TALBOT WORKSHOP TALK OUTLINES/ PREPARATION

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We give guidance for speakers preparing talks for the upcoming Talbot seminar. For the Monday, Thursday, and Friday talks, we have provided a bullet-point description of the topics each speaker should learn and understand in order to prepare his or her talk, though the talk itself will usually not cover all of the topics listed. The speakers for these days are encouraged to contact Kathryn (kathryn.hess@epfl.ch) to discuss which topics should included in the actual talk.

For the Tuesday and Wednesday talks, we give more detailed and directed reading, and we hope to follow up with discussions of talk outlines once the speakers have gone through that directed reading. Contact Dev (dps@uoregon.edu) for further discussion about those days.

1. [Monday] Homotopy-theoretic foundations

1.1. **Introduction and general overview (Kathryn and Dev).**

1.2. **The homotopy theory of operads and their bimodules (K).**

- The definition of (symmetric) operads as monoids in the category of (symmetric) sequences, endowed with the composition monoidal product.
- Important elementary examples of (symmetric) operads, e.g., the associative operad, the commutative operad, the Lie operad and the Poisson/ Gerstenhaber operads.
- Algebras over operads as representations of the operad in the endomorphism operad. (Analogies with group actions.)
- Modules and bimodules over operads: definition and illustrative examples (e.g., arising from algebras, or from a pair formed of an algebra and a bimodule over the algebra)
- Berger and Moerdijk's model category structure on the category of operads in a nice enough monoidal model category, extended by Fresse to (bi)modules over operads.

Possible references: [11], [43], [68], [71].

1.3. **The Boardman-Vogt tensor product (K).**

- Construction of the B-V tensor product of simplicial and topological operads.
- Why B-V tensor products capture interchanging algebraic structures.
- Formula for the B-V tensor product of the associative operad with any other operad.
- Low-arity computations of the B-V tensor product.
- Lifting the B-V tensor product to bimodules and the associated tensor/hom adjunction [37, Proposition 2.3]

Possible references: [14], [15], [16], [37], [42]

Date: March 2, 2015.

1.4. $\mathcal{E}_n$-operads (K).

- Definition of little cubes and little disks operads, explanation of how they detect iterated loop spaces and of their limitations (cf.[62, Section 3]), comparison with Steiner operads.
- The notion of $\mathcal{E}_n$-operad (à la Boardman-Vogt and Fiedorowicz-Vogt).
- Recognition principles when $n = 1, 2, \infty$. ($n = 1$ is Stasheff's thesis.)
- How to prove that the little n-disks operad is indeed an $\mathcal{E}_n$-operad (e.g., [73, Proposition 4.9]).
- Cellular $\mathcal{E}_n$-operads.

Possible references: [10], [15], [21], [33], [41], [42], [62], [73], [82].

1.5. Model categories and derived mapping spaces (K).

- Simplicial localization of categories with weak equivalences.
- Identification of derived mapping spaces in simplicial model categories.
- Quillen's Theorem A.
- The Derived Adjunction theorem [36, Theorem 2.12] and its proof.
- Applications of the Derived Adjunction theorem (e.g., [38, §2]).

Possible references: [32], [36], [38], [34], [35], [53], [70]

2. [Tuesday] Configuration spaces and knot theory

2.1. (Co)homology and compactifications of configuration spaces. .

Background: basically none. (The Leray-Serre spectral sequence is helpful, but not absolutely necessary.)

Directed Reading: Give a thorough reading of [80]. This should be fairly easy going, as it was written for a beginning graduate student audience. The whole paper is relevant to the program of the workshop, and much of it should be addressed in this talk.

Exercise: work out bases for H_6 and H^6 of configurations of four points in $\mathbb{R}^3$.

Then move on to the compactifications paper [76]. Start with the definitions of Section 1, go on to the stratification as treated in Section 3. Go through Section 3.1, making sure your fully understand the first Example. Then read Section 3.2 carefully, referring back to Section 2 as needed. You do not need to have detailed understanding of the proofs in Section 3.3. Then understand the statements of properties listed in Theorems 4.2-4.10.

Exercises:

- Draw representatives for all of the strata of configurations of four points in $\mathbb{R}^3$.
- Explain how the homology classes defined in [80] are given by fundamental classes of strata, and the relations between them are given by fundamental chains as of strata as well.

Finally, move on to operad structure. See how the operad structure is defined in the language of the manifold-theoretic compactifications in the case of the Kontsevich operad in Section 4 of [77]. In particular, understand how to rigorously define maps which "insert a configuration as an infinitesimal configuration." Review this construction as well as get a good summary of what you've learned by reading Sections 5.1 and 5.2 of [61].

