(2)}$ an isotopy invariant on singular knots with two distinct self-intersections, and so on.

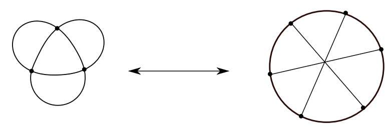
Definition 2.3.1. A knot invariant u is a Vassiliev invariant (or invariant of finite type) of type m if $u^{(m+k)} \equiv 0$ on $\mathcal{K}^{(m+k)}$ for any $k > 0$. Denote by $V_m \subset H^0(\mathcal{K}; \mathbb{R})$ the real vector subspace of Vassiliev invariants of order m.

Example 2.3.2. The Conway polynomial is a knot invariant that assigns a polynomial in $\mathbb{R}[t]$ to every knot. It satisfies the relation $C($  $) - C($  $) = tC(\,)(\,)$. By a combination of this relation and the relation that defines $u^{(k+1)}$ in terms of $u^{(k)}$ one deduces that $C(\mathcal{K}^{(m)}) \subset t^m\mathbb{R}[t]$. Hence the coefficient c_m of t^m in the Conway polynomial is a Vassiliev invariant of type m.

Most of the knot invariants that we know of are Vassiliev. However, it is not known whether all knot invariants are Vassiliev or at least can be approximated by Vassiliev invariants (does the spectral sequence converge?).

Chord diagrams. Let $K \in \mathcal{K}^{(m)}$ be a singular knot with m distinct transverse intersections. Define a *chord diagram* D_K associated to K. A chord diagram is a $2m$-tuple of points on S^1 divided into m pairs, up to reparametrizations of the circle.

Example 2.3.3. The singular trefoil knot on the left has the corresponding chord diagram on the right



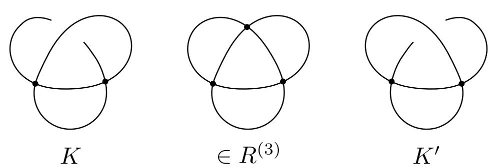
Each of the six points of S^1 that are mapped non-injectively in the immersion $K : S^1 \to \mathbb{R}^3$ corresponds to one of the points on the chord diagram. The pairs that are connected in the chord diagram are the pairs of points on S^1 that map to the same point in $\mathbb{R}^3$.

In general, if a knot has m self-intersections, there are $2m$ points in the domain where it fails to be injective, hence $2m$ points on the outside of the chord diagram. These are connected in pairs corresponding to which points get sent to the same point of $\mathbb{R}^3$. Denote C_m the real vector space generated by chord diagrams with m chords. We have defined an assignment $\sigma : \mathcal{K}^{(m)} \to C_m$, $K \mapsto D_K$ called the symbol. Suppose $u \in V_m$.

Proposition 2.3.4. *$u(K)$ only depends on the symbol D_K.*

Proof. Suppose $D_{K_0} = D_{K_1}$. After an ambient isotopy we can assume that K_0 and K_1 agree in a neighborhood of each of the m transverse intersections. Keeping $\mathbb{R}^3$ fixed near these self-intersections, isotope K_0 to K_1 through a path K_t of singular knots such that $K_t \in \mathcal{K}^{(m)}$ is always a singular knot with m transverse self-intersections except at a finite number of times t_j for which $K_{t_j} \in \mathcal{K}^{(m+1)}$ is an $(m+1)$-singular knot.

Example 2.3.5.

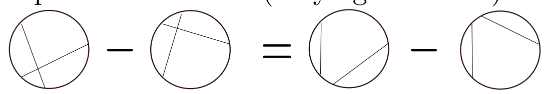


By the Birman-Lin axioms, $u^{(m)}(K_{t_j+\varepsilon}) = u^{(m)}(K_{t_j-\varepsilon}) \pm u^{(m+1)}(K_{t_j})$ at each of the times t_j, and everywhere else $u^{(m)}$ does not change because it is an invariant of $m-$singular knots. However, $u \in V_m$ and therefore $u^{(m+1)}(K_{t_j}) = 0$. It follows that $u(K_0) = u(K_1)$. □

We have described a mapping $V_m \to \mathrm{Hom}(C_m, \mathbb{R})$ which to $u \in V_m$ associates the function $\widehat{u} : C_m \to \mathbb{R}$ defined by the condition that $\widehat{u}(D) = u(K)$ for any $K \in \mathcal{K}^{(m)}$ such that $D_K = D$.

Definition 2.3.6. The quotient $C_m/\{4T, FI\}$ is obtained from C_m by imposing the following two relations:

- (4T Four Term Relation) Any collection of four different chord diagrams which are everywhere identical except for two chords (varying as shown) satisfies the equality:



- (FI Framing Independence) Every diagram with an isolated chord is set to zero. A diagram has an isolated chord if the diagram can be drawn in such a way that one of the chords does not intersect any other chord.

The following is the fundamental theorem of Vassiliev theory.

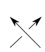
Theorem 2.3.7. *The assignment $u \mapsto \widehat{u}$ induces an isomorphism*

$$V_m/V_{m-1} \xrightarrow{\sim} Hom(C_m/\{4T, FI\}, \mathbb{R})$$

We have almost shown that this mapping is well defined. To prove that for any $u \in V_m$ we have $\widehat{u} = 0$ on $4T$, take any $(m-1)-$ singular knot $K \in \mathcal{K}^{(m-1)}$ and consider $u^{(m-1)}(K)$. Pick a regular point and a self-intersection point on the knot. Isotope the knot so that the regular point and the self-intersection point are very close to each other and both lie in a 2 dimensional plane, with the self-intersection completely contained in the plane and the regular part crossing the plane transversely. If the knot is now isotoped so that the regular point traces a circle in the plane, moving around the self-intersection, the knot will end up back in its starting position after crossing $\mathcal{K}^{(m)}$ exactly four times. Hence the alternating sum of $u^{(m)}$ evaluated on each of the four $m-$singular knots adds up to zero. This shows that $\widehat{u}$ descends to the quotient by the $4T$ relation. To shows that $\widehat{u}$ descends to the quotient by the FI relation, represent a diagram with an isolated chord by a singular knot $K : S^1 \to \mathbb{R}^3$ such that the self intersection point $K(1) = K(-1)$ corresponding to the isolated chord is the unique point of $\mathbb{R}^3$ at which two $3-$balls B_1 and B_2 touch, with $K(S^1 \cap \mathrm{Im}(z) > 0) \subset B_1$ and $K(S^1 \cap \mathrm{Im}(z) < 0) \subset B_2$. Then $u^{(m)}(K)$ is equal to the difference of $u^{(m-1)}$ on the two possible $(m-1)-$singular knots obtained by resolving the self intersection $K(1) = K(-1)$. However, these two resolutions are isotopic and hence $u^{(m)}(K) = 0$.

We have shown that the map $V_m \to \mathrm{Hom}(C_m/\{4T, FI\}, \mathbb{R})$ is well defined and its kernel is clearly V_{m-1}. The hard part of the theorem (and our mission for the rest of this talk) is the proof of surjectivity. We will call functions $w : C_m/\{4T, FI\} \to \mathbb{R}$ weights. The rich algebraic structure related weights and their cousins allows Vassiliev theory to be understood through combinatorics, but this will not be discussed today.

The Linking number. For a two-component link $L : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$, the linking number is defined as

Link$(L) = \#($  $) - \#($  $)$

where in both cases we are counting crossings where the first component of the link is crossing under the second. This is the degree of a map $\deg(S^1 \times S^1 \to S^2)$ defined by considering $(z, w) \in S^1 \times S^1$ as a choice of point z in the first component of L and a choice of point w in the second component of L, then mapping (z, w) to the unit vector pointing from z to w in $\mathbb{R}^3$. Degrees of mappings can always be calculated by integrals, this is the Gauss formula for the linking number.

$$\mathrm{Link}(L) = \int_{S^1\times S^1} (\theta_{12} \circ C_2(L)|_{S^1\times S^1})^* \omega$$

where ω is an $SO(3)$-invariant volume form on S^2, θ_{12} is the map $C_2(\mathbb{R}^3) \to S^2$ sending $(p_1, p_2) \mapsto (p_1 - p_2)/|p_1 - p_2|$ and $C_2(L) : C_2(S^1 \sqcup S^1) \to C_2(\mathbb{R}^3)$ is the functorial extension of the embedding L to the configuration spaces of two points:

$$\begin{array}{ccc} C_2(S^1 \sqcup S^1) & \longrightarrow & C_2(\mathbb{R}^3) \\ \uparrow & & \downarrow{\scriptstyle \theta_{12}} \\ S^1 \times S^1 & & S^2 \end{array}$$

Let us attempt to repeat this strategy for knots (links with one component). Then, given $K : S^1 \to \mathbb{R}^3$ we consider $\theta_{12} \circ C_2(K) : C_2(S^1) \to S^2$. Try to define the self-linking number

$$\int_{C_2(S^1)} (\theta_{12} \circ C_2(K))^* \omega.$$

Problem: The space $C_2(S^1)$ is not compact.

Solution: We have a manifold compactification $C^2[S^1]$ to which the map $\theta_{12} \circ C_2(K)$ extends smoothly (the extension is the Gauss map sending $z \in S^1$ to the unit vector tangent to K at $K(z)$).

Problem: But $C^2[S^1]$ has boundary.

Solution: Since $\partial C^2[S^1] \neq \varnothing$ there is no reason for that integral to be a knot invariant. Indeed it is not.

Note that in some vague and imprecise sense the mapping $\theta_{12} \circ C_2(K)$ corresponds to the chord diagram that has one unique chord pairing two distinct points. This chord is isolated, hence zero by FI. The framing independence prevents such an invariant to exist. This can be fixed by the introduction of framings, but today we are interested in unframed knots.

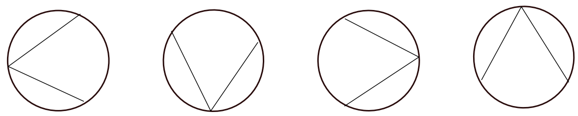
The first Bott-Taubes integral. Continuing on the heuristics in the last paragraph, we seek the first nontrivial weight $w : C_m/\{4T, FI\} \to \mathbb{R}$. For $m = 2$ the space of chords modulo relations is one dimensional generated by the weight that sends $\otimes$ to 1. Consider

$$\begin{array}{ccc} C_4[S^1] \times \mathcal{K} & \xrightarrow{\alpha_{ij}} & S^2 \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{K} & & \end{array}$$

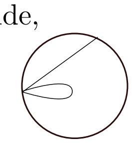
where $\alpha_{ij}\big((z_1, z_2, z_3, z_4), K\big) = \theta_{ij} \circ C_4(K)(z_1, z_2, z_3, z_4)$ and $\theta_{ij} : C_4[\mathbb{R}^3] \to \mathbb{R}$ is defined by $\theta_{ij}(p_1, p_2, p_3, p_4) = (p_i - p_j)/|p_i - p_j|$ on $C_4(S^1)$. For our purposes the fundamental property that makes the manifold theoretic compactifications of configuration spaces important is that the maps θ_{ij} extend smoothly to the boundary. In some vague and imprecise sense the weight under consideration corresponds to the form $\tau_{13} \wedge \tau_{24}$, where $\tau_{ij} = \alpha_{ij}^* \omega$. This is a $4-$form on $C_4[S^1] \times \mathcal{K}$, but one can integrate along the ($4-$dimensional) fibres of π to get a $0-$form $\pi_*(\tau_{13} \wedge \tau_{24})$ on $\mathcal{K}$. For a $0-$form (i.e a function) on $\mathcal{K}$ to be a knot invariant we need it to be locally constant, in other words its exterior derivative must be zero. However, because of Stokes' theorem:

$$d\pi_*(\tau_{13} \wedge \tau_{24}) = \pi_*(d\tau_{13} \wedge \tau_{24}) - (\partial\pi)_*(\tau_{13} \wedge \tau_{24}) = -(\partial\pi)_*(\tau_{13} \wedge \tau_{24}).$$

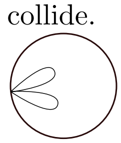
where $\partial\pi : (\partial C_4[S^1]) \times \mathcal{K} \to \mathcal{K}$ is the restriction of π to the boundary. Now, $\partial C_4[S^1]$ is naturally stratified and so we must understand how the different strata (also called faces) contribute to the pushforward of $\tau_{13}\wedge\tau_{24}$ along the fibres of $\partial\pi$. We will think of a picture like  as encoding the location of four points on the circle with each segment representing a map θ_{ij} associated to a pair of points (in this case θ_{13} and θ_{24}). In the boundary of configuration space, if any of the points in  collide we can represent the resulting configuration by a similar (degenerate) picture. For example, consider the principal faces, where only two points collide,



the hidden faces where three points collide,



and the anomalous face, where all points collide.



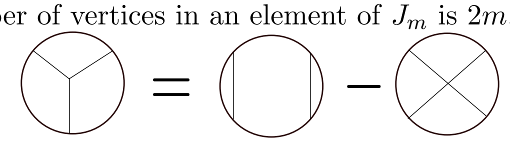
Claim 2.3.8. *In the formula*

$$d\pi_*(\tau_{13} \wedge \tau_{24}) = -\int_{\partial C_4[S^1]} \tau_{13} \wedge \tau_{24} \in \Omega^1(\mathcal{K})$$

only the principal faces give nonzero contributions to the integral.

Proof. The essential observation is that only faces whose image under the map $\alpha_{13}\times\alpha_{24} : C_4[S^1] \to S^2 \times S^2$ have dimension 3 can have nonzero contributions. In the hidden face, the infinitesimal arrangements don't contribute, so the image is at most $2-$dimensional. In the anomalous face, the two maps α_{13} and α_{24} agree, so the mapping $\alpha_{13} \times \alpha_{24}$ factors through the diagonal $S^2 \to S^2 \times S^2$ and is therefore also at most $2-$dimensional. $\square$

Nevertheless, the principal faces do yield nonzero contributions and so we have not constructed a knot invariant yet. To cancel these contributions we are going to draw some inspiration from the land of Jacobi diagrams. Let the vector space of Jacobi diagrams J_m with grading m be generated by chord diagrams in which we allow internal vertices to be trivalent. The total number of vertices in an element of J_m is $2m$. The STU relation is



The following isomorphism holds

$$C_m/4T \xrightarrow{\sim} J_m/STU.$$

Given a semi-simple Lie algebra and a representation, these roads lead to a weight system on chords. If you integrate this using the Kontsevich integral defined below, you get the quantum invariants obtained by quantizing the Lie algebra and solving the quantum Yang-Baxter equation. Tri-valent vertices correspond to the fact that the Lie bracket is tri-valent and the various relations correspond to the Jacobi relation and the bracket/commutator relation.

Now our weight extended to Jacobi diagrams assigns -1 to the unique Jacobi diagram of grading 2 that has a trivalent vertex. By interpreting this diagram in terms of configuration spaces, we define $C_{4,3}[\mathbb{R}^3, K]$ to be the ordered configuration space of 4 points on $\mathbb{R}^3$, 3 of which are restricted to lie on the image of K. We can give manifold theoretic compactifications of these space as in the simpler cases we have already discussed. We still have maps

$$\begin{array}{ccc} C_{4,3}[S^1,\mathcal{K}] & \xrightarrow{\widetilde{\alpha}_{ij}} & S^2 \\ \Big\downarrow{\scriptstyle \widetilde{\pi}} & & \\ \mathcal{K} & & \end{array}$$

so that we can define $\widetilde{\tau_{ij}} = \widetilde{\alpha_{ij}}^*\omega$.

Theorem 2.3.9.

$$\frac{1}{4}\pi_*(\tau_{13}\wedge\tau_{24}) - \frac{1}{3}\widetilde{\pi}_*(\widetilde{\tau}_{14}\wedge\widetilde{\tau}_{24}\wedge\widetilde{\tau}_{34})$$

is a Vassiliev knot invariant of type 2 *whose symbol maps* $\bigotimes \mapsto 1$.

The reason is that if one computes the exterior derivative of the $0-$ form $\widetilde{\pi}_*(\widetilde{\tau}_{14}\wedge\widetilde{\tau}_{24}\wedge\widetilde{\tau}_{34})$ the Stokes formula yields an integral over the fibres of $\partial\widetilde{\pi} : \partial C_{4,3}[\mathbb{R}^3,\mathcal{K}] \to \mathcal{K}$ which as before vanish over the hidden faces and over the anomaly. Hence only principal faces contribute. But all the principal faces are essentially the same as the principal faces that we dealt with when we were studying $d\pi_*(\tau_{13}\wedge\tau_{24})$. The integrals agree. Moreover, in the former case we had 4 such faces and in this latter case we have 3. Hence once we add the weights 1/4 and 1/3, the difference of the two $0-$forms is closed and we obtain a knot invariant.

The Universal Bott-Taubes integral. This strategy can be massively generalized.

Theorem 2.3.10. *Suppose* w *is a weight* $C_m/\{4T, FI\} \to \mathbb{R}$. *Then*

$$K \mapsto \sum_{D\in J_m} w(D)\tau_D(K)/|\operatorname{Aut}(D)| + \textit{anomaly}$$

is a Vassiliev knot invariant of type m *whole symbol corresponds to* w.

If a Jacobi diagram $D \in J_m$ has n univalent vertices on the circle and $m - n$ trivalent internal vertices, then $\tau_D(K)$ will be the result of wedging together the pullbacks of ω by a bunch of maps $C_{m,n}[\mathbb{R}^3,\mathcal{K}] \to S^2$ corresponding to the chords of D, then integrated along the fibres of the map $\mathbb{C}_{m,n}[\mathbb{R}^3,\mathcal{K}] \to \mathcal{K}$, yielding a function $\tau_D(K) \in \Omega^0(\mathcal{K})$. One verifies that the Universal Bott-Taubes integral constructs a knot invariant by computing its exterior derivative. The boundary terms can be grouped in a way analogous to what we did in the previous section in order to show that the exterior derivative of the various τ_D vanish in groups of 3 by the STU relations (after dividing by the possible symmetries). The anomaly is a subtler term that is not know to vanish.

The punchline is that there is a systematic way to use a weight system to balance various integrals on configuration space in such a way that a $0-$form is obtained whose exterior derivative is zero. This is a degree zero cohomology class, i.e a knot invariant. A more careful analysis of the construction shows that the invariant obtained is Vassiliev of finite type m and in fact its symbol is the weight that we started with. Hence the Universal Bott-Taubes integral provides an inverse to the isomorphism of the Fundamental Theorem of Vassiliev Theory.