You are ready to prepare and give the talk when:

- You can explain how the homology-cohomology pairing for configurations in Euclidean space is given by a combinatorial pairing between graphs and trees.
- You can explain why the operad structure on compactifications is related to the combinatorial "choose-two operad", when one looks at unit vectors between points.
- You can explain how compactifications of Euclidean configuration spaces "wear their homology on their sleeves (that is, strata)."
- You can show that the compactification of configurations of $n+2$ points in the interval with two points fixed at the endpoints is the nth Stasheff polytope.

2.2. **Graph-theoretic models for the de Rham theory of configuation spaces.** .

Background needed:

- Familiarity with de Rham theory (for example, from the book by Bott and Tu [20]) and with configuration spaces and their compactifications (the first three sections of [76] – see directed reading in Section 2.1).
- You must also understand what push-forward (a.k.a. wrong-way or shriek map $f_!$) is. There are many models for push-forward maps in cohomology, say for smooth fiber bundles with manifold fibers (and other submersions). There are:
  - Apply Poincare duality to be in homology, apply the natural map, and then apply Poincaré duality again.
  - If a cohomology class is represented by a sub manifold (either through a Thom form in de Rham theory, through its fundamental class in Poincaré duality, or (our favorite) by evaluating on transverse chains of complementary dimension by taking intersection) then the push-forward is defined by taking the image of a representing sub manifold. Note that the codimension changes.
  - Integration over the fiber in de Rham theory.
  - By embedding the bundle (assuming say compactness of base and total space) in Euclidean space, the Pontryagin-Thom construction gives rise to a stable map which coincides with push forwards.

  The de Rham theoretic model through integration over the fiber is important, and one should understand how it is equivalent to both the Poincaré duality definition and the image of representing sub manifold definition. Since push-forwards are common, this is worth understanding.
- And you must of course understand what a model is in (the commutative differential graded algebra formulation of) rational homotopy theory. But don't be worried if you don't know already. This is a great excuse to learn this very fundamental notion, which is explained well in many places.

Directed reading: You are encouraged to team up with the speaker of 3.1, since they will be building directly on your work.

This talk should be grounded in the work of Lambrechts-Volic [61], which fills in details of papers by Kontsevich [56] and Kontsevich-Soibelman [58]. There are many treatments of configuration spaces integrals in the literature going back almost 20 years, but the Lambrechts-Volic treatment is the most definitive. In particular it carefully develops all of the results on compactifications needed to use piecewise semi algebraic forms. These forms themselves are fully developed by Hardt-Lambrechts-Turchin-Volic in

[50], which requires non-trivial results from analysis. The exposition in [61] has both illuminating examples and thorough arguments, but that makes it a longer treatment as well.

Start with a brief read of the first few pages of [61] but then pay close attention starting on page 6 as they discuss the case of configurations in the complex plane, where they give a more direct formality argument due to Arnold. Namely logarithmic forms are closed and satisfy the defining relation, known as the Arnold identity or 3-term relation, at the cochain level. They then follow up (on pages 7 and 8) with the fundamental example of the construction of a form β which cobounds the 3-term relation in all dimensions.

Then take the authors' good advice and focus on Sections 5.1-5.2 and 6.1-6.2 and then the culminating Section 9. Look as well at directed reading for the previous talk for alternate sources for the material in Sections 5.1-5.2. Your goal is to have a thorough understanding of the quasi-isomorphisms between these graph complex and the (semi-algebraic) de Rham theory of these compactifications (Proposition 9.12), at the level of say understanding the statements of the all of the Lemmas which go into the proof of 9.12.

Follow this reading of the more important geometric Section 9 with the more combinatorial/ algebraic quasi-isomorphism of these graph complexes with their cohomology in Section 8. Track through how the quasi-isomorphism works explicitly say for configurations of three points.

Once you have the main idea (which is what most people are content with) there a couple technical points to be aware of. One is that these projection maps between compactified configuration spaces are not submersions, even though the restrictions to interiors clearly are. So one must leave the smooth category and work instead in the semi-algebraic category. This is a significant shift, fully developed in [50]. First one restricts functions to being polynomial functions, but then one demands that the theory be closed under push-forward, and immediately gets forms and functions based on logarithms and polylogarithms. See for example the Wikipedia article on polylogarithms to get a sense for the rich, distinguished theory, which is then implicitly part of this formality story (though there may be less transcendental approaches as well, as there are in similar questions [30]).

The lack of projection maps being submersions is the focus of Example 5.22, on page 44. We find the picture of compactification of two points in the interval given by the last Figure in Section 4 of [76] as more useful; the projection map is then "down" onto an interval, and one can see that the bottom-left corner would need to be "straightened" in this projection, meaning all its tangent vectors would map to zero. See also Example 1 (on page 63) in the Appendix of [58] as a central, very simple example worth working out (just "elementary" calculus).

Another technical point is that one does not have maps of cooperads at the cochain level, but the authors manage to get both formality of individual disk (really, Fulton-MacPherson) operads and relative formality from the tools at hand in Section 10.