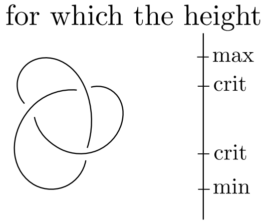
The Kontsevich integral. Why did we take ω to be the $SO(3)$-invariant volume form on S^2? No reason. For example, one can get the combinatorial formula for the linking number by considering instead a 2-form degenerating to a delta function on the north pole. In this section we will use a form that is concentrated on the equator. This leads to the Kontsevich integral.

Think of $\mathbb{R}^3 = \mathbb{C}z \times \mathbb{R}t$ where t is a Morse function on the knot K. Define an integral with values in the completion of $\bigoplus_m C_m$ modulo $4T, FI$.

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{min}<t_1<\cdots<t_{max} \\ t_i \text{ noncritical}}} \sum_{P=(z_i,z_i')} (-1)^{\downarrow P} D_P \bigwedge_{i=0}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \in \widehat{C}/\{4T, FI\}.$$

At height t_1, choose a pairing of points z_1, z_1', at height t_2, choose a pair of points z_2, z_2' etc. This is a pairing P. The diagram D_P is the chord diagram that corresponds to this pairing. The sign $(-1)^{\downarrow P}$ counts the total number of strands through the points z_i, z_i' pointing downwards. The $m-$th integral in the sum is over a subset of the $m-$simplex simplex $t_{min} \leq t_1 \leq \cdots \leq t_m \leq t_{max}$, where we are integrating the pullback of the Arnold forms (generating the cohomology of configuration spaces of points in $\mathbb{C}$) by the maps corresponding to the choices z_i, z_i'.

Example 2.3.11. This is a Trefoil for which the height function t is Morse.



The integral $Z(K)$ does not change if K varies through Morse knots. To see this, first we prove invariance under deformations which do not move the critical points and which preserve the level sets of the height function t. Define the formal KZ connection

$$\Omega_m = \sum_{i<j} D_{ij}\ \omega_{ij}$$

to be a $1-$form on the configuration space $C_{2m}(\mathbb{C})$ with values in a strand version of the space of Jacobi diagrams of degree m. These diagrams are graphs with univalent vertices lying on the picture $\uparrow \ldots^m \uparrow \downarrow \ldots^m \downarrow$ and trivalent interior vertices. The diagram D_{ij} has one unique chord connecting a point on the i−th upwards pointing strand in the picture $\uparrow \ldots^m \uparrow\downarrow \ldots^m \downarrow$ to the j−th downwards pointing strand. The form ω_{ij} is the Arnold form $dz_i - dz_j/(z_i - z_j)$.

Claim 2.3.12. *The Arnold relations on the forms ω_{ij} and the STU relations on the Jacobi diagrams D_{ij} imply that this connection is flat, i.e. $F_\Omega = d\Omega + \Omega \wedge \Omega = 0$.*

Flatness of this connection means that the holonomy of a path in $C_{2m}(\mathbb{C})$ only depends on the endpoints on the path. Tracing the definitions this translates into $Z(K)$ being invariant under horizontal deformations. By factoring non-horizontal deformations into deformations concentrated in a neighborhood of the critical points (needles) followed by horizontal deformations, one can show that $Z(K)$ does not change as K varies through Morse knots. However, the birth or death of two critical points of the height function t, the integral $Z(K)$ will change. Hence we have to normalize in order to get an invariant. The full Kontsevich integral is

$$\widehat{Z}(K) = \frac{Z(K)}{Z(\infty)^{c/2}}$$

where c is the number of critical points of the function t on K. This is a Universal Vassiliev invariant. Indeed, given a weight $w : C_m/\{4T, FI\} \to \mathbb{R}$, one can hit the diagrams in the formula of the Kontsevich integral with w to obtain a knot invariant $K \mapsto w(\widehat{Z}(K)) \in \mathbb{R}$. One can check that this is a Vassiliev invariant of order m, with symbol w. Thus we obtain another inverse for the isomorphism in the Fundamental Theorem of Vassiliev Theory. It has been shown that the Kontsevich and the Bott-Taubes integrals agree, provided that they anomaly vanishes.

Talk 2.4: Embedding calculus (Jeremy Mann)

Embedding calculus, sometimes called manifold calculus, is one flavor of Goodwillie calculus.

Setup: if M and N are smooth manifolds of dimensions $m < n$, we're interested in the space of embeddings $\mathrm{Emb}(M^m, N^n)$. If $m = 1$ and $n = 3$, this is knot theory.

View this as a functor $F = \mathrm{Emb}(-, N) : \mathcal{O}^{op} \to \mathrm{Top}$, where $\mathcal{O}$ is the category of open sets of M. The problem is that this is not a sheaf: being an embedding isn't a local property in the source.

The idea is to approximate F by functors which are "locally determined" (but really, I want "multi-locally determined"). Why is this a useful approximation? A manifold is something that is locally trivial. If they're locally determined, they're designed to exploit nice properties (e.g. handle decomposition) of manifolds. This will allow us to get the homotopy type of the embedding space within a specific range. If we're lucky, this will converge.

Question: approximate in what sense?

Answer: in the sense of Taylor calculus. There's a creepily strong analogy between embedding calculus and Taylor calculus.

Taylor calculus	Embedding calculus
$f : \mathbb{R} \to \mathbb{R}$	$F : \mathcal{O}^{op} \to \mathrm{Top}$
polynomials	polynomial functors
determined by value at 0 and k distinct points	determined by value on $\emptyset$ and $\leq k$ disjoint balls
Taylor series	Taylor tower $T_nF \to T_{n+1}F \to \dots$
$g'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$	$F'(\emptyset) = \mathrm{hofib}(F(B) \to F(\emptyset))$
truncation $f - T_kf$	maps $F \to T_kF$
$L_kf = T_kf - T_{k-1}f$	$L_kF = \mathrm{hofib}(T_kF \to T_{k-1}F)$
$\lvert f - T_kf\rvert \to 0$ as $k \to \infty$?	$\pi_*(\mathrm{hofib}_{k\to\infty}(F \to T_kF)) \to 0$?
$f = \lim_{k\to\infty} T_kf$?	$F \xrightarrow{\sim} \mathrm{holim}_{k\to\infty} T_kF$?

Why is the homotopy fiber a like the norm of the difference? If the homotopy fiber is contractible, the spaces are the same.

Definition 2.4.1. $F : \mathcal{O}^{op} \to \mathrm{Top}$ is *good* if

- it takes isotopy equivalences to homotopy equivalences
- For $\dots \leq V_i \leq \dots$ with $V_i \in \mathcal{O}$ such that $V = \bigcup_i V_i$, we have $F(V) \xrightarrow{\sim} \mathrm{holim}\, F(V_i)$ (F is determined by its value on the interior of compact submanifolds).

What is a linear functor? They had better be computable. They should also be determined by their value at $\emptyset$ and some ball B.

Given $V \in \mathcal{O}$, closed and disjoint $A_0, A_1 \subset V$, I'm going to talk about excision. "We're going to be puncturing holes according to A_0 and A_1." If you're only puncturing two holes, you only need a 2-dimensional diagram to describe this.

$$\begin{array}{ccc} V & \longleftarrow & V - A_1 \\ \uparrow & & \uparrow \\ V - A_0 & \longleftarrow & V - (A_0 \cup A_1) \end{array} \qquad (2.4.1)$$

But if you're puncturing more holes, then you need a higher-dimensional diagram.

Definition 2.4.2. F is linear (or polynomial of degree ≤ 1) if

$$F(V) \xrightarrow{\sim} \mathrm{holim}_{S\subset\{0,1\}} F(V \setminus \cap_{i\in S} A_i).$$

This is saying that F is a homotopy sheaf. It takes homotopy pushouts into homotopy pullbacks. Homology does this, as does stable homotopy.

Example 2.4.3. $\mathrm{Top}(-, X)$ is linear.

Example 2.4.4. $\mathrm{Imm}(-, N)$ is linear. This looks obvious (being an immersion is a local property) but it's actually not obvious because we're asking about it being a *homotopy* sheaf. It's actually highly nontrivial.

It is a theorem that $\mathrm{Imm}(M, N) \hookrightarrow \mathrm{mono}(TM, TN)$ is a weak equivalence; this is *implied by* the fact that $\mathrm{Imm}(-, N)$ is linear.

Let $P_k = P(\{0, \ldots, k\})$ (power set on the set $\{0, \ldots, k\}$).

Definition 2.4.5. A $(k+1)$-cube of spaces is a functor $\chi : P_* \to \mathrm{Top}$.

For example, if N is a manifold with boundary and metric, look at the k-cube $E_k(N)$ defined by $S \mapsto E_S(N) :=$ the space of embeddings of $I - \bigcup_i J_i$ into N that send 0 and 1 to designated points on ∂N, and the speed is constant in each component.

More generally, define $C'_\ell(N)$ as the pullback of

$$\begin{array}{ccc} C'_\ell(N) & \longrightarrow & (STN)^\ell \\ \downarrow & & \downarrow \\ C_\ell(N) & \longrightarrow & N^\ell \end{array}$$

If $F : \mathcal{O}^{op} \to \mathrm{Top}$ and we have $V, A_1, \ldots, A_k$ then consider the $(k+1)$-cube
$$S \mapsto F(V \setminus \cap_{i \in S} A_i).$$
If $k = 2$, then this can be represented as (2.4.1).

Definition 2.4.6. F is polynomial of degree $\leq k$ if $F(V) \to \mathrm{holim}_S F(V \setminus \bigcup_{i \in S} A_i)$ is an equivalence.

Definition 2.4.7. The total fibre of χ is
$$tfib\, \chi = \mathrm{hofib}(\chi(\emptyset) \to \mathrm{holim}_{S \neq \emptyset} \chi(S)).$$

Definition 2.4.8. $F^{(k)}(\emptyset) = tfib(*)$.

Example 2.4.9. $\mathrm{Top}((-)^k, X)$

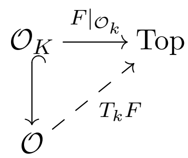
Example 2.4.10. $V \mapsto \mathrm{Top}(\binom{V}{k}, X)$ is polynomial of degree $\leq k$. *What does this notation mean?*

Definition 2.4.11. Define $\mathcal{O}_k \subset \mathcal{O}$ to be the subcategory whose objects are disjoint unions of at most k balls.

Theorem 2.4.12. *Suppose E, F are polynomial of degree $\leq k$. If $\omega : E \to F$ such that $\omega|_{\mathcal{O}_k}$ is an equivalence, then ω is an equivalence.*

We want to take a good functor and associate to it a polynomial of degree $\leq k$ functor, and do it in a way that allows me to compare the two.

Define $T_k F$ as a homotopy right Kan extension.



I get a natural transformation between the two ways of going around this diagram.

$$T_k F(W) = \operatorname{holim}_{\substack{V \subset W \\ V \in \mathcal{O}_k}} F(V)$$

Theorem 2.4.13.

(1) $T_k F$ is polynomial of degree $\leq k$.
(2) If F is polynomial of degree $\leq k$, then $T_k F \simeq F$.
(3) $T_k F \xrightarrow{T_k(\eta_k)} T_k T_k F$ is an equivalence (i.e. applying T_k twice doesn't change anything).

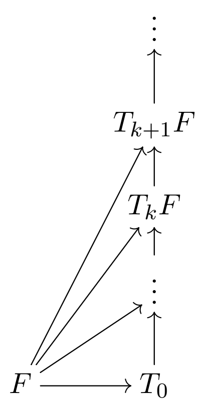
Example 2.4.14. $\mathrm{Imm}(B, N) \simeq \mathrm{Emb}(B, N)$. Note that immersions are a local property. You can show that $T_1 \mathrm{Emb}(-, N) = \mathrm{Imm}(-, N)$.

Example 2.4.15.

$$\begin{array}{ccc} T_1\mathrm{Emb}_\partial(I, N) & \longrightarrow & \mathrm{Emb}(I - *, N) \\ \downarrow & & \downarrow \\ \mathrm{Emb}^\partial(I - *, N) & \longrightarrow & \mathrm{Emb}(I - **, N) \end{array}$$

(∂ means the intervals have to get sent to fixed things) *still confused about this...*

Inclusions $\mathcal{O}_{k-1} \hookrightarrow \mathcal{O}_k$ give rise to the Taylor tower:



We want the connectivity of $T_kF \to T_{k+1}F$ to go to infinity as $k \to \infty$ (this is analogous to "the remainder going to zero" in Taylor calculus).

Theorem 2.4.16. *L_kF is homogenous of degree k, i.e. $T_{k-1}L_k(V) \simeq *$ is polynomial of degree $\leq k$.*

You should be thinking of $L_kf = f'(0)\frac{x^k}{k!}$. The fibers are equivalent to $F^{(k)}(\emptyset)$.

Theorem 2.4.17. *Take M, N as before but assume $n - m \geq 3$. (These estimates don't work in the case of knot theory.) Let $F(V) = \mathrm{Emb}(V, N)$. Then $F(V) \to T_kF(V)$ is $(k(n-m-2)-1-m)$-connected. In particular, it $\to \infty$ as $k \to \infty$.*

This is not known in the classical knot case. Surgery arguments break down if the codimension is too small.

Theorem 2.4.18. *Let $\dim N \geq 4$. $\mathrm{Emb}^\partial(I, N) \to T_k\mathrm{Emb}^\partial(I, N)$ is $(k-1)(\dim N - 3)$-connected.*

Theorem 2.4.19. *$\mathrm{Emb}^2(I, N) \to \mathrm{holim}_k\, T_k\mathrm{Emb}(I, N)$ is an equivalence. This is also $\simeq \mathrm{holim}_k\, \mathrm{holim}_S\, E_S(N)$.*

(If we're taking a holim, we don't want the indexing category to have a terminal object, because then the holim is the terminal object.)

The proof is a clever combination of transversality, general position, and dimension-counting arguments.

The kth approximation is sort of saying "what's the closest I can get to embedding with only looking at k points at a time."

Talk 2.5: Cosimplicial models for spaces of long links and long knots (Kim Nguyen)

What we've seen today:

- Fulton-MacPherson compactification $C_n[M]$
- Embedding calculus $\mathrm{Emb}^\partial(I, M) = \mathrm{holim}(T_0\mathrm{Emb}(I, M) \leftarrow T_1\mathrm{Emb}(I, M), \dots)$

Plan for this talk:

- Mapping space model
- Cosimplicial models for $\mathrm{Emb}(I, M)$
- Some spectral sequences

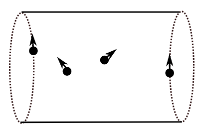
Emb^∂ meant you fix two endpoints and two tangent vectors. We'll have to modify the Fulton-MacPherson operad to account for this. Consider a subspace $C_n[M, \partial] \subset C_{n+2}[M]$: we're given two points in the boundary, and the subspace of the configuration space consists of these extra two points $(y_0, x_1, \ldots, x_n, y_1)$. We also have tangent vectors we want to get in the game.

Define $C'_n(M)$ as the pullback

$$\begin{array}{ccc} C'_n(M) & \longrightarrow & TM^{\times n} \\ \downarrow & & \downarrow \\ C_n(M) & \longrightarrow & M^{\times n} \end{array}$$

Definition 2.5.1. Let $AM_n(M)$ denote the space of aligned maps $\widetilde{C}^1_k[I, \partial] \to C'_n[M, \partial]$, where $\widetilde{C}^1_k[I, \partial]$ is the component of those configurations where the order agrees with the order in I. *I'm not following the discussion of what an aligned map is.*

Recall: elements of $C^1_T[M, \partial]$ look like



I want to describe a map $\mathrm{Emb}(I, M) \to AM_n(M)$. Start with an embedding f; this gets sent to a map $C_k(I, \partial) \to C_k(M, \partial)$ sending $(x_1, \ldots, x_k) \mapsto (f(x_1), \ldots, f(x_k))$.

Theorem 2.5.2. $AM_k(M) \simeq T_k\mathrm{Emb}(I, M)$

Rough sketch. $AM_k(M) \simeq \mathrm{holim}_{P_0(\underline{n+1})} D_k[M]$ where $D_n[M]$ is a functor $P_0(n+1) \to$ Top sending $S \mapsto C'_{\#S-1}[M, \partial]$. Induced maps are given by "diagonals." Here $P_0(\underline{n+1})$ is the power set poset, where 0 means that the empty set is not allowed.

$D_n[M]$ is levelwise weakly equivalent to the cubical functor $E_k(M)$ that punctures the knot in k places. □

Cosimplicial model. We want a cosimplicial space $X : \Delta \to \mathrm{Top}$ such that $\mathrm{Tot}\, X^f \simeq \mathrm{Emb}(I, M)$. Totalizations are dual to geometric realizations. One way to write it down is as the space of natural transformations $Nat(\Delta^\bullet, X)$. I need to take the fibrant replacement, because I need things to be homotopy-invariant. Recall that geometric realizations are not homotopy-invariant unless things are "really cofibrant".

There's a way to go from a cubical diagram to a cosimplicial diagram.

Definition 2.5.3. Define $G_n : P_0(\underline{n+1}) \to \Delta_n \subset \Delta$ by sending $S \mapsto [\#S - 1]$, and $S \subset S' \mapsto [\#S - 1] \cong S \subset S' \cong [\#S' - 1]]$

Theorem 2.5.4. *Given* $X : D \to \mathrm{Top}$*, then we have* $\mathrm{Tot}^f_n\, i_* X \simeq \mathrm{holim}_{P_0(n+1)} X \circ G_n$.

We get a tower $\mathrm{Tot}_0 X \leftarrow \mathrm{Tot}_1 X \dots$ where $\mathrm{Tot}_n X = Nat(i_*\Delta, i_n X)$. n^{th} truncation is composition with $i_* : \Delta_n \hookrightarrow \Delta$.

Sketch of proof. We need two ingredients:

- Quillen's theorem A: suppose we have $F : C \to D$ such that $|F \downarrow d| \simeq *$ for all $d \in D$ (this condition is called (left?) cofinal). Then for $G : D \to \mathrm{Top}$, $\mathrm{holim}_D G \simeq \mathrm{holim}_C G \circ F$.
- $\mathrm{holim}_{\Delta_n} i_n X \simeq \mathrm{Tot}_n X^f$.

We need to show that G_n is *(left? right?)* cofinal. Exercise: find a nice simplicial complex that is $\simeq$ the geometric realization of this comma category, and show that this is contractible.

$\mathrm{Tot}_1 X = \mathrm{Nat}(i\Delta, i_1 X)$

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{f_0} & X_0 \\ \updownarrow & & \downarrow \\ \Delta^1 & \xrightarrow{f_1} & X_1 \end{array}$$

Exercise: show that this is a weak equivalence: $\mathrm{Tot}_n X \simeq \mathrm{holim}(X_0 \overset{d_0}{\to} X_1 \overset{d_1}{\leftarrow} X_0)$. ☐

The cosimplicial model is $C^* [M] : \Delta \to \mathrm{Top}$ sending $[n] \mapsto C'_n [M, \partial]$; the cosimplicial maps are "doubling" and "forgetting."

We have $D_n [M] = i_n C^* [M] \circ G_n$, so

$$\mathrm{holim}_{P_0(n+1)} D_k [M] \simeq \mathrm{holim}_{\Delta_n} i_k C^* [M] \simeq \mathrm{Tot}_n C^* [M] .$$

Conclusion:

$$\mathrm{Tot}\, C^\bullet [M]^f \simeq \mathrm{Emb}(I, M).$$

Now that we have a cosimplicial space, we can write down spectral sequences.

Suppose we have a tower of fibrations

$$\begin{array}{ccccccc} X_0 \longleftarrow & X_1 & \longleftarrow & X_1 & \longleftarrow X_2 \longleftarrow \dots \\ & \uparrow & & \uparrow & \\ & F_0 & & F_1 & \end{array}$$

Apply homotopy groups. Since these are fibrations, we get connecting maps

$$\begin{array}{ccccccc} \pi_*(X_0) & \longleftarrow & \pi_*(X_1) & \longleftarrow & \pi_*(X_2) & \longleftarrow & \dots \\ & \searrow^{\partial} & \uparrow & \searrow^{\partial} & \uparrow & & \\ & & \pi_*(F_0) & & \pi_*(F_1) & & \end{array}$$

This gives rise to an exact couple, and from this a spectral sequence. For a cosimplicial space $X : \Delta \to \mathrm{Top}$ we get a spectral sequence associated to the tower of fibrations

$$\mathrm{Tot}_0 X^f \leftarrow \mathrm{Tot}_1 X^f \leftarrow \mathrm{Tot}_2 X^f \leftarrow \dots$$

so you get a spectral sequence for X which converges (sometimes) to $\pi_*(\mathrm{Tot}\, X^f)$.

This also works for homology. The forgetful functor $TopAb \to Top$ has a left adjoint $\mathbb{Z}$ (free topological abelian group). This has the property that $\pi_*(\mathbb{Z}X) \cong \widetilde{H}_*(X, \mathbb{Z})$ (Dold-Thom).[1] Make a new cosimplicial space $\Delta \xrightarrow{X} \text{Top} \xrightarrow{\mathbb{Z}} \text{Top}$, and take the homotopy spectral sequence to compute the homology of your original cosimplicial space.

But there's a subtlety: in principle when we compute this, you want $\pi_*(\mathbb{Z}\,\text{Tot}\,X)$ instead of $\pi_*(\text{Tot}\,\mathbb{Z}X)\ldots$

Theorem 2.5.5. *There is a spectral sequence converging to* $\pi_*(\text{Emb}(I, M))$ *with* E_1*-page*

$$E_1^{-p,q} = \bigcap \ker s_*^j \subset \pi_\varphi(C_p'\,[M, \partial])$$

(where the s_j*'s are the degeneracies) with differential* $d_1 = \sum(-1)^i d_*^i$ *(alternating sum of coface maps). There is a spectral sequence converging to* $H^*(\text{Emb}(I, M))$ *where*

$$E_1^{-p,q} = \text{coker} \sum (s_i)^* : H^q(C_{p-1}'\,[M, \partial]) \to H^q(C_p'\,[M, \partial])$$

and the differential is given by $d_1 = \sum(-1)^i(d^i)^*$*. This is the Sinha spectral sequence.*

[1] Qiaochu mutters something about basepoints being needed.

Day 3: Formality and its consequences

Talk 3.1: Formality of models for spaces of long knots
(John Berman)

This talk is about the rational homotopy type of the space of long knots.

Let's first recall some notation. If $X^\bullet$ is a cosimplicial space, then the homotopy totalization $ho\,\mathrm{Tot}(X^\bullet)$ is equivalently the fibrant replacement $\mathrm{Tot}(X^\bullet)^f$, or $\mathrm{holim}(\Delta \xrightarrow{X} \mathrm{Top})$.

Recall from yesterday that compactified configuration spaces $C^*[M]$ give rise to a cosimplicial space, which induces spectral sequences converging to $H_*\mathrm{Emb}(I, I^d)$, $H^*\mathrm{Emb}(I, I^d)$, or $\pi_*\mathrm{Emb}(I, I^d)$, depending on what we want to calculate.

Umut talked about formality of D_n with real coefficients.[2] This implies that these spectral sequences collapse with $\mathbb{Q}$-coefficients, provided $d \geq 4$. Unfortunately, most knot theorists like to set $d = 3$, but then the Goodwillie tower may not converge to the right space, and the spectral sequence may not converge.

If you have never worked with spectral sequences, this one may be a little daunting. The best thing we can possibly hope for is that there are no differentials beyond the E^r page. In this case, $E^r = E^\infty$. We're working with vector spaces, so there are no extensions to resolve, at least if we are only interested in additive structure.

Theorem 3.1.1 (Sinha, Lambrechts-Volic-Turchin). *The Sinha spectral sequence for rational homology collapses at the E_2 page, and*

$$H_*(\Omega^2 S^{d-1}, \mathbb{Q}) \otimes H_*(\mathrm{Emb}(I, I^d); \mathbb{Q}) = HH_*\mathrm{Pois}_{d-1}.$$

We know that the spectral sequence arises from configuration spaces, so we might ask if it arises from the little disks operad. Indeed, it does! Plan:

- Show that multiplicative operads give rise to a cosimplicial space;
- The Fulton-MacPherson operad gives rise to a cosimplicial space, and hence a Sinha spectral sequence;
- Formality of the Fulton-MacPherson operad implies that the cosimplicial space is formal as a diagram, and hence the spectral sequence collapses.

Then we will try to compute the rational homology and homotopy of the embedding space.

Definition 3.1.2. A non-Σ operad (i.e. no action of Σ_n) $\mathcal{O}$ is multiplicative when it comes with an operad map $\mathrm{Ass} \to \mathcal{O}$.

In particular, the map $\mathrm{Ass}(n) \to \mathcal{O}(n)$ gives preferred multiplications.

[2] It turns out that D_n is formal with $\mathbb{Q}$-coefficients, but that's extremely recent.

Given a multiplicative operad $\mathrm{Ass} \to \mathcal{O}$, we get a cosimplicial space

$$\mathcal{O}(0) \rightrightarrows \mathcal{O}(1) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \mathcal{O}(2) \qquad \dots \tag{3.1.1}$$

Where do the maps come from? We have preferred elements $m \in \mathcal{O}(2)$ and $e \in \mathcal{O}(0)$. Given $x \in \mathcal{O}(2)$, you get four things in $\mathcal{O}(3)$: $m(-, x(-,-))$, $m(x(-,-),-)$, $x(m(-,-),-)$, $x(-, m(-,-))$, etc. Similarly, you can get two things in $\mathcal{O}(1)$ by $x(e,-)$ and $x(-,e)$. In general, an element $x \in \mathcal{O}(n)$ induces $n+2$ elements of $\mathcal{O}(n+1)$ given by multiplication on the left or right outside of x, or multiplication of two adjacent inputs to x. It also induces n elements of $\mathcal{O}(n-1)$ given by inputting one identity $e \in \mathcal{O}(0)$.

Unfortunately, D_d is not quite multiplicative (it is only multiplicative up to homotopy). Possible fixes:

(1) take a strict multiplicative replacement. This will be the Kontsevich operad. Recall that the FM operad kept track of direction and relative distance. We will just keep track of direction now. Consider the map $f : \mathrm{Conf}(n, \mathbb{R}^d) \to (S^{d-1})^{\binom{n}{2}}$ sending $\frac{(x_1,\dots,x_n) \mapsto \{x_i - x_j}{|x_i - x_j|\}_{ij}}$. Define $K_d(n) = \overline{\mathrm{im}(f)}$. This forms an operad, called the Kontsevich operad, which is an E_d-operad. The map $\mathrm{Ass} = K_1 \to K_d$ gives rise to a multiplicative structure. The Bousfield-Kan spectral sequence is, in this case, the Sinha spectral sequence.

(2) In order to prove collapse, FM_d is not multiplicative, but there is a map of ∞-operads $\mathrm{Ass} \to FM_d$. This induces an ∞-map $N\Delta \to \mathrm{Top}$. (Basically, just use ∞-language instead of trying to get strict models for things.) This still requires some work to make rigorous, and I am not aware of anywhere it is written up.

The spectral sequence is

$$E^1_{-p,q} = H_q(FM_d(p), \mathbb{Q}) \implies H_*(\overline{\mathrm{Emb}}(I, I^d), \mathbb{Q}).$$

We need to take a quick digression on this $\overline{\mathrm{Emb}}(I, I^d)$. Define

$$\overline{\mathrm{Emb}}(I, I^d) := \mathrm{hofib}\big(\mathrm{Emb}(I, I^d) \xrightarrow{f} \mathrm{Imm}(I, I^d) \simeq \mathrm{Hom}(S^1, S^{d-1}) = \Omega S^{d-1}\big).$$

The map $\mathrm{Emb}(I, I^d) \to \Omega S^{d-1}$ takes $g \mapsto (t \mapsto g'(t))$ (take the unit tangent vector). This is nullhomotopic: a nullhomotopy comes from

$$\Delta^2 \times \mathrm{Emb}(I, I^d) \to \Omega S^{d-1} \text{ taking } (t_0 \le t_1), g \mapsto \frac{g(t_1) - g(t_0)}{|g(t_1) - g(t_0)|}.$$

So $\overline{\mathrm{Emb}}(I, I^d) = \mathrm{Emb}(I, I^d) \times \Omega^2 S^{d-1}$, which is no longer so intimidating.

Recall that the homology of an E_d-operad is the Poisson $(d-1)$-operad. The equatorial embedding $D_1 \hookrightarrow D_d$ induces a canonical map $\mathrm{Ass}_{\mathrm{Vect}_k} = H_*(D_1, \mathbb{Q}) \to H_*(D_d, \mathbb{Q}) = \mathrm{Pois}_{d-1}$, so Pois_{d-1} is a multiplicative operad, and hence we get a cosimplicial vector space $\mathrm{Pois}_{d-1}(0) \to \dots$ as in (3.1.1).

The Dold-Kan correspondence says that there is a Quillen equivalence of model categories between cosimplicial $\mathbb{Q}$-vector spaces and (positive-degree) cochain complexes over $\mathbb{Q}$. The associated cochain complex to the Poisson cosimplicial object above looks like

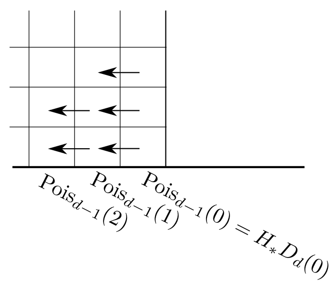
$$\mathrm{Pois}_{d-1}(0) \xrightarrow{d_0 - d_1} \mathrm{Pois}_{d-1}(1) \xrightarrow{d_0 - d_1 + d_2} \dots$$

Definition 3.1.3. Hochschild homology $HH_*\text{Pois}_{d-1}$ is defined to be the cohomology of the above cochain complex.

At least in theory, this is computable: we know what all the terms and differentials are.

(Why is this called Hochschild homology? If A is an algebra, you get a multiplicative operad $\text{Ass} \to \text{End}_A$. If you do this same construction, you get the Hochschild *co*homology of A.)

The E_1 page looks like



The E_2 page is $HH_*\text{Pois}_{d-1}$.

We want to see that the spectral sequence collapses at E_2.

Recall: not only was FM_d formal, but also relatively formal: the maps $FM_1 \to FM_d$ are formal as maps of diagrams. That is, we can replace these maps of operads (over rational spaces) by maps of operads over commutative graded algebras. Let's quickly review the notion of formality in rational homotopy theory:

Definition 3.1.4. A diagram $I \to \text{Top}$ is *formal* if a lift exists (up to homotopy)

$$\begin{array}{ccccc} & & & & CGA_{\mathbb{Q}}^{op}(\text{no differential}) \\ & & & \nearrow & \downarrow \\ I & \longrightarrow & \text{Top}_{\mathbb{Q}} & \simeq & CDGA_{\mathbb{Q}}^{op} \end{array}$$

Recall that $\text{Top}_{\mathbb{Q}}$ is Top modulo maps that induce equivalences on $\pi_* \otimes \mathbb{Q}$.

Definition 3.1.5. $I \to \text{Top}$ is stably formal if there is a lift (up to homotopy)

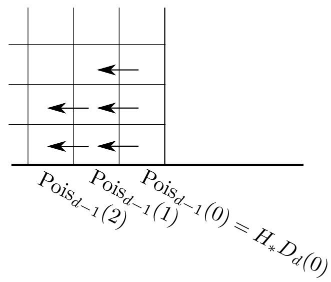
$$\begin{array}{ccccc} & & & & GrVect_{\mathbb{Q}} \\ & & & \nearrow & \downarrow \\ I & \longrightarrow & \text{Top}_{\mathbb{Q}} & \xrightarrow[H_*(-,\mathbb{Q})]{} & Ch_*(\mathbb{Q}) \end{array}$$

Theorem 3.1.6. *If we have a cosimplicial space $X^* : \Delta \to \text{Top}$ and this is formal (as a diagram), then the Bousfield-Kan spectral sequence collapses at E_2.*

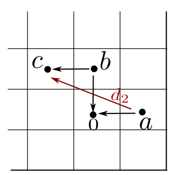
Corollary 3.1.7. *As a result, all we need to show to prove collapse is that the assignment from multiplicative operads to cosimplicial spaces preserves formality. (We won't talk about this, but it's not too bad. There are subtleties, especially if you don't want to use higher categorical language.)*

The E_0 page is C_*X^i (with vertical d_0 differentials).

The E_1 page looks like



By formality, the vertical d_0's are zero. I claim this implies that all differentials $d_{\geq 2}$ are zero. For example, if a survives to the E_2 page, then $d_1(a) = 0$ at the E_1 page, so $d_1(a) = d_0(b)$ at the E_0 page, and $d_2(a)$ is constructed from b. But since $d_0 = 0$, we just have $d_1(a) = 0$, so we can choose $b = 0$, so $d_2(a) = 0$.



So $d_{\geq 2} = 0$, and the spectral sequence collapses at $E^2 = HH_*\mathrm{Pois}_{d-1}$.

$$\begin{array}{ccc} E^2 & = & E^\infty \\ \| & & \| \\ HH_*\mathrm{Pois}_{d-1} & & H_*(\overline{\mathrm{Emb}}(I, I^d); \mathbb{Q}) \\ & & \| \\ & & H_*(\mathrm{Emb}(I, I^d); \mathbb{Q}) \otimes H_*(\Omega^2 S^{d-1}, \mathbb{Q}) \end{array}$$

The LHS is computable (if a little difficult). Also $H_*(\Omega^2 S^{d-1}, \mathbb{Q})$ is computable: rationally, being a loop space is enough to be an infinite loop space, so by a recognition principle, this is just the free commutative algebra on $H_*(S^{d-3}, \mathbb{Q})$.

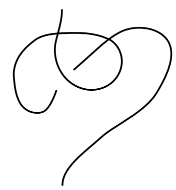
Remark 3.1.8. $H_*(\mathrm{Emb}(I, I^d); \mathbb{Q})$ depends (up to grading) only on the parity of d (for $d \geq 4$). Because (rationally) embedding spaces are loop spaces, to compute rational homology, it's enough to know the $\mathbb{Q}$-homotopy type.

The little disks operad is also coformal (paper only written in the last few years; result maybe older?). This can be used to work out rational homotopy groups.

Finally, I want to say a word about actually computing the rational homology. In principle, this amounts to computing some terms of Poisson operads, which are well-understood: elements of Pois_{d-1} look like products of iterated brackets $[[X_3, X_4], X_1][X_2, X_3] \in \mathrm{Pois}_{d-1}(5)$, where the grading of a term involving n iterated brackets is $(d-1)^n$.

Example 3.1.9. $\alpha = [X_1, X_3][X_2, X_4]$ lives in the fourth column (decreasing indexing) and has grading $2d - 2$.

α might live in $H_*\mathrm{Emb}(I, I^d)$. In fact it does, and we can describe it topologically. Pick any knot



with 2 crossings. We want to resolve both crossings. The crossing lives in $\mathbb{R}^2$, and the orthogonal complement is $\mathbb{R}^{d-2}$. Send $S^{d-3} \to \mathrm{Emb}(I, I^d)$ sending $t \mapsto$ what happens when you resolve one strand through point of S^{d-3} in the orthogonal complement of the 2-dimensional crossing space. To resolve *both* crossings, take $S^{d-3} \times S^{d-3} \to \mathrm{Emb}(I, I^d)$. This represents a nonzero homology class, which has to do with doing Bott-Taubes in higher dimensions. In fact, it is the homology class for α, a product of two brackets. For iterated brackets, the topological realization of the homology classes are not yet fully understood.

Talk 3.2: Deformation quantization (Felix Wierstra)

In physics we have classical mechanics and quantum mechanics. They are kind of different.

Classical mechanics	Quantum mechanics
Manifold X	Hilbert space
Observables $A = C^\infty(X)$	operators on a Hilbert space
Time evolution $\{-,-\}$	commutator $[-,-]$

Idea: what if the quantum observable algebra is a deformation of the classical one? You have to let go of the interpretation, and just think about the algebra.

Definition 3.2.1. A Poisson algebra is a commutative algebra with a Lie bracket $\{-,-\}$ satisfying the Jacobi identity $\{f,g\}h + g\{f,h\}$.

We would like to find a "star-product" $\star : A[[\hbar]] \otimes A[[\hbar]] \to A[[\hbar]]$, written $f \star g = f \cdot g + \beta_1(f,g)\hbar + \sum \beta_i(f,g)\hbar^i$, and the Poisson bracket is defined to be $\{f,g\} = \beta_1(f,g) - \beta_1(g,f)$.

Question: do star products exist?

Disclaimer: for simplicity, I will completely ignore the action of gauge groups.

First we translate the problem into differential graded Lie algebras.

The Maurer-Cartan algebra is

$$MC(L) = \{\gamma \in L^1 \,:\, d\gamma + \frac{1}{2}[\gamma, \gamma] = 0\}.$$

Define the algebra of polyvector fields

$$T^n_{poly}(X) = \Gamma(X, \Lambda^{n+1} T_X)$$

with differential zero. The bracket is the Schouten-Nijenhuijs bracket, which is a complicated formula that is just taking the Lie bracket on vector fields and extending it. $MC(T_{poly}(X))$ corresponds to Poisson structures.