You know you're ready to give the talk when:

- You can explain all of the signs in the graph complexes.
- You can explain why the "principal face contributions" of integration of tautological forms correspond to the differential in the graph complexes.
- You can explain why the graph complexes are quasi-isomorphic to their cohomology.
- You can explain why semi-algebraic forms are needed in this story.

2.3. **Finite type knot invariants and the Kontsevich and Bott-Taubes configuration space integrals.** .

Background needed: Basically none. You'll need the basic definitions of knot theory, including Reidemeister moves, which should be part of the vocabulary of every mathematician much less every topologist.

Directed Reading: For convenience, we will stick mostly to the book by Chmutov, Duzhin and Mostovoy [28] though we will refer to elsewhere in the literature some topics are treated as we go along. Start with whatever background you want to see in the first two Chapters. The exercises on the Jones polynomial (2.4.1 and 2.4.2) in particular are accessible to undergraduates and something "everyone should do at some point." The exercises on quantum sl_2 and sl_N invariants, namely 37 and 38 in the Exercises section, are worth doing for people wanting a concrete grounding in the relationship between representation theory and topology. This relationship is actively being developed through topological field theories and related topics.

Chapter 3 gives a detailed introduction to finite-type invariants, which are also known as Vassiliev invariants. Definition 3.1.3 and Example 3.1.4 are basic, the latter indicating how all skein-theoretic invariants are (after a change of coordinates, in most cases) of finite type. Move on to Section 3.3, which makes the definition concrete, and then Sections 3.4 on chord diagrams (popularized by Bar-Natan [8] and 3.6 on some well-known invariants which are finite-type after a change of variables. Then Section 3.7 which develops actuality tables (first made clear in the paper of Birman and Lin [13]) makes the theory concrete.

When you are reading the basic definitions, especially through actuality tables, try to bring back Vassiliev's original picture on the space of knots [88]. (See also the technically clearer development with "plumber's knots" [47].) Think of the space of all smooth maps as something like a high-dimensional Euclidean space, and knots with double points as walls between different knot type chambers. Those walls then intersect in places with higher singularities – for example, two double points. The actuality tables are recursively giving differences in the value of an invariant as one "crosses walls." Instead of listing values for all possible walls (= isotopy classes of knots with double points), the values are given at representative ones, and then values on "walls between walls" are needed. The theory is effective because one assumes that values for complicated enough singularities are zero, so the process terminates. This is the singularity theory point of view.

Not all possible assignments of values to "walls" are possible, with the primary restriction given by the four-term relations on chord diagrams. These are covered in Section 4.1 of [28]. Then go on to read Section 4.2, which proves the easy half of Kontsevich's fundamental theorem.

But before reading the full proof of the fundamental theorem (in Chapter 8), switch to the paper by Bott and Taubes [19]. We emphasize this point of view because it connects so strongly with the graphical models for cochains of the little disks operad from the previous talk. Moreover, the perspective (choice of word intentional) deeply informs a lot of subsequent work, especially on the calculus of embeddings. Pay close attention to the example of linking number, and then read Sections 1 and 2. Note that the program outlined at the end of Section 2 has essentially been completed and will be discussed in further talks.

Instead of Sections 3 and 4 here, one can instead refer to Chapters 5,6 and 9 of [61] for details about compactifications of configuration spaces and push forward of forms. See talk preparation in previous section. In the end, understand how Bott and Taubes define an integral which computes the first nontrivial finite-type invariant (of degree two), and which would define all of them over the real numbers but for possible "anomaly terms" (which to this day are not known to vanish, but can be swept under the rug in a sense to yield a universal invariant).

Exercise: Use forms which are concentrated in neighborhoods of points in S^2 to see that this Bott-Taubes integral counts pairs of interleaved crossings of a knot, along with triples of points which lie on some translation of the standard coordinate axes.

Finally, read Chapter 8 of [28], which gives Kontsevich's proof of fundamental theorem of Vassiliev invariants (over the real numbers), namely that all weight systems with real coefficients are realized by some knot invariant. This integral is one of the main "recent" applications of configuration spaces to geometry, though in light of the workshop we do not have a rephrasing in any kind of operadic language. Note that the Kontsevich integral also plays a role in constructing associators, which are part of the Grothendieck-Teichmuller story.

You are ready to give a talk when:

- You can show that after a change of variables, the coefficients which occur in the changed Jones polynomial are all finite type.
- You can explain why all possible values in actuality tables cannot be realized because of the "four term relation."
- You can explain how Bott-Taubes integrals must be finite type, directly from the definition.
- You can see why informally the Kontsevich integral is a case of the Bott-Taubes integrals (also due in part to Kontsevich) where the basic forms chosen on S^2 are complex and concentrated around the equator.

2.4. **Embedding calculus.** .

Background needed: all very basic - definitions of embeddings and immersions, and homotopy limits and colimits including the elementary exact/ spectral sequences for homotopy groups of homotopy limits and homology groups of homotopy colimits.

Directed Reading: Start with the expository article by Weiss [91]. Focus on the Introduction, and understand the reformulation of the Smale-Hirsch theorem on the homotopy type of spaces of immersions [81, 52]. Then go through one light reading of the rest of [91] to get a sense for language and perspective, though not details.

Instead of focusing on the generalities of embedding calculus, we will have you develop a fairly thorough understanding in the case of knots. First, you should understand the Blakers-Massey theorem, if you do not already. Read Chapters 4-6 of the Munson-Volic text [67] (which is available on the authors' web pages), going back further if needed (e.g. Section 3.2 if you haven't seen homotopy pull-backs before). The proof in the two-cube case, in Section 4.4 (especially the dimension counting of 4.4.1 and if people want the more homotopy-theoretic coordinate counting of 4.4.2), and especially the applications of Section 4.3 are the highlights. Outside of those, knowing statements in the general case, treated in Chapters 5 and 6 is the main need. But again, Sections 4.3 and 4.4 are the critical ones. For those who haven't seen homotopy limits and colimits before, also read Chapter 8 of [67] (or content yourself with Definition 1.2 of [78]).

Now, you are ready to understand the statements of the Goodwillie-Klein excision theorems, which make the embedding calculus work. You can also understand proofs in the easy case where the manifold being embedded has dimension roughly half of the dimension of the ambient manifold. The goal is to read the fairly elementary Appendices of [48]. Appendix A (aka Chapter 8) recounts some of the same ideas which you read in the Munson-Volic book. Appendix B is the main goal, in the case where the domain manifold is one-dimensional – that is, spaces of knots – and combined with an easy argument which shows that excision statements follow from disjunction statements. In higher-order excision statements for embeddings and similar functors such as Conjecture B (on page 5) of [48], diagrams in which the domain manifold varies are shown to be highly Cartesian. [Exercise: why is this rightly called excision?] In disjunction statements such as Proposition 9.7, diagrams in which the range manifold varies are shown to be highly Cartesian. [Exercise: why is this rightly called disjunction?]

You should go through these by starting with the "punctured knot" cubical diagram $\overline{\mathcal{E}_n}(M)$ of Definition 3.3 of [78]. Take fibers in one direction (the maps are fibrations) and verify that the resulting cube, which has the same total fiber as the original, is now a disjunction cube. Then go through the proof of Proposition 9.7 of [48] in this case, referring back to previous material as needed. Focus on the case $n = 3$ – that is, a three-dimensional diagram, obtained by punching various combinations of up to three holes. You should be able to see that if the ambient manifold has dimension five, the original cube gets more highly connected, which means the space of embeddings is effectively modeled by the homotopy limit of the rest of the cube (a.k.a. the punctured cube, or the sub-cubical diagram defined by the rest).

With the concrete case of knots in hand, you should re-read Weiss's Bulletin article [91] as well as further developments such as [49], [92].

You know you're ready to give the talk when:

- You see immediately how mapping spaces are 1-excisive.
- You can check that the section spaces over configurations are 2-excisive.
- You can prove the key step in the Blakers-Massey theorem at least for squares using some kind of dimension counting.
- You understand how dimension counting – along with a fair amount of cubical homotopy theory – is ultimately all that is needed to model spaces of knots.

2.5. **Cosimplicial models for the spaces of long knots and of long links.** .

Background: Now is one of the first times that various ideas which other speakers develop more in isolation are put together. One should become familiar with both compactifications of configuration spaces. This is discussed for the first Tuesday talk in Section 2.1, starting with the paragraph which begins "Then move on to the compactifications paper [76]." One should also understand the language of cubical diagrams. It is also helpful if you are familiar with the Eilenberg-Moore spectral sequence for the cohomology of a loop space.

Directed reading: You will mainly be following [78]. Read the first four sections fairly closely, perhaps filling in background materials with treatments as outlined for previous talks. (Optional: go through preparation for the previous talk in order to understand the "easy version" of Goodwillie's cutting estimates.) Then read the statement of Theorem 5.4 and look ahead to its proof through a zig-zag of equivalences on page 23 of the published version. It is your choice whether to look closely at the details of each of these equivalences, but you should at least be able to see how they are defined geometrically. This let's us move on from sub cubical diagrams of "punctured knots" to configuration spaces and evaluation maps. The configuration spaces need to be compactified, so go through the directed reading of 2.1 as well to gain a better understanding.

Then, in Section 6 we go from sub cubical diagrams to cosimplicial diagrams. Go through this section carefully. This is quite general, and should be a technique which is better known. Indeed, a good follow-up to this reading is Section 5 of [22] where we translate back and forth between cosimplicial and cubical models in order to do detailed analysis of group structure and layers in the Goodwillie-Weiss tower for classical knots.

You may also find it helpful to review the results of [78] through papers which summarize it in their introductions, such as [77], [60].

Then before getting to spectral sequences, one should review the Eilenberg-Moore spectral sequence for cohomology of loop spaces. There are five approaches of which one should be aware.