There is the Lie algebra of polydifferential operators $D_{poly}(X)$, a subcomplex of the Hochschild complex, given by

$$D^n_{poly}(X) = \left\{ f \in \mathrm{Hom}(A^{\otimes n+1}, A) \,:\, f \text{ is a polydifferential operator} \right\}$$

(polydifferential operators are just differential operators in each variable; they are polynomials in differential operators). Its differential is the Hochschild differential. Note the degree shift ensures the bracket is in degree zero. Since this is a subcomplex of the Hochschild complex, these correspond to small deformations. The Mauer-Cartan algebra corresponds to the algebra you get after deforming is associative. $MC(D_{poly}(X))$ corresponds to star products.

Theorem 3.2.2 ("Formality conjecture")**.** *D_{poly} and T_{poly} are quasi-isomorphic.*

This was proven by Kontsevich. He constructed an explicit L^∞ morphism $T_{poly} \to D_{poly}$. The formulas come from ideas in string theory.

Since T_{poly} has zero differential, it is the homology of D_{poly}. But this is not completely trivial...

We have a map $U_1^{(0)} : T_{poly} \to D_{poly}$ that sends $(\xi_0 \wedge \cdots \wedge \xi_n) \mapsto (f_0 \otimes \ldots \otimes f_n \mapsto \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} sgn(\sigma) \prod_{i=0}^k \xi_{\sigma(i)})(f_i)$.

Theorem 3.2.3 (Hochschild, Konstant, Rosenberg)**.** *$U_1^{(0)}$ is a quasi-isomorphism of complexes. $U_1^{(0)}$ does not commute with brackets.*

Definition 3.2.4. An L_∞ morphism $U : L_1 \to L_2$ is a sequence of maps $U_i : \Lambda^i L_i \to L_2[1-i]$ (that's a degree shift) satisfying certain identities.

Kontsevich gave specific formulas for $\mathbb{R}^d$, so restrict to $X = \mathbb{R}^d$.

Definition 3.2.5. An admissible graph Γ is a directed graph such that:

(1) we have two types of vertices $\{1, \ldots, n\}$ (type 1), $\{\overline{1}, \ldots, \overline{m}\}$ (type 2).
(2) every edge starts at a type 1 vertex
(3) define $star(k)$ to be the set of outgoing edges of a type 1 vertex k. Label these $e_k^1, \ldots, e_k^{\#star(k)}$. These are related to the type 1 vertices from before, but don't think about that because you'll get confused.

Recall, we're trying to construct a map from $T_{poly} \to D_{poly}$. To each graph we associate a polydifferential operator: define the map $U_\Gamma : \bigotimes_{\gamma_1 \otimes \ldots \otimes \gamma_n} T_{poly} \to D_{poly}$ as follows (where the γ_i's are poly vector fields). To each type 1 vertex we attach

$$\left\langle \psi_i, \left\langle \gamma_i, dx^{I(e_i')} \otimes \ldots \otimes dx^{I(e_i^{\#star(i)})} \right\rangle \right\rangle$$

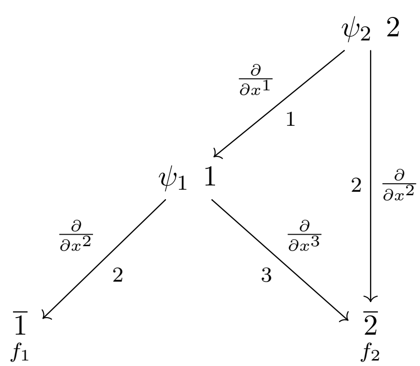
where I is a map from edges to $\{1, \ldots, d\}$. To type 2 vertices we associate the inputs. We decorate the edges with partial derivatives: put $\frac{\partial}{\partial x^{I(i)}}$ on each edge. Then our map does

$$U_\Gamma(\gamma_1 \otimes \ldots \otimes \psi_n)(f_1, \ldots, f_m) = \sum_{I : E_\Gamma \to \{1, \ldots, d\}} \Phi_I$$

where

$$\Phi_I = \prod_x \Big(\prod_e \frac{\partial}{\partial x^{I(e)}} \Big) \psi_i$$

Example:



Here $\psi_1 = \langle \psi_1, dx^2 \otimes dx^3 \rangle$.

The L_∞ morphism $U_n : \bigotimes^n T_{poly} \to D_{poly}$ is

$$U_n = \sum_{m \geq 0} \sum_{\substack{\text{graph } \Gamma \text{ with } n \text{ type 1,} \\ m \text{ type 2 vertices}}} W_\Gamma \cdot U_\Gamma$$

where W_Γ are weights to be defined below.

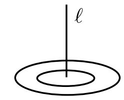
Theorem 3.2.6. *This is an L_∞ quasi-isomorphism.*

Define

$$\mathrm{Conf}_{n,m} = \{P_1, \ldots, P_n : g_1, \ldots, g_m \; : \; P_i \in H, g/j \in \mathbb{R}, P_i \neq P_j\}$$

Draw these graphs in the upper half-plane. Then there is an action of $G^{(1)}$ = translation by $\mathbb{R}$ and scaling by $\mathbb{R}$. Define $C_{n,m}$ to be $\mathrm{Conf}_{n,m}/G^{(1)}$.

We want to integrate, so we want to compactify first in the style of Fulton-MacPherson.

$\overline{C}_{2,0}$ can be drawn as . Say φ is an angle map if it measures the angle with ℓ. 
Each edge gives a function φ_* on $C_{n.m}$. $\bigwedge_{e\in E_\Gamma} d\varphi_e$ is a volume form on $C_{n,m}$. Then

$$W_\Gamma = \text{const.} \int_{\overline{C}_{n,m}} \bigwedge_e d\varphi_e.$$

"Certain conditions" correspond to some integral over configuration spaces, and that integral is zero by Stokes' theorem. $\int_{C_{n,m}} d \bigwedge_e d\varphi_e = 0$. (In $C_{n,m}$ we use graphs with $2n+m-3$ edges.) Use Stokes' theorem to say $\int_{\partial \overline{C}_{n,m}} \bigwedge_e d\varphi_e = \int_{C_{n,m}} d(\bigwedge_e d\varphi_e) = 0$. So the vanishing of the "certain condition" follows from Stokes' theorem.

Tamarkin's proof of the formality conjecture. We wanted to show that T_{poly} is quasi-isomorphic to D_{poly}. These algebras contain much more structure than we've been considering above. Recall that D_{poly} is a subcomplex of the shifted Hochschild complex. We want to give it more structure. At this point it's important to shift everything back in degree, so our new T_{poly} is old $T_{poly}[-1]$ and new D_{poly} is old $D_{poly}[-1]$.

Theorem 3.2.7 (Deligne conjecture)**.** *There is an E_2-action on the Hochschild complex $CH^*(A)$ such that this descends to a Gerstenhaber algebra on $H^*(A)$.*

T_{poly} has a wedge product and is also a Gerstenhaber algebra.

Let P be an operad in chain complexes. Its minimal model $P_\infty \to P$ is a quasi-free operad[3] such that it's quasi-isomorphic to P and "is as small as possible." (Over $\mathbb{Q}$, minimal models always exist. Characteristic $\neq 0$ might be harder.) Minimal models are unique up to quasi-isomorphism. If two algebras are quasi-isomorphic, then they have the same minimal model. For example, if $P = Ass$, then P_∞ is an A_∞-operad.

Theorem 3.2.8 (Homotopy transfer theorem)**.** *Let A be a dg P-algebra and let $p : A \to H(A)$ be the map from A to its homology $H(A)$. The map p is in general not a quasi isomorphism, but we can define a P_∞-structure on $H(A)$ such that*

$$A \xrightarrow{\sim} H(A) + P_\infty\text{-structure}$$

is a quasi-isomorphism of P_∞ algebras.

So we have an E_2-structure on D_{poly}, and a G-structure (Gerstenhaber) on $T_{poly} \cong H(D_{poly})$. But we don't know if the map between them is a quasi-isomorphism. We could look at $D_{poly} \to H(D_{poly})$, and transfer the structure of the minimal model $(E_2)_\infty$ to $H(D_{poly})$.

Theorem 3.2.9 (Tamarkin)**.** *E_2 is formal.*

[3]meaning the underlying operad in vector spaces is free (no relation among the operad structure maps), but it has a differential, and the homology is far from free

By uniqueness of minimal models, the $(E_2)_\infty$-structure is the same as the G_∞-structure. T_{poly} is also a G_∞-structure. Now we have two G_∞-structures on T_{poly}. To show formality, we have to show that these two structures are the same. If $T_{poly} \ncong H(D_{poly})$, then there are certain obstructions.

Day 4: Delooping embedding spaces

Talk 4.1: Rational homotopy of spaces of long embeddings (Robin Koytcheff)

Look at $\overline{\mathrm{Emb}}(M, \mathbb{R}^n)$, where for this entire talk, M will be an open subset of $\mathbb{R}^m$, viewed as a fixed subspace of $\mathbb{R}^n$. (Recall that $\overline{\mathrm{Emb}}$ is the homotopy fiber from embeddings to immersions.) We're also going to consider the "long" version $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ (where c means the embedding is fixed outside of a compact set). If $m = 1$, $\mathrm{Emb}_c(\mathbb{R}^1, \mathbb{R}^n)$ is your standard space of long knots.

Reference: Arone–Turchin (a.k.a. Tourtchine), "On the rational homology ...." (published in *Geometry & Topology*).

Theorem 4.1.1 (Arone–Turchin)**.** *Assume $n - m \geq 3$. Then*

$$\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) = \mathrm{hInfBimod}_{\mathcal{D}_m}(\mathcal{D}_m, \mathcal{D}_n)$$

where $\mathrm{hInfBimod}_{\mathcal{D}_n}$ *means the derived mapping space in the category of infinitesimal bimodules over the little disks operad $\mathcal{D}_n$.*

There is a similar result for $\overline{\mathrm{Emb}}(M, \mathbb{R}^n)$. Proposition 4.1.6 at the end of this section should makes clear what the precise statement is in this case.

Is the RHS any easier than the LHS? (Yes: it leads to calculations of rational homology and rational homotopy.)

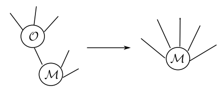
Motivation and consequences:

- For $m = 1$, this recovers the cosimplicial model for the knot space.
- $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n); \mathbb{Q})$ "depends only on the parity of m and n" (well, it's built out of the same pieces if your manifolds have the same parity).
- $H_*(\overline{\mathrm{Emb}}(M, \mathbb{R}^n); \mathbb{Q})$ depends only on $H_*(M; \mathbb{Q})$ if $2m + 1 < n$. (Recall that in this setting, M is an open subset of $\mathbb{R}^m$.)
- In the next talk, we'll see a double delooping of $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$, and that will use the structure of maps of modules over operads.

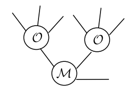
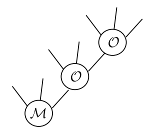
Plan:

(1) Modules over operads
(2) Modules over operads as functors
(3) Specialize to the operad $\mathcal{D}_m$
(4) Taylor tower
(5) Sketch proof of result above

Recall, a right-module M over an operad $\mathcal{O}$ is a sequence of spaces $\{M(n)\}_{n \in \mathbb{N}}$, together with maps $M(k) \times \mathcal{O}(\ell) \to M(k + \ell - 1)$

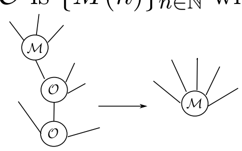


such that the following hold:

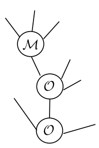
- the order of insertions in this diagram doesn't matter: 
- the order of insertions in this diagram doesn't matter: 
- an identity condition.

An infinitesimal right module is one where you have to do the insertions one at a time. This is exactly the same as a right module.

An infinitesimal left module M over $\mathcal{O}$ is $\{M(n)\}_{n\in\mathbb{N}}$ with maps $\mathcal{O}(k)\times M(\ell)\to M(k+\ell-1)$

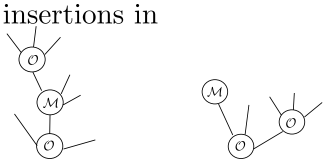


such that the following hold:

- the order of insertions in this diagram doesn't matter: 
- an identity condition.

A (non-infinitesimal) left module is one where you can plug multiple M-trees into an $\mathcal{O}$-slot. Neither of these conditions implies the other one.

An infinitesimal bimodule M over $\mathcal{O}$ is M with both infinitesimal left- and right-module structures such that the order of insertions in

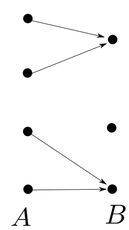


doesn't matter. Sometimes the word "infinitesimal" is replaced by "weak" or "linear."

Let $\mathcal{F}$ be the category of finite sets, with all maps. Define $\mathcal{F}(\mathcal{O})$ to be the category of finite sets A, B, ... with morphisms

$$\mathrm{Hom}(A,B)=\coprod_{\alpha:A\to B}\bigotimes_{b\in B}\mathcal{O}(\alpha^{-1}(b)).$$

Example:



where the summand corresponding to the indicated map $A \to B$ is $\mathcal{O}(2) \times \mathcal{O}(2) \times \mathcal{O}(0)$.

Proposition 4.1.2. *A right module over $\mathcal{O}$ is a (contravariant) functor from $\mathcal{F}(\mathcal{O})$.*

Let Γ be the category of pointed finite sets and pointed maps. Out of this we can build the category $\Gamma(\mathcal{O})$ of pointed sets S, T. Then $\mathrm{Hom}(S, T)$ is the same as for $\mathcal{F}(\mathcal{O})$.

$\widetilde{\Gamma}(\mathcal{O})$ is the same as $\Gamma(\mathcal{O})$, but with "composition at the basepoint reversed." (I'll say what this means in the case of the little disks operad.)

Proposition 4.1.3. *An infinitesimal bimodule over $\mathcal{O}$ is a (contravariant) functor from $\widetilde{\Gamma}(\mathcal{O})$.*

Let $\mathcal{M}$ be the category of open subsets of $\mathbb{R}^m$, where the morphisms are affine-linear (the composition of a scaling and a translation, a.k.a. "standard") embeddings on each component.

We can describe the k-th space in the little disks operad as $\mathcal{D}_m(k) = s\mathrm{Emb}(\bigsqcup_k D^m, D^m)$ (standard embeddings). We can also think of $\mathcal{D}_m$ as the operad $\mathrm{End}(D^m)$ of endomorphisms of D^m in the category $\mathcal{M}$. Then $\mathcal{F}(\mathcal{D}_m)$ is the category of finite sets A, B, ... where morphisms in $\mathrm{Hom}(A, B)$ are standard embeddings $A \times D^m \to B \times D^m$.

Proposition 4.1.4. *We can identify $\widetilde{\Gamma}(D_m)$ with the category of finite pointed sets $S, T, \dots$, where*

$$\mathrm{Hom}(S, T) = s\mathrm{Emb}(S\backslash\{*\} \times D^m \sqcup (\mathbb{R}^m\backslash D^m), T\backslash\{*\} \times D^m \sqcup (\mathbb{R}^m\backslash D^m)).$$

Call $\mathbb{R}^m\backslash D^m$ the "antiball." Notation: Arone and Turchin write $s\mathrm{Emb}(S \boxtimes D^m, T \boxtimes D^m)$.

Idea of this proposition: for the anti-ball, an embedding $\alpha : D^m \hookrightarrow D^m$ gives rise to $\alpha^{-1} : \mathbb{R}^m\backslash D^m \hookrightarrow \mathbb{R}^m\backslash D^m$.

When discussing the Taylor tower we were looking at functors $F : \mathcal{O}(\mathcal{M})^{op} \to \mathrm{Top}$. Now we also want to consider $F : \widetilde{\mathcal{O}}(\mathbb{R}^n)^{op} \to \mathrm{Top}$, where $\widetilde{\mathcal{O}}(\mathbb{R}^n)$ is the category of open sets with bounded complement and inclusions (i.e., an object is an open set that contains some antiball). This variation is required to handle "long embeddings," i.e. those which are standard outside a compact set.

Recall that for $F : \mathcal{O}(M)^{op} \to \mathrm{Top}$, we defined $T_k F := \mathrm{holim}_{U \in \mathcal{O}_k} F(U)$, the degree-$k$ polynomial approximation to F. Here $\mathcal{O}_k \subset \mathcal{O}(M)$ is the subcategory of disjoint unions of at most k balls. (See Talk 2.4 for the definition of "degree-k polynomial functor.") We had an equivalence $F(U) \xrightarrow{\sim} T_kF(U)$ for $U \in \mathcal{O}_k$, and this property together with being polynomial of degree k, uniquely characterize T_kF.

For $F : \widetilde{\mathcal{O}}(\mathbb{R}^n) \to \mathrm{Top}$, define $T_kF := \mathrm{holim}_{U \in \widetilde{\mathcal{O}}_k} F(U)$, where $\widetilde{\mathcal{O}}_k \subset \widetilde{O}(\mathbb{R}^n)$ is the subcategory of disjoint unions of at most k balls and one antiball. As in the previous case, this T_kF is polynomial of degree k, and we have an equivalence $F(U) \xrightarrow{\sim} T_kF(U)$ for every $U \in \widetilde{\mathcal{O}}_k$. Once again, these properties uniquely characterize T_kF.

Definition 4.1.5. A functor $F : \mathcal{O}(M)^{op} \to \mathrm{Top}$ is *context-free* if it factors through $\mathcal{M}$. A functor $F : \widetilde{\mathcal{O}}(\mathbb{R}^n)^{op} \to \mathrm{Top}$ is *context-free* if it factors through $\widetilde{\mathcal{M}}$, where instead of open sets, the objects are disjoint unions of open sets, one of which has bounded complement (i.e. contains an antiball).

The term "context-free" means you don't care which manifold the open subsets live in.

For example, $\mathrm{Emb}(-, N) : \mathcal{O}(M)^{op} \to \mathrm{Top}$ is context-free.

An example of a functor that is *not* context-free is the space of sections of a bundle over some fixed manifold.

One can show that $\overline{\mathrm{Emb}}(-, \mathbb{R}^n) : \mathcal{O}(M)^{op} \to \mathrm{Top}$ and $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^n) : \widetilde{\mathcal{O}}(\mathbb{R}^n) \to \mathrm{Top}$ are (weakly equivalent to) context-free functors. Recall that $\mathcal{M}$ (resp. $\widetilde{\mathcal{M}}$) contains $\mathcal{F}(\mathcal{D}_m)$ (resp. $\widetilde{\Gamma}(\mathcal{D}_m)$) as a subcategory. Hence $\overline{\mathrm{Emb}}(-, \mathbb{R}^n)$ (resp. $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^n)$) is a right module (resp. infinitesimal bimodule) over $\mathcal{D}_m$.