(1) The Adams-Hilton construction, which can be rephrased as producing a cubical complex (that is, a CW complex built from cubes, just as a simplicial complex is built from simplices) which is weakly homotopy equivalent as a subspace to the loop space. Here one sees the cup coproduct as part of the face data. This is developed nicely in the fourth section of the survey paper by Carlsson-Milgram [23]. It is also outlined briefly as motivation for the bar construction in the expository paper [79].

(2) Given a loop $f : \mathbb{I} \to X$ one can evaluate it at n times to get an n-tuple of points in X. That is, there is an evaluation map $\Delta^n \times \Omega X \to X^n$. If one pulls back a cochain from X^n and then pushes it forward along the projection from $\Delta^n \times \Omega X$ to ΩX one obtains a cochain on ΩX. By the boundary formula for push forward (perhaps most easily understood in the de Rham setting), the additional terms which occur correspond to faces in Δ^n, which in turn correspond to diagonal maps $X^{n-1} \to X^n$. These cochains then naturally fit into a bicomplex. In de Rham theory, these are Chen integrals [27], which occur in number theory (multiple zeta and gamma functions) and differential geometry and physics (parallel transport).

(3) The following is my perspective, so there isn't a reference at the moment - contact Dev for more details.
We can think about cochains defined as sub manifolds. This can be done through Thom forms on tubular neighborhoods in de Rham theory. Or over the integers it can be done more directly by saying that the value of W on some $\sigma : \Delta^n \to X$ which is transversal to W is the count of the preimages of W (assuming it has codimension n and that the map σ is smooth and transversal; this is just counting intersection with the image of σ if σ is an immersion). In either case, coboundary corresponds to taking the boundary of W (Stokes theorem).
Then W also defines a cochain on the loop space, taking $\Delta^n \to \Omega X$ which by adjointness gives $\Delta^n \times \mathbb{I} \to X$ and counting the number of preimages of W in $\Delta^n \times \mathbb{I}$. This means counting instances where some time t at which a loop is in W. But now given W and V we can count instances where there is some $t_1 \leq t_2$ so that the loop is in W at time t_1 and in V at time t_2. Call this $[W|V]$. The boundary has a couple different terms, one coming from "running off the boundary" of either W or V and another because t_1 and t_2 can approach each other, in which case one winds up in the intersection, which represents the cup product (up to a coboundary). That is $\delta[W|V] = [\partial W|V] \pm [W|\partial V] \pm [W \cap V]$.

(4) The loop space is a homotopy pullback, of two inclusions of a point into X (at the base point). If we replace this diagram say with path spaces over X then we can restate the Eilenberg-Moore model for cochains on the loop space as the homotopy push out of cochains on PX with itself over cochains on X. Standard homotopy theory yields a spectral sequence $\mathrm{Tor}_{H^*(X)}(k; k)$ converging to $H^*(\Omega X; k)$ where k is some field.

(5) For a cosimplicial space $X^\bullet$, the spectral sequence for homotopy groups of $\mathrm{Tot} X^\bullet$ is a tautology. Recall as well that the homotopy groups of $\mathbb{Z}Y$, the free abelian group on Y, are the homology groups of X. Because $\mathbb{Z}$ is a functor, we can apply it to a cosimplicial space to get $\mathbb{Z}X^\bullet$. Then the generalized Eilenberg-Moore spectral sequence for homology of a cosimplicial space (see [75]) follows from the remarkable statement that with some assumptions on $X^\bullet$ (e.g. a "vanishing line") $\mathbb{Z}\mathrm{Tot}X^\bullet \simeq \mathrm{Tot}\mathbb{Z}X^\bullet$. (This is remarkable as Tot is given by a limit and the free abelian

group functor by a colimit.)

The cosimplicial model for ΩX is $X^{S^1_\bullet}$, obtained by taking based maps from the standard simplicial model for S^1 into X. Applying the generalized Eilenberg-Moore spectral sequence gives the classical Eilenberg-Moore spectral sequence for cohomology loops on X. (And applying the generalized Eilenberg-Moore spectral sequence to the cosimplicial space defined by maps from the simplicial set $Y_\bullet$ to X gives the Anderson spectral sequence for the cohomology of the space of maps from $|Y_\bullet|$ to X.

This point of view is developed in Section 9.6 of [67], which gives many references including [2].

Once you feel like you understand the Eilenberg-Moore spectral sequence for a loop space, we can build on the second of these (that is, (2) above) to see how cochains or cohomology of configuration spaces will give rise to cohomology of knot spaces, as done in Section 2 of [78]. And we can use the fifth of these (that is, (5) above – so make sure to read Section 9.6 of [67]) to write down a spectral sequence, which is done in Section 7 of [78]. A key ingredient is understanding of homology and cohomology of configuration spaces, so you should go through the directed reading 2.1 above. For more calculations, take a look at [86].

In the de Rham theoretic setting, the evaluation map perspective of (2) can be used to produce cohomology classes (as opposed to spectral sequences, which technically only produce classes in an associated graded). The ideas were initially due to Bott and Taubes [19], though they focused on degree-zero cohomology of the classical knot space (that is, knot invariants). The program in higher dimensions carries through more cleanly – see the paper by Cattaneo, Cotta-Ramusino-Longoni [24]

You know you're ready to give the talk when:

- You can sketch a picture and/or give a simple explanation of each of the zig-zag of equivalences on page 23 of [78].
- You can see the second (n=2) case of Theorem 6.5 through a fiber sequence of two-dimensional sub cubical diagrams analogous to the fiber sequence given on the first half of page 25 of [78].
- The Eilenberg-Moore spectral sequence for (co)homology of a loop space isn't a black box to you!
- You can explicitly write down the spectral sequence for cohomology of knots in $\mathbb{R}^4$ in the rectangle between bidegrees $(-4, 9)$ and $(0, 0)$.

Bonus reading: to see how operads fit in with this story, you are in a good position to read [77], as summarized for example at the beginning of [60].

3. [Wednesday] Formality and its consequences

3.1. **Formality of models for spaces of long knots.** .

Background: one should be familiar with formality of spaces rationally, including for example that those spaces have no Massey products and that their homotopy groups are combinatorially determined. This is addressed in an introductory way in the textbook of Bott and Tu [20] and thoroughly in the book by Felix-Halperin-Thomas [40]. We are now getting to the point when previous talks are also part of the needed background.

Directed reading: This talk requires familiarity with the topics outlined in 2.1, 2.2, and 2.5 above. Start with the directed reading for 2.2 (which requires knowledge of 2.1), as we would like this talk to reinforce the formality aspect of the story more while the next talk will reinforce ideas from Goodwillie-Weiss calculus. You are encouraged to work with the speaker of 2.2.

To understand the connection between formality and spectral sequence collapse results, review how the Eilenberg-Moore spectral sequence as given by the (co)bar complex is the spectral sequence of a bicomplex. (See the material on the Eilenberg-Moore spectral sequence from the previous talk.) Then note that if a chain complex with multiplication - that is, a DGA - has zero differential then this spectral sequence of a bicomplex will collapse, as in general the spectral sequence of a bicomplex with one differential zero will collapse.

Next, understand the fundamental relationship between homotopy limits and cosimplicial spaces. See for example Section 9.3 of [67] (which is available on the authors' web pages), filling in with previous material (mainly the beginning of Chapters 8 and 9) as you need. Review as well the standard Bousfield-Kan spectral sequence for a cosimplicial space, which is treated in Section 9.6 of [67].

Once that is in hand, you can understand the basic idea for how formality of a diagram leads to spectral sequence collapse. With the definitions in place, this is given in the straightforward Proposition 3.2 of [60]. Pascal Lambrechts in particular needs to be recognized for realizing that one could prove collapse through formality, and pushing through that program through in the extensive work of [50], [61], and [60].

Now read the introduction to [60], if you haven't already. After operads were in the background in the material for yesterday's talk, they should be brought more to the fore here. The connection between knot spaces and operads was first seen combinatorially by Turchin [85], which motivated a space-level conjecture by Kontsevich then established in some sense in [77]. Give a close reading of [77], except perhaps for the very last subsection 7.2. Make note of how operad insertion maps are used as structure maps in the cosimplicial model for knots (modulo immersions).

Put this together at least at a conceptual level by understanding how the formality of the Kontsevich operad would imply collapse of the spectral sequence for rational homology of long knot spaces (modulo immersions). Then recognize that the formality work done is for the Fulton-MacPherson operad. Go over the resolution of this discrepancy (using "fanic diagrams") in [60]. If you know ∞-categories, ponder if they could give an alternate resolution of this "up to homotopy" technical issue.

Once one has he collapse result, the homology of knot spaces becomes a purely algebraic/ combinatorial matter. To see some work along these lines, read [86]. This work can also be extended to give rational homotopy groups [4] [59].

3.2. **Formality and deformation quantization.** This topic is not as aligned with the rest of the program as any of the other topics. We include it because Kontsevich's inspirational paper [56] (and his talks following it, including at the "Millenial" conference at UCLA in 2000) was a precursor to much of the work in this program, as well as much other work beyond it. In particular, [56] sets forth many of the ideas which have then in the past decade been connected to the worlds of Goodwillie-Weiss calculus and homotopy theory of operads. Moreover, we suspect that there is something which the mathematical community (and in particular young researchers) can gain from making stronger connections between these ideas

There are two approaches to Kontsevich's celebrated deformation quantization theorem. The first due to Kontsevich in [57] is more direct, essentially giving recipes - as values of integrals - for structure constants. The second is due to Tamarkin [83] in dimension two, following a basic idea of Tsygan, but then has been extended to all dimensions and ruminated on and refined by many, including Kontsevich himself in [56]. Again, this paper has been inspiration for a wide range of further work including many topics of the workshop, and the ingredients of these seminal papers overlap with topics of the workshop as follows.