Proposition 4.1.6. *For any U in $\mathcal{O}(M)$, resp. $\widetilde{\mathcal{O}}(\mathbb{R}^m)$,*

$$T_k\overline{\mathrm{Emb}}(U, \mathbb{R}^n) \simeq \underset{\mathcal{D}_m}{\mathrm{hRMod}}_{\le k}(s\mathrm{Emb}(-, U), \mathcal{D}_n)$$
$$T_k\overline{\mathrm{Emb}}_c(U, \mathbb{R}^n) \simeq \underset{\mathcal{D}_m}{\mathrm{hInfBimod}}_{\le k}(s\mathrm{Emb}(-, U), \mathcal{D}_n).$$

Idea of proof: the proofs of the two statements are similar, so we sketch the proof of the first statement. First, the functor $F := \overline{\mathrm{Emb}}(-, \mathbb{R}^n)$ is equivalent as a right module over $\mathcal{D}_m$ to $\mathcal{D}_n$. Then the right-hand side above can be written as a holim of spaces of the form $\mathrm{holim}_{V,Y} \mathrm{Map}(s\mathrm{Emb}(V, U), F(Y))$. Considering this space as the value of a functor G at U, one can show that G is a degree-k polynomial functor. By the (enriched) Yoneda Lemma, $G(U)$ is equivalent to $F(U)$ for any $U \in \mathcal{O}_k$. Thus by the two properties that characterize T_kF, G is equivalent to T_kF, meaning that $T_kF(U) \simeq G(U)$ for any $U \in \mathcal{O}(M)$.

Putting $U = M$ (resp. $U = \mathbb{R}^m$) and $k = \infty$ and using that $s\mathrm{Emb}(-, \mathbb{R}^m)$ is equivalent to $\mathcal{D}_m$ yields Theorem 4.1.1.

Talk 4.2: Double delooping theorem for spaces of long knots (Rebecca Wei)

Let $\mathcal{A}$ denote the associative operad.

Theorem 4.2.1 (Dwyer, Hess). *Given a map of nonsymmetric simplicial operads $\omega : \mathcal{A} \to \mathcal{O}$ (with $\mathcal{O}_0 \simeq \mathcal{O}_1 \simeq *$), there's a natural weak equivalence*

$$\Omega^2 \operatorname{Map}^h_{Op}(\mathcal{A}, \mathcal{O})_\omega \simeq \operatorname{holim} \mathcal{O}^\bullet$$

where $\mathcal{O}^\bullet$ is the cosimplicial space associated to $\mathcal{O}$.

Context:

- Sinha showed that for $\mathcal{O} = K_m$ (the Kontsevich operad, which is like the Fulton-Macpherson operad where you only care about relative directions and not relative distances), the RHS is weakly equivalent to $\overline{\operatorname{Emb}}_c(I, I^m)$. Note that long knots have a multiplication, namely concatenation of the compact support part.
- Given $\omega : \mathcal{A} \to \mathcal{O}$, McClure-Smith defined $\mathcal{O}^\bullet$ and showed that the totalization is an E_2-algebra.

Exercise/ example 4.2.2. If $\mathcal{O} = \operatorname{End}_X$, then $(\mathcal{O}^\bullet, d = \Sigma(-1)^i d^i)$ is the Hochschild cochain complex.

Exercise/ example 4.2.3. If (X, e) is a topological monoid, show that $\mathcal{O}_n = X^n$ defines an operad with multiplication whose totalization is $\simeq \Omega_e X$ (loop space based at the identity).

Outline:

(1) Proof sketch of main theorem
(2) Computation and example related to main theorem
(3) More details of the proof

Proof sketch of main theorem. The proof depends on a more general theorem.

Theorem 4.2.4 (Fiber sequence theorem). *Let $(C, \diamond, e_\diamond)$ be a model category with monoidal product $\diamond$. Let $\omega : R \to S$ be a map of monoids.*

Suppose C, C_S (category of right S-modules), $C_{R,S}$ (category of R-S bimodules), Mon_C (category of monoids in C) have compatible model category structures.[4]

Then, under some conditions I won't describe, there is a natural fiber sequence

$$\Omega \operatorname{Map}^h_{\operatorname{Mon}_C}(R, S)_\omega \to \operatorname{Map}^h_{C_{R,R}}(R, R) \to \operatorname{Map}^h_C(e_\diamond, S)$$

(derived mapping space based at the map ω).

These conditions hold in all the examples we care about, but they can be really hard to check.

We want to apply this in two different settings.

[4] Kathryn: this is not a strong hypothesis.

Consider C = graded spaces, $\diamond$ = composition product (written $*$); monoids are operads. Then $e_\diamond = *$ in level 1. Use $R = \mathcal{A}$, $S = \mathcal{O}$. The conditions work, so the theorem says that

$$\Omega \operatorname{Map}^h_{Op}(\mathcal{A}, \mathcal{O})_\omega \simeq \operatorname{Map}^h_{A,A}(\mathcal{A}, \mathcal{O})$$

(the last term in the sequence is contractible).

We could also think about this where C = right $(\mathcal{A}, \mathcal{O})$-modules, $\diamond$ = the graded cartesian product (written $\odot$), i.e. $(X \odot Y)_m = \coprod_{i_1+i_2=m} X_{i_1} \times Y_{i_2}$. Monoids w.r.t. the graded cartesian product are the same as left $(\mathcal{A}, *)$-modules.

Notation: write $\mathcal{A}$ for $(\mathcal{A}, \text{composition product})$, and $A = (\mathcal{A}, \odot)$.

In both settings, our map ω is the given map $\mathcal{A} \to \mathcal{O}$. The fiber sequence theorem says

$$\Omega \operatorname{Map}^h_{\mathcal{A}\text{-}\mathcal{A},*}(\mathcal{A}, \mathcal{O}) \simeq \operatorname{Map}^h_{A\text{-}A,\odot,\text{mods } \mathcal{A},*}(\mathcal{A}, \mathcal{O})$$

and hence

$$\Omega^2 \operatorname{Map}^h_{Op}(\mathcal{A}, \mathcal{O})_\omega \simeq \operatorname{Map}^h_{A\text{-}A,\odot,\text{mods } \mathcal{A},*}(\mathcal{A}, \mathcal{O}).$$

Computation. $\operatorname{holim} \mathcal{O}^\bullet \simeq \operatorname{Map}^h_{A\text{-}A,\text{mods } \mathcal{A}}$.

Idea: give a simplicial resolution for A in $(A\text{-}A\text{-}bimods_\odot, \text{mods } \mathcal{A})$ whose diagonal is weakly equivalent to A.

We'll write down the cosimplicial resolution.

$$H_n A = A^{\odot n+2}$$

$$(H_n A)_m = \coprod_{i_1+\dots+i_{n+2}=m} A_{i_1} \times \ldots \times A_{i_{n+2}}$$

and one of the face maps takes that to $A_{i_1+i_2} \times A_{i_3} \times \ldots \times A_{n+1} \in (H_m A)_m$. The degeneracy maps are insertions of A_0 in the appropriate places.

There is a diagonal map diag : $ssSet \to sSet$; then $\text{diag} X_{.,.} \simeq \operatorname{hocolim} X_{.,\bullet}$.

$\text{diag} HA \overset{\simeq}{\rightrightarrows} A$ because we have an extra degeneracy in $\rightrightarrows \coprod_{i_1+i_2=m} A_{i_1} \times A_{i_2} \to A_n$.

So:

$$\begin{aligned} \operatorname{Map}^h(A, \mathcal{O}) &\simeq \operatorname{Map}^h(\text{diag} HA, \mathcal{O}) \\ &\simeq \operatorname{Map}^h(\operatorname{hocolim} HA, \mathcal{O}) \\ &\simeq \operatorname{holim} \operatorname{Map}^h(HA, \mathcal{O}) \end{aligned}$$

Claim 4.2.5. $\mathcal{O}^\bullet \simeq \operatorname{Map}^h(H_\bullet A, \mathcal{O})$

I can take the definition of $(A^{\odot n})_m$ and stick in an extra factor of the point, written $*_n$ in front of each piece of the coproduct. This shows $(A^{\odot n})_m = (*_n \circ A)_m$ where $*_n$ is a graded space empty in all levels except for a point in level n. So,

$H_n A = \text{Free}_{A\text{-}A,\odot}(A^{\odot n}) = \text{Free}_{A\text{-}A,\odot}(\text{Free}_{\text{mods-}\mathcal{A},\circ}(*_n))$

$$\begin{aligned}\text{Map}^h_{A\text{-}A,\text{mod-}\mathcal{A}}(HA, \mathcal{O}) &\simeq \text{Map}^h_{\text{mod}-\mathcal{A}}(\text{Free}(A^{\odot n}), \mathcal{O}) \\ &\simeq \text{Map}^h_{\text{grSpaces}}(*_n, \mathcal{O}) \simeq \mathcal{O}_n = \mathcal{O}^n\end{aligned}$$

Example. The example pertains to a variant of the fiber sequence theorem.

Definition 4.2.6. A pointed right S-module M is called *distinguished* (d) if $S \cong e_\diamond \diamond S \to M \diamond S \to M$ is an equivalence.

A right S-module is called *potentially distinguished* (pd) if there is a zig-zag of weak equivalences between S and M. (This might not necessarily be pointed.)

Examples? S is distinguished!

Theorem 4.2.7 (Prelooped fiber sequence theorem)**.** *Given some conditions, there is a natural fiber sequence*

$$M(C^d_{R,S}) \to M(C^{pd}_{R,S}) \to M(C^{pd}_S)$$

where M is the nerve of the subcategory of weak equivalences (i.e. where the morphisms are weak equivalences).

M is a moduli space – it tells you about "weak automorphisms".

Consider C = symmetric sequences in spaces, $\diamond = \circ$, R = a cofibrant operad, $S = \text{End}_X$. Then the monoids are Σ-operads.

Choose $M \in M(C^{pd}_{\text{End}_X})$. This means we have a zigzag of weak equivalences of right S-modules $M_n \times X^n \overset{\sim}{\to} M'_n \times X^n \overset{\sim}{\leftarrow} \ldots \overset{\sim}{\leftarrow} (\text{End}_X)_n \times X^n$. In particular, we have the above zigzag for n=0. We claim that once we choose the zigzag for n=0, the space of choices of zigzags for higher n is contractible.

Claim 4.2.8. $M(C^{pd}_{\text{End}_X}) \sim M(\mathit{Spaces})_X$ *(i.e. this is the only choice I had to make)*

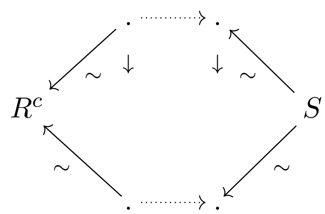
More details of the proof.

Theorem 4.2.9. $\text{Map}^h(R, S)_\omega \simeq \mathcal{M}(C^d_{R,S})$

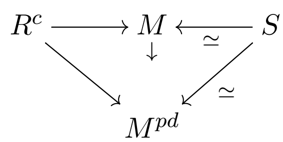
We assume that we have an adjunction Free: $C_{R,S} \leftrightarrows \text{Mon}_{R,S}$: Forget, and that $\text{Free}^h(X) := \text{Free}(X^c)$ preserves distinguished objects. (This preservation of distinguished objects is the key technical condition to check for all the categories to which we want to apply the fiber sequence theorem.) Then we have $\text{Free}^h \circ \text{Forget} \to\leftarrow \mathbb{1}$, $\text{Forget} \circ \text{Free}^h \to\leftarrow \mathbb{1}$, which is enough to show that $\mathcal{M}(\text{Mon}^d_{R,S}) \simeq \mathcal{M}(C^d_{R,S})$.

We make assumptions sufficient to show that $\mathcal{M}(\mathrm{Mon}^d_{R,S}) \simeq \mathcal{M}(\mathrm{Mon}^d_{R^c,S})$ where $R^c \overset{\simeq}{\twoheadrightarrow} R$ is a cofibrant replacement in Mon_C.

When R is cofibrant, our hammock localization diagrams



simplify to



Talk 4.3: Right-angled Artin operads and their resolutions
(Dimitri Zaganidis)

Theorem 4.3.1. *If P, Q are cofibrant simplicial operads and Z is a $P \otimes Q$-bimodule, there is a natural weak equivalence*

$$\mathrm{Map}^h_{P\otimes A}(P \otimes Q, Z) = \mathrm{Map}^h_P(P, \mathrm{Map}^h_Q(Q, \gamma_\bullet Z)).$$

Recall $\gamma_n(X)$ is a symmetric sequence given by $\gamma(X)(k) = \gamma(n \cdot k)$. The Σ_k action is by the forgetful map $\Sigma_n \times \Sigma_k \hookrightarrow \Sigma_{nk}$.

Recall: if M is a P', Q' bimodule, you can take the Boardman-Vogt tensor product $-\widetilde{\otimes} M : \mathrm{Bimod}_{P,Q} \to \mathrm{Bimod}_{P\otimes P', Q\otimes Q'}$ and that has a right adjoint $\mathrm{Map}_{P',Q'}(M, \gamma_\bullet(-))$.

The theorem looks like a derived adjunction statement. But note that this is $P \otimes Q$, not $P \widetilde{\otimes} Q$.

The motivation will be in Inbar's talk, where it is used to prove that

$$\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1} \mathrm{Map}^h_{Op}(D_m, D_n)$$

. We will discuss the proof of the theorem in a simple case, where P, Q are free. This does not imply that $P \otimes Q$ is free, though. They are, however, all right-angled Artin operads.

Strategy of the proof:

(1) Solve the problem when P, Q are free. P, Q, $P \otimes Q$ are right-angled Artin operads. Come up with a resolution of right-angled Artin operads as bimodules over themselves. Since we only know about $\widetilde{\otimes}$ for free things, this resolution should be in free bimodules.
(2) Construct a resolution of an operad by free operads.
(3) Combine the above using bisimplicial techniques.

Right-angled Artin operads. As a warm-up, let us consider the case of right angled monoids.

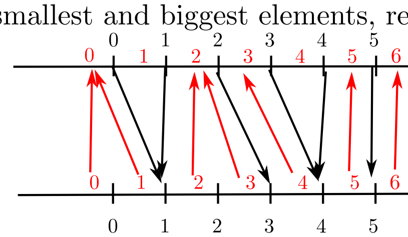
Right-angled Artin monoids. Consider the map non-directed graphs $\to$ Monoids sending $G = (V, E) \mapsto M_G = F(V)/\sim$, where $vw \sim wv$ iff $\{v, w\} \in E$. Right-angled monoids are monoids arising in this way.

If $W \subset V$ is a clique (subgraph that is a complete graph), we call it spherical.

If W is a spherical subset, define $\lambda_W = \prod_{w\in W} w$.

Preliminary 4.3.2. I will give a model for $\Delta[1]$. First, I claim that $\Delta_+^{op} \cong \Delta_{-\infty,+\infty}$. The first category is the category of finite ordinals (possibly empty), with non decresing maps. The second category is the category of non empty finite ordinals, with non decresing maps that preserve smallest and biggest element (which can be equal). The maps sends the ordinal $n = \{0, \ldots n-1\}$ to $n+1 = \{0, \ldots n\}$.

On morphisms, it takes a map f depicted in black to a morphism f^{op} pointing in the other direction and that preserves smallest and biggest elements, represented in red:



$\Delta[1]_n = \{f : [n] \to [1]\} = \{f : [2] \to [n+1] \ : \ f \in \Delta_{-\infty,+\infty}\} = \{0, \ldots, n\}$.

Define

$$\mathcal{X}_n = \{(W, f) \ : \ W \subset V \text{ spherical}, f : W \to \{1, \ldots, n\} \subset \{0, \ldots, n+1\}\}$$

and define $R_n(G) = F_{M_G}(\mathcal{X}_n) = M_G \times \mathcal{X}_n \times M_G$ (free M_G-bimodule).

If $\varphi : [m] \to [n]$, then you get an induced map $R(G)(\varphi) : R_n(G) \to R_m(G)$. To define it, by the universal property of the free bimodule, it is enough to define a map $\mathcal{X}_n \to R_m(G)$.

Let $(W, f) \in \mathcal{X}_n$ that is, f lands in $\{1, \ldots, n\}$. Morally, want you want to do this by postcomposing f by φ^{op}. But this is not well defined, because in general, $\mathrm{im}(\varphi^{\mathrm{op}} \circ f) \not\subseteq \{1, \ldots, m\}$.

What we do instead, is restricting to the sub-spherical subset $W' = (\varphi^{\mathrm{op}} \circ f)^{-1}\{1, \ldots, m\}$.

More precisely, we define

$$R(G)(\varphi)(W, f) = \lambda_{(\varphi^{\mathrm{op}}\circ f)^{-1}\{0\}} \cdot (W', \varphi^{\mathrm{op}} \circ f) \cdot \lambda_{(\varphi^{\mathrm{op}}\circ f)^{-1}\{m+1\}}$$

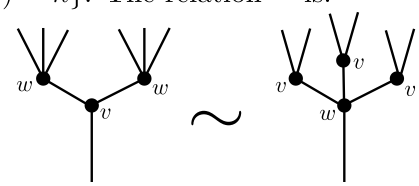
We have the following proposition, which indicates that $R_\bullet(G)$ is indeed a resolution of M_G.

Proposition 4.3.3. $|R(G)| \simeq M_G$.

Let us start now the real work and go on to Right-angled Artin Operads.

Right-Angled Artin Operads.

Definition 4.3.4. Consider the map from graded graphs to operads sending a graph $G = (V, E)$ with grading $g : V \to \mathbb{N}$ to $F_{Op}(V_n \cdot \Sigma_n)/\sim$ (where $F_{op}(-)$ denotes the free operad on $-$), where $V_n = \{v \in V \,:\, g(v) = n\}$. The relation $\sim$ is:



Let $\mathcal{O}_{G,g} := F_{Op}(V_n, \Sigma_n)/\sim$.

Notice that free operads on free Σ-sequences are right-angled Artin operads, and so is their Bordman-Vogt $\otimes$-product. Indeed, we have the following proposition:

Proposition 4.3.5. *Suppose* (G, g) *and* (G', g') *are graded graphs. Then*

$$\mathcal{O}_{G,g} \otimes \mathcal{O}_{G',g'} \cong \mathcal{O}_{(G*G',g\sqcup g')}$$

where

$$G * G' = (V \sqcup V', E \sqcup E' \sqcup (V \times V')).$$

Let $\mathcal{X}_n(k) = \{(W, f) \,:\, W \xrightarrow{f} \{1, \ldots, n\}, g(W) = k\}$, where $g(W) = \prod_{w \in W} g(w)$.