- Configuration space integrals are a key ingredient in the first approach to deformation quantization in [57] as well as formality of the disks operad as established in our directed reading for 2.2 above. Indeed, these configuration space integrals were initially developed by Kontsevich, who in turn was inspired by Feynman integrals and other constructions from quantum field theory, as developed by Witten and others (as for example occur in Bar-Natan's thesis under Witten).
- Formality of the little disks operad (or, really, the Fulton-MacPherson operad) is a key ingredient both in embedding calculus collapse results and in the Tamarkin proof of deformation quantization.
- Another ingredient of the Tamarkin proof is the Deligne Conjecture, which in the embedding calculus context was used to show that spaces of knots are two-fold loop spaces (first in [77]).

- Because of the essential(?) role of configuration spaces integrals in these stories, and their connection to certain period integrals (and polylogarithms, etc.) Kontsevich noticed a connection between operads and motives (hence the title), which brings the Grothendieck-Teichmülcer group into the picture. This is now being realized within the world of operads and their homotopy theory, as our program will discuss on Friday.

We give two directed readings, a more thorough one based roughly on [57] and a briefer one based roughly on [56], as we suggest only the first be a basis for a talk. Though the second paper has been much more influential, including to the results highlighted in this program, the program it outlines for establishing the deformation quantization result requires a fair amount of machinery coming from homotopy theory of algebras over operads. In a single talk, the first approach is more self-contained and provides a clearer connection to part of the story through configuration space integrals. The second approach would demand more of a sustained development.

Reading of Kontsevich's First Proof

Background needed: It would be helpful to know at least the basic story in broad terms of quantum mechanics and the Lagrangian formulation of classical mechanics. There are a number of possible sources we're aware of – entire books devoted to "physics for mathematicians", John Baez's "This week's finds" (very influential for a generation), the n-Lab, etc. But unlike for other material, we will not give you much guidance as to how to fill in this "cultural" background. Also, the seminal paper by Gerstenhaber [45] is an important introduction to deformation theory. There are certainly more complete, modern introductions as well. An example of similar ideas which might be more familiar to some topologists would be the relationship between extensions of algebras and group cohomology as explained in the first Chapter of [1]. Directed reading: Start with the survey of Karaali [55]. This starts at an elementary level and outlines the story.

Then, go back and read the material in 2.2 on graphical models for cochains of configuration spaces, which you should go through fairly thoroughly, and the material on the Kontsevich integral in knot theory, which you can just focus on the integral itself and its invariance (rather than its prominent role in finite-type knot theory) if you'd like.

Next, learn about the Hochschild-Kostant-Rosenberg Theorem, which is developed for example in Chapter 9 of Weibel's book [90] or Ginzburg's Lectures on Noncommutative Geometry [46]. The latter has a perspective very much in line with Kontsevich's approach.

Then finally dive into [57]. Read the preliminaries, which should be familiar after reading [55], just to set notation. Then work carefully through the examples in section 1.4, checking that they do satisfy the requirements of a deformation. Next, carefully work through the definitions of Section 2, which should be familiar after learning about configuration space integrals for formality and for knot theory.

Give a light reading through Section 4.5. Then Section 4.6 gives a smooth version of the Hochschild-Kostant-Rosenberg Theorem, which is the "zeroth part" of the main result. Switch to reading Voronov's account [89] of how the main result, a sort of formality for the Hochschild complex of functions on a manifold, implies deformation quantization.

If you have looked closely at compactifications of configuration spaces, then Section 5 which compactifies configurations in the upper half-plane is straightforward, though it is *not* giving the usual construction. Exercise: what are the strata (for small number of points) in this case, and in particular how do they differ from those of the standard construction.

You might now hopefully be able to follow the arguments of Section 6, which establishes the result for open subsets of Euclidean space, with which we can be content. If you cannot get the signs to work out, you're in good company, as a follow-up paper by Arnal, Manchon and Masmoudi [3] was needed to sort things out.

Reading of the second proof, due to Tamarkin, Kontsevich, Tsygan, and others

Again the main reference, which sets forth ideas which are still being mined by many to this day, is [56]. But rather than a "direct" construction of a suitable map , this line of reasoning which first developed in [83] introduces a host of structures (e.g. transfer of structure, and "G_∞") to make use of formality of the little disks operad [84] (that is, the same result we go through in 2.2 above). While configuration space integrals are common to both proofs, the Tamarkin proof also makes use of the Deligne conjecture, as proved by a number of people including McClure and Smith [63]. (The review of this paper by Voronov is very good.) The McClure-Smith work generalizes [64], pointing to the existence of deloopings of embedding spaces.