Define $R_n(G, g) = F_{\mathcal{O}_{G,g}}(\mathcal{X}_n) = \mathcal{O}_{G,g} \circ \mathcal{X}_n \circ \mathcal{O}_{G,g}$. Then a map $\varphi : [n] \to [m]$ induces a map $R_n(G, g) \to R_m(G, g)$, in a similar fashion (replace $\cdot$ by $\otimes$).

This is also a resolution of the operad as a bimodule over itself, since we have the following proposition:

Proposition 4.3.6. $|R_\bullet(G, g)| \simeq \mathcal{O}_{G,g}$.

Proposition 4.3.7.

$$R_n(G, g) \widetilde{\otimes} R_n(G', g') \cong R_n(G * G', g * g').$$

Proof. This involves the $\square$ product of symmetric sequences, which is defined by

$$(X \square Y)(n) = \coprod_{k,l : kl = n} X(k) \times X(l) \times_{\Sigma_k \times \Sigma_l} \Sigma_n.$$

Recall that if $M = P \circ X \circ P$ and $M' = P \circ X' \circ P$ are free bimodules over P, $M \widetilde{\otimes} M' = P \circ (X \square X') \circ P$, by definition.

Remark that if $\mathcal{X}_n, \mathcal{X}'_n$ and $\mathcal{X}''_n$ are respectively the symmetric sequences used to define $R_n(G,g)\widetilde{\otimes}R_n(G',g') \cong R_n(G*G', g*g')$, then

$$\mathcal{X}_n \square \mathcal{X}'_n = \mathcal{X}''_n.$$

□

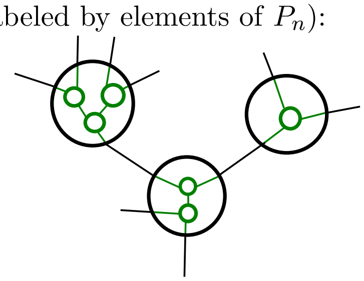
Now let's describe how you take a free resolution of operads.

Free resolution of operads. There is an adjunction F : symmetric sequences $\leftrightarrows Op : U$ (where U is the forgetful functor).

Definition 4.3.8. Let P be an operad. Define $G_k(P) = (FU)^{k+1}(P)$.

This is kind of like how you build cofibrant replacement of a category.

Here is a picture of an element of $G_1(P)_n = (FU)^2(P)_n$ (tree (black) labelled by elements of $FU(P)$, that is, (green) trees labeled by elements of P_n):



If you want to talk about $(FU)^k(P)_n$, iterate this, so you have more nested trees.

Fact 4.3.9. *If P is a simplicial operad, $G_\bullet P$ is a bisimplicial operad.* $\operatorname{diag}(G_\bullet P) = W(\Delta[1], P)$ *(the W-construction, a cofibrant replacement of P).*

Intuitively, this fact can be understood in the following way: Given a element of the free operad on P, giving a $m+1$-fold nesting of the operations is equivalent to giving lengths from 1 to $m+1$ to the inner edge, like in the W-construction.

Let us now give a sketch of the proof of the theorem we stated in the beginning of the talk:

$$\begin{aligned}
\operatorname{Map}^h_{P\otimes Q}(P\otimes Q, Z) &\overset{\substack{\text{base}\\ \text{change}}}{\simeq} \operatorname{Map}^h_{\operatorname{diag}(G_\bullet(P)\otimes G_\bullet Q)}(\operatorname{diag}(G_\bullet(P)\otimes G_\bullet(Q)), \operatorname{diag} c_\bullet Z)\\
&\overset{\substack{\text{diag is}\\ \text{Q. equiv}}}{\simeq} \operatorname{Map}^h_{G_\bullet(P)\otimes G_\bullet(Q)}(G_\bullet(P)\otimes G_\bullet(Q), c_\bullet(Z))\\
&\overset{\substack{\text{operads}\\ \text{are RAAO}}}{\cong} \operatorname{Map}^h_{\mathcal{O}(G^{P_{\bullet,\bullet}})\otimes\mathcal{O}(G^{Q_{\bullet,\bullet}})}(\mathcal{O}(G^{P_{\bullet,\bullet}})\otimes\mathcal{O}(G^{Q_{\bullet,\bullet}}), c_\bullet(Z))\\
&\overset{\substack{\text{cof.}\\ \text{repl.}}}{\simeq} \operatorname{Map}_{\mathcal{O}(G^{P_{\bullet,\bullet}})\otimes\mathcal{O}(G^{Q_{\bullet,\bullet}})}(|R(G^{P_{\bullet,\bullet}})|\widetilde{\otimes}|R(G^{Q_{\bullet,\bullet}}))|, c_\bullet(Z))\\
&\overset{\substack{\text{enhanced}\\ \widetilde{\otimes}-\text{hom}}}{\simeq} \operatorname{Map}_{\mathcal{O}(G^{P_{\bullet,\bullet}})}(|R(G^{P_{\bullet,\bullet}})|, \operatorname{BiMap}_{\mathcal{O}(G^{Q_{\bullet,\bullet}})}(|R(G^{Q_{\bullet,\bullet}})|), \gamma_\bullet(c_\bullet(Z)))
\end{aligned}$$

$$\simeq \ldots \text{(diagonal magic)}$$
$$\simeq \operatorname{Map}^h(P, \operatorname{Map}^h_Q(Q, \gamma_\bullet(Z)))$$

TALK 4.4: THE ITERATED DELOOPING THEOREM FOR SPACES OF LONG EMBEDDINGS (Inbar Klang)

Theorem 4.4.1.
$$\overline{\operatorname{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1} \operatorname{Map}^h_{Op_\Sigma}(D_m, D_n)$$
for $n \geq m+3$.

Theorem 4.4.2. *For* $n \geq m+3$*,*
$$\operatorname{Map}^h_{D_m\text{-}lin.}(D_m, D_n) \simeq \overline{\operatorname{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$$
(D_m*-linear is the same as the infinitesimal bimodules from earlier).*

Remark 4.4.3. A map of P-bimodules from $P \to M$ (where M is a P-bimodule) gives a P-linear bimodule structure on M. Why? Imagine trees with M on only one of their leaves, you can plug in $\mathbb{1}_P$ into all the other slots.

In the $m = 1$ case: in Rebecca's talk, we saw the double delooping theorem, which said:

Theorem 4.4.4.
$$\overline{\operatorname{Emb}}_c(\mathbb{R}, \mathbb{R}^n) \simeq \Omega^2 \operatorname{Map}^h_{Op}(A, K_n)$$

What's the difference? This is with nonsymmetric operads. Also, this is simplicial operads and the theorems for this talk are on topological operads. These are not hard to fix, because of some good adjunctions:
$$|-| : sSet \leftrightarrows \text{Top} : S_\bullet$$
$$\cdot\Sigma : Op \leftrightarrows Op_\Sigma : U$$
Recall:

Theorem 4.4.5 (Fiber sequence theorem)**.** *Let C be a monoidal category, and $\omega : R \to S$ a map of monoids. Then under some assumptions, there is a natural fiber sequence*
$$\Omega \operatorname{Map}^h_{\text{Mon}}(R, S) \to \operatorname{Map}^h_R(R, S) \to \operatorname{Map}_C(I, S)$$
where $\operatorname{Map}_R =$ *maps of R-bimodules.*

The first loop in the $m = 1$ case came about by taking C to be the category of non-symmetric sequences with composition product, and $S(1)$ contractible. There were equivalences
$$\Omega \operatorname{Map}^h_{Op}(R, S) \simeq \operatorname{Map}^h_R(R, S)$$
We can apply the symmetric, topological operad version of this with $R = D_m$ and $S = D_n$ to obtain the first delooping in the iterated delooping theorem. The second loop in the

$m = 1$ case was obtained by applying the fiber sequence theorem to sequences with $\odot$ (graded cartesian product), $S(0)$ contractible, and $R = \mathcal{A}$ (the associative operad). We got $\Omega \operatorname{Map}^h_{\mathcal{A}}(\mathcal{A}, S) \simeq \operatorname{Map}^h_{\mathcal{A}\text{-linear}}(\mathcal{A}, S)$. Maybe you remember a different category instead of $\mathcal{A}$-linear. It ends up being the same thing as this. We'll phrase it this way because that's convenient for this theorem.

Now imagine all of the above with symmetric sequences and Σ-operads instead of nonsymmetric sequences and non-Σ-operads.

Now here's where the other loops in the iterated delooping theorem come from: Suppose B is an E_n-operad, M is a B-bimodule, $\varphi : B \to M$ is a map of B-bimodules, and $M(0) \simeq *$. We'll see that

$$\Omega^m \operatorname{Map}^h_B(B, M) \simeq \operatorname{Map}^h_{B\text{-linear}}(B, M) \tag{4.4.1}$$

Recall there was a fixed linear embedding (e.g. the standard equatorial embedding) which induced a map $\varphi_{m,n}$ of operads from D_m to D_n. This makes D_n into a D_m-linear bimodule (see Remark 4.4.3), so taking $B = D_m$, $M = D_n$,

$$\begin{aligned} \operatorname{Map}^h_{D_n\text{-linear}}(D_m, D_n) &\simeq \Omega^m_{\varphi_{m,n}} \operatorname{Map}^h_{B_n}(D_m, D_n) \\ &\simeq \Omega^{m+1}_{\varphi_{m,n}} \operatorname{Map}^h_{Op_\Sigma}(D_m, D_n) \end{aligned}$$

Which gives the iterated delooping theorem.

Our goal is to prove by induction (4.4.1). In the $m = 1$ case, if B is an E_1 operad, then $\varphi : B \to M$ is a map of bimodules, $M(0) \simeq *$, then

$$\Omega \operatorname{Map}^h_B(B, M) \simeq \operatorname{Map}^h_{B\text{-lin}}(B, M)$$

(this follows from the earlier $m = 1$ result, plus the adjunctions stated above).

Now do the inductive case. The main ingredient is the $\otimes$-Hom adjunction. But first, "recall":

Theorem 4.4.6. *If P is a cofibrant enough E_k-operad and Q is a cofibrant enough E_ℓ-operad, then $P \otimes Q$ is an $E_{k+\ell}$-operad.*

(Note: this is really nonobvious. Here's an example of a failure of this when things aren't cofibrant enough: if A is the associative operad, then $A \otimes A =$ the commutative operad. This is not an E_2-operad!)

Fortunately we have the W construction, a cofibrant replacement. This allows us to turn the little disks operad into something that satisfies the above theorem.

So pretend that $D_1 \otimes D_n = D_{n+1}$. To be precise about this, there would need to be W's in everything.

Now we can show (4.4.1) by induction on m.

$$\begin{aligned} \Omega^{m+1} \operatorname{Map}^h_{D_{m+1}}(D_{m+1}, M) &\simeq \Omega^{m+1} \operatorname{Map}^h_{D_1 \otimes D_m}(D_1 \otimes D_m, M) && \text{pretending + base change} \\ &\simeq \Omega^{m+1} \operatorname{Map}^h_{D_1}(D_1, \operatorname{Map}^h_{D_m}(D_m, \gamma_\bullet M)) && \otimes\text{-Hom} \end{aligned}$$

Aside about $\mathrm{Map}^h(D_m, \gamma_\bullet M)$: $\mathrm{Map}^h_{D_m}(D_m, \gamma_k M)$ is a simplicial set. If replacing k with $\bullet$, this is a simplicial set at each arity, therefore $\mathrm{Map}^h(D_m, \gamma_\bullet M)$ is a symmetric sequence. This is a D_1-bimodule. We're also pretending that simplicial sets are topological spaces: for example, D_m really means the singular simplicial set of D_m.

$$\simeq \Omega \,\mathrm{Map}^h_{D_1}(D_1, \Omega^m \,\mathrm{Map}^h_{D_m}(D_m, \gamma_\bullet M))$$

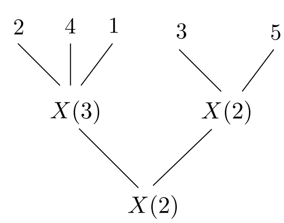
The reason you can do this is $D_1 \simeq * \times \Sigma$, and if you have maps from a point to something, loops can move inside.

$$\begin{aligned} &\simeq \Omega \,\mathrm{Map}^h_{D_1}(D_1, \mathrm{Map}^h_{D_m\text{-lin}}(D_m, \gamma_\bullet M)) && \text{induction hypothesis} \\ &\simeq \mathrm{Map}^h_{D_1\text{-lin}}(D_1, \mathrm{Map}^h_{D_m\text{-lin}}(D_m, \gamma_\bullet M)) && m = 1 \text{ case} \\ &\simeq \mathrm{Map}^h_{D_1 \otimes D_m\text{-lin}}(D_1 \otimes D_m, M) && \otimes\text{-Hom adjunction} \\ &\mathrm{Map}^h_{D_{m+1}\text{-lin}}(D_{m+1}, M) && \text{pretending + base change} \end{aligned}$$

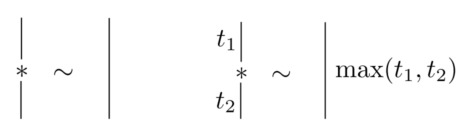
Reminder about γ: we had the divided powers functor $(\gamma_n X)(m) = X(nm)$.

Boardman-Vogt W construction. What is a free algebra? For a vector space V, we can take $TV = \bigoplus_{n \geq 0} V^{\otimes n}$ as the free associative algebra on V.

What is a free operad? For a pointed symmetric sequence X (i.e. distinguished point at arity 1), the free operad on X consists of planar rooted trees with internal vertices labelled by X; i.e. $F(X)(n) =$ such trees with n leaves, and internal vertices of valence k are labelled by elements of $X(k)$, modulo unit and equivariance. Picture:



Definition 4.4.7. The W-construction on a topological operad $\mathcal{O}$ is the same as the free operad on $\mathcal{O}$, except edges have length in $[0, 1]$. If an edge has length 0, then that is the same as the one where you compose the two nodes that are distance zero from each other. Also,



Idea: this is a mapping cylinder in an operad-y way, which is why we expect cofibrancy out of these things. It is a cofibrant replacement if $\mathcal{O}$ is well-pointed. In simplicial sets, replace $[0, 1]$ with $\Delta[1]$.

Day 5: The Grothendieck-Teichmüller group

Talk 5.1: The profinite Grothendieck-Teichmüller group (Massimiliano Ungheretti)

If you look up the definition of the Grothendieck-Teichmüller group, you are told something like this:

Definition 5.1.1. Let $\widehat{GT}$ be the subset of $(\lambda, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2'$ satisfying some conditions...

During this talk I will try to give you some idea of what this group is and where the conditions come from. Let's start by saying what these hats mean.

Profinite groups.

Definition 5.1.2. Let C be a category. We want to construct a category Pro$C$ in which the objects are inverse limits of objects in C. So, an object in Pro$C$ should be viewed as $\varprojlim_I G_i$ for $G_i \in C$ where I is a poset such that for all $a, b \in I$ there exists c such that $a \leq c$ and $b \leq c$. More generally, we can let I be a cofiltered category. Morally, the morphisms should be $\mathrm{Hom}_{\mathrm{Pro}C}(\varprojlim G, \varprojlim H) = \varprojlim_J \varinjlim_I \mathrm{Hom}_C(G_i, H_j)$. See Artin & Mazur's book on étale homotopy theory.

Rather than look at the formal definition of Pro$C$, let's do the example of C being the category of finite groups, thought of with discrete topology.

Definition 5.1.3. A profinite group G is a topological group $G = \varprojlim G_i$ where G_i are finite.

These show up when you're doing infinite Galois theory and the étale fundamental group.

Definition 5.1.4. Let G be any (discrete) group. We can associate to it the profinite completion $\widehat{G} = \varprojlim_{\substack{|G:H|<\infty \\ H \leq G}} G/H \subset \prod G/H$. This is a profinite group by definition.

Example 5.1.5.

- The p-adic integers are $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.
- The profinite completion of $\mathbb{Z}$ is $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$, with maps $\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ iff $m \mid n$.
- We have $\mathbb{Z} \subset \widehat{\mathbb{Z}} \subset \prod \mathbb{Z}/n\mathbb{Z}$ by viewing an integer λ as a family $\lambda_n = (\lambda \text{ modulo } n)$.
- If G is profinite, it is not necessarily the case that $\widehat{G} \cong G$.
- The map $G \to \widehat{G}$ is injective iff G is residually finite.

The étale fundamental group $\pi_1^{\text{ét}}$ is a functor from certain schemes to profinite groups. For now, all you need to know is that when X is a nice scheme over $\mathbb{C}$ (i.e. finite type), then $\pi_1^{\text{ét}} X = \widehat{\pi}_1 X^{an}$. This talk will not mention base-points, even though they do matter a lot.

Proposition 5.1.6. *When a group G is profinite, the power map (not necessarily a homomorphism) $\mathbb{Z} \times G \to G$ sending $(\lambda, g) \mapsto g^\lambda$, extends to powers $\lambda \in \widehat{\mathbb{Z}}$.*

While you could use that $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$, let's instead see what happens in terms of elements. To see what these profinite powers are, we first define them on finite groups and see that they extend to profinite groups by showing a compatibility along group homomorphims.

If $\lambda \in \mathbb{Z}$ and $g \in G = \varprojlim G_i$, then $(g^\lambda)_i = (g_i)^\lambda \in G_i$. If G_i is finite, then g_i^λ only depends on λ modulo $|G_i|$. Using that $\lambda \in \mathbb{Z} \subset \widehat{\mathbb{Z}}$ as a family $\lambda_n = \lambda$ modulo $n \in \mathbb{Z}/n\mathbb{Z}$, we have that $g_i^\lambda = g_i^{\lambda_{|G_i|}}$. Now for $\lambda \in \widehat{\mathbb{Z}}$, we promote this description to a definition: $g_i^\lambda := g_i^{\lambda_{|G_i|}} = g_i^{\lambda_{|g_i|}}$ (here $\lambda_{|G_i|}$ denotes the projection of λ to the $\mathbb{Z}/|G_i|\mathbb{Z}$ piece). For any map $\varphi : G_i \to G_j$, we have a commuting square (of maps of sets):

$$\begin{array}{ccc} G_i & \xrightarrow{(-)^\lambda} & G_i \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ G_j & \xrightarrow{(-)^\lambda} & G_j \end{array}$$

This coherence implies that the power map extends to profinite groups by acting on each finite G_i in the described way.

Outer Galois actions. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. This group contains a lot of information but is hard to study. For example, the only obvious elements you can write down are the identity and complex conjugation.

Let X be a scheme over $\mathbb{Q}$ (i.e. it has a map $X \to \operatorname{Spec} \mathbb{Q}$). We can take the pullback of this

$$\begin{array}{ccc} X_{\overline{\mathbb{Q}}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{\mathbb{Q}} & \longrightarrow & \operatorname{Spec} \mathbb{Q} \end{array}$$

which is called "base change." If you think in terms of polynomials and solution sets, then the $X_{\overline{\mathbb{Q}}}$ knows about solutions in $\overline{\mathbb{Q}}$, but forgot about the polynomials only having coefficients in $\mathbb{Q}$. This gives rise to a SES in fundamental groups.

$$\pi_1^{\text{ét}} X_{\overline{\mathbb{Q}}} \to \pi_1^{\text{ét}} X_{\mathbb{Q}} \to \pi_1^{\text{ét}} \operatorname{Spec} \mathbb{Q}.$$

Audience: You can think of this as Mayer-Vietoris, where $\pi_1^{\text{ét}} \operatorname{Spec} \overline{\mathbb{Q}} = 0$.

By a version of the Lefschetz principle, $\pi_1^{\text{ét}} X_{\overline{\mathbb{Q}}} = \pi_1^{\text{ét}} X_{\mathbb{C}} = \widehat{\pi}_1 X^{an}$.

In general, for a SES of groups, i.e. $N \leq G$ some normal subgroup, you get a map $G/N \to \mathrm{Out} N$ (automorphisms modulo inner automorphisms, i.e. conjugation automorphisms). This is done by picking a representative, conjugating by that representative and then showing that, up to conjugation by N, the automorphism didn't depend on the representative. In our case, we get a map $G_{\mathbb{Q}} = \pi_1^{\text{ét}} \operatorname{Spec} \mathbb{Q} \to \operatorname{Out}(\widehat{\pi}_1 X)$.

Let $\mathcal{M}_{g,n}$ = the moduli space of curves of genus g with n marked points. Points are iso classes of Riemann surfaces of type (g, n). Modular (complex) dimension $dim\mathcal{M}_{g,n} = 3g - 3 + n$. So $dim\mathcal{M}_{0,4} = 1$. In fact, as $PGL_2\mathbb{C}$ acts triply transitively on $\mathbb{P}^1$, $\mathcal{M}_{0,n} = (Conf_n\mathbb{P}^1)/PGL_2\mathbb{C} \cong (\mathbb{P}^1\backslash\{0,1,\infty\})^{n-3}\backslash\Delta$. This is especially nice for $\mathcal{M}_{0,4} = \mathbb{P}^1\backslash\{0,1,\infty\}$. This has étale fundamental group $\widehat{F_2}$ (profinite completion of the free group on 2 generators). Its automorphisms are S_3, which are exactly those Möbius transformations that permute the removed points. This will be important later on.

So this scheme gives us an outer Galois action $\mathbb{G}_{\mathbb{Q}} \to \operatorname{Out}\pi_1^{\text{ét}}\mathcal{M}_{0,4} = \operatorname{Out}\widehat{F}_2$. In fact, Belyi showed there is a lift $G_{\mathbb{Q}} \hookrightarrow \operatorname{Aut} \widehat{F}_2$. Write

$$F_2 = \langle x, y, z \,:\, xyz = 1\rangle$$

(where the generators are loops around the three removed points).

Fun fact 5.1.7. $\overline{\mathcal{M}}_{0,n+1}$ is an awesome operad. (It's almost the Fulton-MacPherson operad.)

History. In *Esquisse d'un Programme* (recommended read), Grothendieck sketches some ideas that have now grown into the study of GT, the Grothendieck-Teichmuller group. He said that the rich structure of $\mathcal{M}_{g,n}$ (erasing points /gluing) should be reflected in towers of fundamental groupoids $\widehat{T}_{g,n}$ and that this structure is respected by an action of $G_{\mathbb{Q}}$. Look at the first non-trivial such groupoid, $\widehat{T}_{0,4}$, containing $\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus \{0,1,\infty\}) = \widehat{\mathbb{Z}}$. The Galois action on this will determine some necessary conditions for automorphisms of $\widehat{\mathbb{Z}}$ to really come from $G_{\mathbb{Q}}$. With hard work, it might be possible to get enough necessary conditions to really say something about $G_{\mathbb{Q}}$, possibly even finding sufficient conditions. He then states what is called the *two level principle*: knowing the Galois action in modular dimensions 1 and 2 is enough to build the action on all of $\widehat{T}$.

Then came Drinfeld: In a paper on quasitriangular quasi-Hopf algebras, a version of the Grothendieck-Teichmuller group pops up for the first time. Here the conditions (0-3) that we'll see later come from the pentagon and hexagon relations. From his construction, parts of which we'll see in a talk later today, $\widehat{GT}$ is automatically seen to act on braid groups. Drinfeld does talk about how this group is related to Grothendieck's vision and says that $G_{\mathbb{Q}}$ should be a subgroup. Later Ihara gave a geometric description, arriving at the same conditions (0-3). From his construction he is able to show that $G_{\mathbb{Q}}$ injects into $\widehat{GT}$.

Schneps's version of Ihara's story.

Fact 5.1.8. *The $G_{\mathbb{Q}}$ action preserves inertia subgroups up to conjugation. For now, inertia subgroups are copies of $\langle x\rangle = \{x^\lambda\} = \widehat{\mathbb{Z}} \subset \widehat{F}_2$, profinitely generated by loops around* 0, 1, *or* ∞. *Concretely, $\langle x\rangle \mapsto h^{-1}\langle x\rangle h$ for some $h \in \widehat{F}_2$. This h can be different for the different generators and σ's.*

Let's see what information this buys us. Let $\sigma \in G_{\mathbb{Q}}$. Using conjugation, you can arrange the action such that it sends $x \mapsto x^{\lambda}$, $y \mapsto f^{-1}y^{\mu}f$, and $z \mapsto g^{-1}z^{\nu}g$, for some $\lambda, \mu, \nu \in \widehat{\mathbb{Z}}$, $f, g \in \widehat{F_2}$ all depending on the σ. We have $xyz = 1$ which implies $x^{\lambda}f^{-1}y^{\mu}fg^{-1}z^{\lambda}g = 1$ if we want σ to induce an automorphism. In the abelianization, this implies $x^{\lambda}y^{\mu}z^{\nu} = 1$ so $\lambda = \mu = \nu$. Also note that we didn't really need g because the action on z can be computed using $z = (xy)^{-1}$.

For each σ, the pair (λ, f) is almost unique. But, if we had changed f to $y^{\alpha}f$ this would define the same automorphism. We can get rid of this freedom by requiring $f \in \widehat{F}'$ (derived subgroup, where you take all commutators). It turns out that the f is then uniquely determined by σ.

We can compare this action to the action on $\mathbb{P}^1 \backslash \{0, \infty\}$. This has $\pi_1^{\text{ét}} = \widehat{Z}$, generated by a small loop around 0. So we have a map $\chi : G_{\mathbb{Q}} \to Aut\widehat{Z} = \widehat{Z}^{\times}$. The comparison (see Ihara) tells you that our $\lambda = \chi(\sigma)$. This coincides with what is called the cyclotomic character, which evaluates the action on roots of unity:

If you have $\sigma \in G_{\mathbb{Q}}$ and an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$, identify $\mathbb{Z}/n\mathbb{Z}$ with the n^{th} roots of unity. σ sends roots of unity to roots of unity. If you stare at it, you realize that it has to take an element to some power of that element. That power is the cyclotomic character.

Exercise 5.1.9. Every pair $(\lambda, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2'$ gives rise to an element of End $\widehat{F}_2$ by $x \mapsto x^{\lambda}$ and $y \mapsto f^{-1}y^{\lambda}f$. The multiplication on $\widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2'$ making this

$$G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2' \to \operatorname{End} \widehat{F}_2$$

a homomorphism of monoids is

$$(\lambda, f) \cdot (\mu, g) = (\lambda\mu, fF(g))$$

where F is the element of End $\widehat{F}_2$ associated to (λ, f).

Let's go back to the definition.

Definition 5.1.10. Let $\widehat{GT}$ be the subset of $(\lambda, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2'$ satisfying:

(0) The induced endomorphism F is invertible

(1) $\theta(f)f = 1$

(2) $\omega^2(fx^m)\omega(fx^m)fx^m = 1$, where $m = (\lambda - 1)/2$.

(3) $\rho^4(\widehat{f})\rho(\widehat{f})\rho^2\widehat{f}\rho(\widehat{f})\widehat{f} = 1$ (Here $\rho \in \operatorname{Aut}(\mathcal{M}_{0,5})$. This comes from modular dimension 2. $\pi_1^{\text{ét}}\mathcal{M}_{0,5}$ is the mapping class group of the 5-punctured sphere. This is generated by Dehn twists.)

We have now made sense of what the $\widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2'$ is, and what (0) means. As for the remaining conditions, they reflect the compatibility of the Galois action with the Aut $\mathcal{M}_{0,n}$ action on $\pi_1^{\text{ét}}$ (groupoid). For example, $S_3 = \operatorname{Aut} \mathcal{M}_{0,4}$ as mentioned earlier. Let $\theta \in S_3$ be the element that swaps the generators x and y. This is given by the Möbius transformation $\theta : t \mapsto 1 - t$. Compatibility with this implies condition (1). Similarly, there's another generator $\omega : t \mapsto (1 - t)^{-1}$ which cyclically permutes x, y, z. Compatibility with this implies condition (2). Note that these change basepoints, which is why we need to be careful and use $\pi_1^{\text{ét}}$ groupoids with tangential basepoints, which was developed by Serre I believe. We will not do this now, see Ihara. If you play a similar game with $\mathcal{M}_{0,5}$ plus the map to $\mathcal{M}_{0,4}$, you get condition (3).

Proposition 5.1.11. *If $[\theta, F] = 1 \in \mathrm{Out}\widehat{F}_2$ then condition (1) holds.*

Proof. (Proof not in lecture) F and θ commuting in the outer automorphisms means that there is some $\gamma \in \widehat{F_2}$ for which $inn(\gamma)F\theta = \theta F$ in $\mathrm{Aut}\,\widehat{F}_2$. Evaluating both sides of this on x and y gives

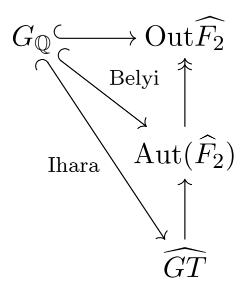
$$F\theta(x) = F(y) = f^{-1}y^{\lambda}f$$
$$\theta F(x) = \theta(x^{\lambda}) = y^{\lambda}$$
$$F\theta(y) = F(x) = x^{\lambda}$$
$$\theta F(y) = (\theta(f))^{-1}x^{\lambda}\theta(f)$$

The first two equations imply that $y^{-1}f^{-1}y^{\lambda}f\gamma = y^{\lambda}$. But which elements can commute with y^{λ}? Only powers of y can! So $f\gamma = y^k, k \in \widehat{\mathbb{Z}}$. The second two equations similarly imply that there is an m such that $\theta(f) = x^m\gamma = x^m f^{-1}y^k$. In the abelianization we have $\theta(f) = f = id$, so $m = k = 0$. The conclusion is that $\theta(f) = f^{-1}$ which is condition (1). ☐

Proposition 5.1.12. *By magic or intimidation, this is a group.*

(You need to check that all these relations are respected by the multiplication rule. The fact that inverses exist is artificially imposed using condition 0)

Injection $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. Because of the recipe we gave for $G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2'$ plus the calculations implying (1-3), we know that $G_{\mathbb{Q}} \to \widehat{GT}$. Belyi showed injectivity of $G_{\mathbb{Q}} \hookrightarrow \mathrm{Out}\widehat{F}_2$, meaning that $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. This was all put together by Ihara, confirming a prediction Drinfeld made when defining GT.



Example 5.1.13. What is the image of complex conjugation? The cyclotomic character is -1. I claim $f = 1$, because y has to be mapped to its own inverse. To see this, draw $\mathbb{P}^1$ and the generators and act by complex conjugation on the picture.

Actions of $G_{\mathbb{Q}}$ and $\widehat{GT}$. $G_{\mathbb{Q}}$ maps to $\mathrm{Out}\widehat{\pi}_1 X$ for most reasonable X's, which is expected to usually be injective. Almost by definition, $\widehat{GT}$ acts on $\pi_1^{\text{ét}}$ of a few moduli spaces. Whenever you can write down the action of $G_{\mathbb{Q}}$ in terms of $(\lambda_\sigma, f_\sigma)$, you are in business and can try to extend this action to $\widehat{GT}$. This can be done for example for braid groups (configuration spaces) and genus zero mapping class groups (moduli spaces of curves). When done correctly, the action has all sorts of coherence for these families.

If $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ was an iso, this would be great. This is an open problem though.

Nakamura describes the action in terms of (λ, f) on mapping class groups of surfaces with any genus, and 0 or 1 marked point. That doesn't necessarily show that $\widehat{GT}$ acts, as you have to check the defining conditions of $\widehat{GT}$ are respected. The action of $G_{\mathbb{Q}}$ gives an extra condition, which could be added as a fourth condition, but no one knows if it's implied by the $\widehat{GT}$ axioms. If it is not, then $G_{\mathbb{Q}} \neq \widehat{GT}$, but the smaller group that satisfies the extra condition still might be.

Talk 5.2: Little 2-disks and the profinite Grothendieck-Teichmüller group (Chris Owens)

Goal:

Theorem 5.2.1. *The group of homotopy automorphisms of profinite completions of D_n is isomorphic to $\widehat{GT}$.*

Definition 5.2.2. A space X is π-finite if it has a finite number of connected components, and $|\pi_n X| < \infty$ and almost all zero.

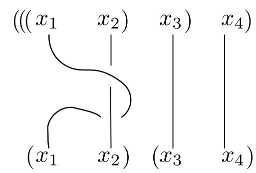
This is an interesting condition because I'm interested in minimal models for X: I want a map from a π-finite version of X to all the other π-finite ones. But there is no initial object.

Artin-Mazur pro-category of $Ho(\mathbf{Spaces})$. This is a replacement for our non-existent universal object. There are several ways to do this; one was constructed by Quick. The profinite completion $\widehat{\mathrm{Set}}$ is equivalent to the category of compact Hausdorff totally disconnected spaces where the morphisms are continuous maps.

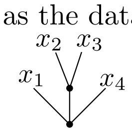
Little 2-disks. We said that the n^{th} space of the little 2-disks operad had homotopy type of $\mathrm{Conf}_n(\mathbb{R}^2)$; it's equivalent to the classifying space of the pure braid group K_n. This is the kernel in $K_n \to B_n \to \Sigma_n$ where B_n is the ordinary braid group; it consists of braids that end where they started.

Operads in groupoids. Because of the above, we have that there is an operad in groupoids that is a model for the little 2-disks operad. We can talk about operads in groupoids in the first place, because the category of groupoids is symmetric monoidal w.r.t. the cartesian product, where the arity-0 element is the point. The classifying space functor $B : \mathrm{Cat} \to sSet$ is symmetric monoidal.

Let PaB denote the operad of parenthesized braids, where in arity n the objects are we have parenthesized words in $\{x_1, \ldots, x_n\}$, and the morphisms are braids (ignoring the parentheses) with strands labelled by $x_1, \ldots, x_n$.



Alternatively, you can view $x_1(x_2\ x_3)x_4$ as the data of a tree



or as balls, where x_2 and x_3 are smaller than x_1 and x_4 (this kind of ties in with the "infinitesimal points" idea of the Fulton MacPherson operad).

It has really good categorical properties, and it provides a way to encode the braided monoidal structure on a category in a weak sense. There is a functor $\varphi : PaB \to \mathrm{End}_C$.

Grothendieck-Teichmüller. This is based on a paper of Horel, who looked at Drinfeld's papers and wrote them in operadic language.

Theorem 5.2.3 (Drinfeld)**.** *The monoid $\widehat{GT}$ is the monoid of endomorphisms of $\widehat{PaB}$ which induce the identity on $\mathrm{Obj}\widehat{PaB}$.*

Let S be the category of simplicial sets, $\widehat{S}$ the profinite completion in the category of groupoids, the category G of groupoids, and its profinite completion $\widehat{G}$. There are classifying space functors (which are right Quillen) $B : G \to S$, $B : \widehat{G} \to \widehat{S}$, and profinite completion functors $\widehat{(-)} : S \to \widehat{S}$ and $\widehat{(-)} : G \to \widehat{G}$ which are left Quillen. The category $\widehat{G}$ has a functorial path object $C \to C^{I(1)}$ in $Op\widehat{G}$. (Here $I(1)$ is the completion of the category [1].)

In the following, π will mean the homotopy category.

Theorem 5.2.4. *$\widehat{GT} \to \mathrm{End}_{\pi\mathrm{Op}(\widehat{G})}(\widehat{PaB})$ is an isomorphism induced by $\widehat{GT} \to \mathrm{End}(\widehat{PaB})$.*

Important issue: profinite completion doesn't work well with products. If you take levelwise completion of an operad, it's most likely not an operad anymore. It's a weak operad. But, this is OK for groupoids.

Horel's original approach was via dendroidal sets. But he recently revised the paper and used weak operads instead. These are defined via algebraic theories of operads.

If we have weak operads, we need to know they have the same homotopy type as what we started with:

$$\mathrm{End}_{\pi\mathrm{Op}\widehat{G}}(PaB) \cong \mathrm{End}_{\pi W\mathrm{Op}\widehat{G}}(N)$$

where N is the dendroidal nerve.

The takeaway is that, even on the space level, $\widehat{GT}$ is the automorphisms of the parenthesized braid operad.

There should be an action of the absolute Galois group: think of configurations of points in $\mathbb{R}^2$ as points in $\mathbb{C}$, and that's a variety. But you should only see this action up to completion. You can do this over other fields.

There's a similar result for the pro-ℓ completion.

Talk 5.3: Little disks and the pro-unipotent Grothendieck-Teichmüller group (Joshua Wen)

A Drinfeld story. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. $t \in \mathfrak{g} \otimes \mathfrak{g}$ is an r-matrix if it satisfies the classical Yang-Baxter equations:

- $[t_{12}, t_{13}] + [t_{12}, t_{23}] + [t_{13}, t_{23}] = 0$

(Don't worry about what t_{ij} means, or read your favorite book on quantum groups.) If you have one of these things, you can make a knot invariant. Drinfeld cared because they're good for physics.

Given $\mathfrak{g}$ and an r-matrix t, you can get a quantum universal enveloping algebra A, and $\mathrm{Rep}(A)$ is braided monoidal. t corresponds to the twisting isomorphism $V \otimes W \to W \otimes V$ on the $\mathrm{Rep}(A)$ side.

There's also an associativity axiom:

$$\alpha : ((V_1 \otimes V_2) \otimes V_3) \xrightarrow{\cong} (V_1 \otimes (V_2 \otimes V_3))$$

When this is an equality on the nose, this is a strict braided monoidal category; Kirillov et al. say they don't know any good examples of non-strict ones.

In conformal field theory, there's a God-given connection called the KZ connection. We need $t \in \mathfrak{g} \otimes \mathfrak{g}$ and an action of $\mathfrak{g}$ on V. You end up with a connection on $\mathrm{Conf}_r(\mathbb{C}) \times V^{\otimes r}$:

$$w_{KZ} = \sum_{1 \le i \le r} t_{ij} \otimes d\log(z_i - z_j).$$

If t is an r-matrix, then w_{KZ} is flat. Parallel transport only depends on the homotopy class of the path, so I get an action of the fundamental group.

Drinfeld studied the monodromy $\pi_1(\mathrm{Conf}_r\mathbb{C})$ acting on $V^{\otimes r}$. He found these associativity isomorphisms and "Drinfeld's associators." He defined the pro-unipotent Grothendieck-Teichmüller $GT(\mathbb{Q})$ group with an action on associators, and showed that these associators are a torsor for it.

Theorem 5.3.1 (Fresse)**.** *Let $\widehat{R}_2$ be a cofibrant-fibrant model for the rationalization of D_2. Then*

$$h\,\mathrm{Aut}_{sSetOp}(\widehat{R}_2) \simeq GT(\mathbb{Q}) \rtimes S^1_{\mathbb{Q}}$$

where $S^1_{\mathbb{Q}}$ is the rational circle (a $K(\mathbb{Q},1)$). Also,

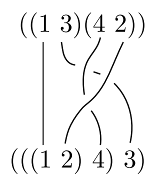
$$\mathrm{Aut}_{hsSetOp}(\widehat{R}_2) = GT(\mathbb{Q}).$$

Fact 5.3.2.

$$E_2(r) \simeq \mathrm{Conf}_r(\mathbb{C}) \simeq K(PaB, 1)$$

where PBr is the pure braid group.

Parenthesized braid operad. The parenthesized braid operad PaB is an operad in groupoids, where in the r^{th} level the objects are fully parenthesized words in $\{x_1, \dots, x_r\}$ (e.g. $((x_1x_2)x_3)$). Alternatively, think of this as planar binary trees with r labelled leaves. The morphisms between two words a, b are isotopy classes of braids linking x_i to x_i (where you don't care about the parentheses).



This is an operad. We will show that $\mathrm{Aut}(g) \simeq PBr$, and $B(PaB(r)) \simeq E_2$. *What's g?*

Algebras for PaB in Cat are braided monoidal categories.

Malcev completion of group(oid)s. Let G be a group, $g \in G$. We want to be able to take rational powers of g. How? If we can write $g = \exp(\xi)$ and the product $a\xi$ for $a \in \mathbb{Q}$ makes sense.

Hopf trivia: $\mathbb{Q}[G]$ is a Hopf algebra with $\Delta(g) = g \otimes g$. Let H be a Hopf algebra. The grouplike elements are $\mathbb{G}(H) = \{x \in H : \Delta(x) = x \otimes x\}$, the primitives are $\mathbb{P}(H) = \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}$, and the augmentation ideal is $\mathbb{I}(H) = \ker \varepsilon$.

You can show that $\mathbb{G}(H)$ is a group, $\mathbb{I}(H)$ is an ideal, and $\mathbb{P}(H)$ is a Lie algebra.

We have a filtration $H = \mathbb{I}^0(H) \supset \mathbb{I}^1(H) \supset \mathbb{I}^2(H) \supset \dots$. Define $\widehat{H} = \lim_s H/\mathbb{I}^s(H)$.

Definition 5.3.3. The Malcev completion of G is $\widehat{G} := \mathbb{G}\widehat{\mathbb{Q}[G]}$.

This is left adjoint to the forgetful functor from Malcev-complete groups down to groups.

Proposition 5.3.4. *There are inverse bijections*

$$\exp : \mathbb{P}(\widehat{H}) \leftrightarrows \mathbb{G}(\widehat{H}) : \log .$$

$\mathbb{P}(\widehat{H})$ is filtered from H and complete. The grouplikes $\mathbb{G}(\widehat{H})$ have a filtration $F_n\mathbb{G}(\widehat{H}) = \{g \in \mathbb{G}(\widehat{H}) : g - 1 \in F_n(\widehat{H})\}$. This is complete.

In particular,

$$F_n\mathbb{G}(\widehat{H})\Big/F_{n+1}\mathbb{G}(\widehat{H})$$

is a $\mathbb{Q}$-module. The associated graded is a Lie algebra.

Let $\mathscr{G}$ be a groupoid. I can contemplate $\mathbb{Q}[\mathscr{G}]$, which has the same objects as $\mathscr{G}$, and morphisms $\mathrm{Hom}_{\mathbb{Q}[\mathscr{G}]}(x, y) = \mathbb{Q}[\mathrm{Hom}_{\mathscr{G}}](x, y)$ ("morally free" vector space – it needs to be compatible with composition). The coproduct is still $\Delta\varphi = \varphi \otimes \varphi$. Let $\mathbb{I}(\mathbb{Q}[\mathrm{Hom}(x, y)]) = \ker \varepsilon$ be the augmentation ideal, and define the filtration $\mathbb{I}^n(\mathbb{Q}[\mathrm{Hom}(x, y)]) =$ the set of

$$x_0 = x \xrightarrow{\varphi_1} x_1 \xrightarrow{\varphi_2} \dots \qquad x_{n-1} \xrightarrow{\varphi_n} y = x_n$$

such that $\varphi_i \in \mathbb{I}(\mathbb{Q}[\mathrm{Hom}(x_i, y_i)])$.

If I have $G = \{x_1, \dots, x_n \,:\, \text{relations } w_1 = 1, \dots, w_m = 1\}$. The Lie algebra $\mathbb{P}(\widehat{\mathbb{Q}[G]})$ is the free Lie algebra on $\xi_1, \dots, \xi_n$ modulo the ideal generated by $\log w_i \exp \xi_i$.

Notice that $B(PaB_{\mathbb{Q}}) = K(BPaB_{\mathbb{Q}}, 1)$ (here $PaB_{\mathbb{Q}}$ is the rationalization, as in rational homotopy theory).

Definition 5.3.5. $GT(\mathbb{Q})$ is the group of operad automorphisms of $PaB_{\mathbb{Q}}$ that fix objects.

Fact 5.3.6. *It's enough to consider $\varphi : PaB \to PaB_{\mathbb{Q}}$, by adjunction nonsense. Such φ is determined by*

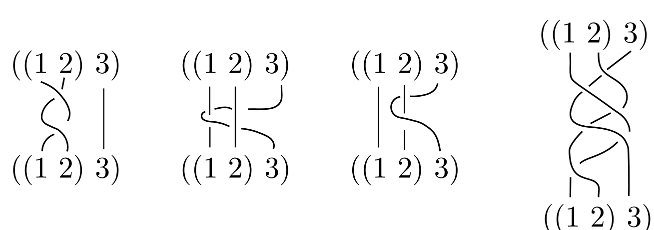
τ α

and $\varphi(\tau) = \tau^a$ for $a \in \mathbb{Q}^\times$.

$\varphi(\tau)$ and $\varphi(\alpha)$ satisfy similar relations for braided monoidal categories.

$$\varphi(\alpha) = \alpha \circ \mathrm{Mor}_{PaB_{\mathbb{Q}}}(((1\ 2)\ 3), ((1\ 2)\ 3))$$

generated by



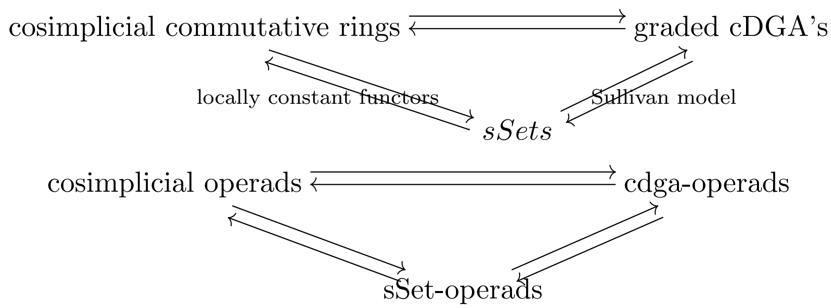
a_{12} a_{13} $\alpha^{-1}a_{23}\alpha$ c

(c is central.) (Here Mor represents a specific morphism, not the set of them.)

$\varphi(\alpha) = \alpha \circ c^\lambda F(a_{12}, a_{23})$ where $F(\dots) \in \mathbb{F}(a_{12}, a_{23})_\mathbb{Q}$ *miscopied something?*. If you play with the relations, you get $\lambda = 0$.

The hexagon relation gives $F(x,y)F(z,x)F(y,z) = 1$, $xyz = 1$.

Rational homotopy for operads. In the usual case for simplicial sets, you have adjunctions



$(D_2)_\mathbb{Q} = GC^*_{CE}(\widehat{p}) = B(PaB_\mathbb{Q})$.

Question: how do all the GT's relate?

$$\begin{array}{ccc} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \hookrightarrow & \widehat{GT} \\ & & \downarrow \\ & & GT_\ell \\ & & \downarrow \\ GT(\mathbb{Q}) & \longrightarrow & GT(\mathbb{Q}_\ell) \text{ pro-}\ell \end{array}$$

Talk 5.4: Recent developments and future directions (Kathryn Hess)

A final word on GT.

Conjecture 5.4.1. There exists some operad E in schemes/$\mathbb{Q}$ such that the étale homotopy type recovers the profinite completion of an E_2-operad. So you get an action of $G_\mathbb{Q}$ on $\widehat{E}_2$.

A "geometric" proof of iterated delooping. This is about work by Boavida de Brito and Michael Weiss posted on the arxiv on February 5, 2015.

Here's their main result.

Theorem 5.4.2. *If $n - m \geq 3$, there is a homotopy fiber sequence*
$$\mathrm{Emb}_\partial(D^m, D^n) \hookrightarrow \mathrm{Imm}_\partial(D^m, D^n) \to \Omega^m \operatorname{Map}^h_{Op}(D_m, D_n).$$
So the homotopy fiber $\overline{\mathrm{Emb}}(D^m, D^n)$ is $\simeq \Omega^{m+1} \operatorname{Map}^h_{Op}(D_m, D_n)$.

For a smooth manifold M, they construct an ∞-category $\underline{\mathrm{Con}}(M)$ whose objects are configurations of points in M, and the morphisms are paths of configurations, where paths can merge. This comes with a functor to finite sets (actually, it's a functor to the nerve).

They actually prove a stronger result.

Theorem 5.4.3. *For all $k \geq 1$, there exists a homotopy fiber sequence*
$$T_k\mathrm{Emb}_\partial(D^m, D^n) \to \mathrm{Imm}_\partial(D^m, D^n) \to \Omega^m \operatorname{Map}^h_{\mathrm{Fin}}(\underline{\mathrm{Con}}(\mathbb{R}^m; k), \underline{\mathrm{Con}}(\mathbb{R}^n))$$
(recall that Imm *is the first level in the Taylor tower).*

So not only is the embedding space an m-fold loop space; every approximation is, too.

Theorem 5.4.2 is a consequence of a "jazzed-up Hirsch-Smale":

Theorem 5.4.4. *If $n - m \geq 3$, there is a homotopy pullback*
$$\begin{array}{ccc} \mathrm{Emb}(M, N) & \longrightarrow & \operatorname{Map}^h_{\mathrm{Fin}}(\underline{\mathrm{Con}}(M), \underline{\mathrm{Con}}(N)) \\ \downarrow & & \downarrow \\ \mathrm{Imm}(M, N) & \longrightarrow & \Gamma \end{array}$$
Here Γ is a space of sections of $E \xrightarrow{\pi} M$, where
$$\pi^{-1}(x) = \{(y \in N, \underline{\mathrm{Con}}(T_xM) \xrightarrow{F} \underline{\mathrm{Con}}(T_yN)) \text{ over } \mathrm{Fin}\}.$$
The vertical maps are basically forgetful maps.

We also need the following:

Theorem 5.4.5.
$$\operatorname{Map}^h_{Op}(D_m, D_n) \simeq \operatorname{Map}^h_{\mathrm{Fin}}(\underline{\mathrm{Con}}(\mathbb{R}^m), \underline{\mathrm{Con}})(\mathbb{R}^n)$$

Theorem 5.4.6 ("Alexander trick")**.**
$$\operatorname{Map}^h_{\mathrm{Fin}_*,\partial}(\underline{\mathrm{Con}}(D^m), \underline{\mathrm{Con}}(D^n)) \simeq *$$

I want to talk about a few models for $\underline{\mathrm{Con}}(M)$. A complete Segal space is a simplicial space where the n^{th} space is a bunch of iterated homotopy pullbacks involving X_0 (objects) and X_1 (morphisms). This encodes the fact that the composition map is only associative up to homotopy.

Particle model (Andrade). $\mathrm{Map}(\underline{k}, M)$ is stratified by equivalence relations on $\underline{k} = \{1, \dots, k\}$. A stratum consists of the maps that factor through the n^{th} equivalence relation. Let C_m be the topological category with $(C_M)_0 = \coprod_{k\geq 0} \mathrm{Conf}_k(M)$. If $f \in \mathrm{Conf}_k(M)$ (a map from k points into M that happens to be an embedding) and $g \in \mathrm{Conf}_\ell(M)$, then the morphisms

$$(C_m)_1(f, g) = \coprod_{v:\underline{k}\to\underline{\ell}} = \{\text{reverse exit paths from } f \text{ to } gv\}$$

where reverse exit paths are paths that respect the strata (i.e. points can collide but not un-collide).

Compactification approach. There's also an approach due to Fulton-MacPherson, Axelrod-Singer, and Sinha.

Start with a topological category FM. The objects are given by $(FM)_0 = \coprod_{k\geq 0} \mathrm{Conf}_k[M]$.

Rational formality of D_n. This is work of Fresse-Willwacher, posted on the arxiv on April 6, 2015.

From now on, we're working over $\mathbb{Q}$. Warning: I'm going to be un-careful about basepoints.

"Recall": $H_* D_n = Pois_{n-1}$ for $n \geq 2$. This is generated by two operations of arity 2: the commutative multiplication $\mu \in Pois_{n-1}(2)_0$ (degree 0) and $\lambda = \{-, -\} \in Pois_{n-1}(2)_{n-1}$.

$Pois_{n-1}$ is also equipped with an involution J_* induced by the given involution J on D_n.

Remark 5.4.7. $Pois_{n-1}$ is an operad in graded cocommutative coalgebras, i.e. it's a cocommutative Hopf operad. The dual will be called $Pois_{n-1}^c$; it is a commutative Hopf cooperad, i.e. a cooperad in commutative graded algebras.

Strategy: build a rational model for D_n from $Pois_{n-1}^c$. However, that's naïve. The real problem is a homotopy invariance problem. This is very far from cofibrant. (We've moved out of the world of ∞-categories, so we have to care about this.)

Solution: there exists an operad P_n in graded Lie algebras such that, $Pois_{n-1}^c \xrightarrow{\sim} C^*_{CE}(P_n)$ (Chevalley-Eilenberg complex). The RHS is (quasi)-free, so we can do homotopy theory.

Recall: there is an adjunction $A^* : sSet^{op} \leftrightarrows cDGA : G_\bullet$. This gives an equivalence on the rational homotopy categories. We want to end up with a model for little disks. Apply $G_\bullet$ levelwise to $C^*_{CE}(P_n)$ to get a simplicial operad.

You can understand the elements of this as Maurer-Cartan elements associated to $P_n \otimes \Omega^*(\Delta^\bullet)$.

The point is that $G_\bullet(C^*_{CE}(P_n))$ is a really good model for D_n operads.

Theorem 5.4.8. *Let P be a simplicial operad with $P(1) = *$ and all $P(r)$ "behave well" w.r.t. $\mathbb{Q}$-localization. If $H_*P \cong Pois_{n-1}$ (as operads) for some $n \geq 3$, and*

> *if $4 \mid n$, there exists an involution $J : P \to P$ such that H_*J corresponds to the action of J_* on the little disks operad*

then there exists a cofibrant replacement $P \xleftarrow{\sim} R$, such that $R_{\mathbb{Q}} \xrightarrow{\sim} G_\bullet(C^*_{CE}(P_n))$.

Slogan: this is a rational recognition principle for E_n-operads, for $n \geq 3$. These are intrinsically formal: as soon as the homology is the right thing, they're all rationally equivalent (to $G_\bullet(C^*_{CE}(P_n))$).

In the topological version, this works for $n = 2$ as well because this case is taken care of by other means by Drinfeld.

Theorem 5.4.9. *Let $n \geq 2$. Then there exists $R \xleftarrow{\sim} D_n$ such that $R_{\mathfrak{q}} \xrightarrow{\sim} |G_\bullet(C^*_{CE}(P_n))|$. You can replace D_n by any E_n-operad.*

Moreover, if $n - m \geq 2$, the equatorial inclusion $\varphi_{m,n} : D_m \to D_n$ fits into a diagram

$$\begin{array}{ccccc}
D_m & \xleftarrow{\sim} & R_m & \xrightarrow{-_{\mathbb{Q}}} & |G_\bullet(C_{CE}(P_m))| \\
\downarrow{\scriptstyle \varphi_{m,n}} & & \downarrow & & \downarrow \\
D_n & \longleftarrow & R_n & \xrightarrow{-_{\mathbb{Q}}} & |G_\bullet(C_{CE}(P_n))|
\end{array}$$

commuting up to operad homotopy.

Slogan: formality of $\varphi_{m,n}$.

Remark 5.4.10 (Turchin-Wallwacher)**.** $\varphi_{n-1,n}$ is not formal.

Proof strategy. The idea is to reduce the problem to pure algebra. They prove analogous theorems for differential graded commutative Hopf cooperads satisfying analogous hypotheses. Then they recover the simplicial/ topological case by applying an operadic extension of $A^* : sSet^{op} \to cDGA$.

Now I'll talk about the algebraic solution. Choose a cofibrant replacement $R \xrightarrow{\sim} P$ of the dg commutative Hopf cooperad. Now we study obstructions to the existence of $R_{\mathbb{Q}} \to C^*_{CE}(P_n)$. This naturally lies in come bi-cosimplicial bi-derivation complex. Its cohomology can be computed by a certain bicomplex built from the Harrison complex and the cooperadic cobar construction.

You can reduce this to the computation of the homology of a graph complex – a complex where in each degree it's spanned by graphs. This is Kontsevich's graph complex; the graphs are (at least) trivalent. We have a differential that takes a vertex and splits it, putting an edge in between. They show that the cohomology is zero, so there are no obstructions, and you win. □