As all of the set-up through refined development of the theory of algebras over operads is not our primary expertise. So we leave you with two good expository articles, a more focussed and detailed one by Hinich [51] and a later one with a wider perspective by Dolgushev, Tamarkin and Tsygan [29]. Reading these and work they cite along with [56] gives a full current understanding of this celebrated result in mathematical physics.

3.3. **Further applications in knot theory, and a time to address loose ends.** We include this as a place-holder for the schedule, one which doesn't require any participant preparation. Dev will make the connection between formality of the disks operad and knot theory more apparent. He will also indicate a current project (joint with Ryan Budney, Jim Conant, and Robin Koytcheff (a participant)) to extend this work beyond the what has been the very fruitful choice to work over the real numbers and work over the integers instead. Finally, we can all take moment to collect questions and address some of them, as well as address topics which were not developed to people's satisfaction for lack of time.

4. [Thursday] Delooping embedding spaces

4.1. **Rational homology of spaces of long embeddings (K).** There are a few different papers which give models for more general embedding spaces.

(1) Arone, Lambrechts and Volic in [5] define for models of embeddings of arbitrary manifolds into Euclidean spaces and prove a formality/ collapse result. Moreover they prove the striking result that when a manifold embeds in a Euclidean space of dimension d, then the rational homology of the space of embeddings into $\mathbb{R}^{2d+1}$ depends only on the rational homology of M! This is also proved in [6].
(2) Munson and Volic consider links in [66], showing that the Goodwillie-Weiss theory naturally gives rise to mutli-cosimplicial models, which then can be "restricted to the diagonal" to produce cosimplicial models. Collapse of the associated spectral sequence (through formality) has not been established.
(3) Arone and Turchin in [6] give a model for the Goodwillie-Weiss tower of embeddings in terms of infinitesimal bimodules over the little disks operad, and again apply operad formality to show collapse.
(4) Turchin in [87] gives a more geometric take on very similar results, using the Fulton-MacPherson operad instead of the disks operad. In this setting, the evaluation map gives a comparison.
(5) Boavida and Weiss in [17] give a description of the embedding calculus tower in terms of configuration categories, which also rely on derived spaces of operad maps. They also give a delooping result (of the sort to be established in the Thursday talks), through a neat "Alexander trick" kind of result.

We choose to focus on [6]. Here is a brief outline of topics we would like to emphasize.

- Foundations of the embedding calculus. Reference the reading for 2.4 Embedding Calculus above.
- Infinitesimal (aka linear) bimodules over operads.
- The description of modules and infinitesimal bimodules over operads as diagrams.
- The proof of Theorem 0.1 in [6].
- The proof of Theorem 0.3 in [6], and its computational consequences.

Possible references: [6, Theorem 0.1], [18], [87].

The final answers for rational homology of spaces of "long embeddings" in Euclidean spaces seem to coincide with the production of cohomology classes through differential forms [72]. In the high-dimensional case, both the agreement of combinatorics and the fact that the de Rham classes realize the real cohomology are open. In the one-dimensional case, it has been shown that the combinatorics of [**?**] agrees with that of the embedding calculus (and Vassiliev) calculation, but it is open whether the de Rham classes realize this isomorphism.

4.2. The Double Delooping theorem for spaces of long knots (K).

- Operads with multiplication, their associated cosimplicial spaces [64], and the mapping space interpretation [36, Section 8].
- The proof of the Fiber Sequence Theorem [36, Theorem 3.11] and various applications (e.g., [36, Sections 6 and 7]).
- The proof of left properness for the category of nonsymmetric simplicial operads [36, Proposition 4.3].
- The bar resolution of an operad as a bimodule over itself and its promotion to a resolution as operads [36, Section 5].

Possible references: [36], [64]

4.3. Right-angled Artin operads and their resolutions (K).

- Right-angled Artin groups: examples, the Salvetti complex, and applications.
- Right-angled Artin monoids and their "spherical" resolutions as bimodules over themselves.
- Generalization of the "spherical" resolution to bimodules and linear bimodules over right-angled Artin operads.
- Boardman-Vogt tensor products and right-angled Artin operads.
- Proof of the enriched simplicial adjunction for bimodules and linear bimodules [39, Theorems 1.1 and 1.2].

Possible references: [25], [26], [39]

4.4. The Iterated Delooping theorem for spaces of long embeddings (K).

- The Boardman-Vogt W-construction and its application to creating cofibrant $\mathcal{E}_n$-operads.
- The Boardman-Vogt tensor product as a tool for creating $\mathcal{E}_n$-operads (e.g., [21, Theorem C], [42, Additivity Theorem]).
- The inductive proof of the Iterated Delooping Theorem.

Possible references: [12], [21], [33], [38], [42],

5. [Friday] The Grothendieck-Teichmüller group

5.1. The profinite Grothendieck-Teichmüller group (K).

- Definition of the profinite G-T group via the profinite completion of $\widehat{F}_2$, the free group on two generators.
- Interpretation of the profinite G-T group in terms of outer automorphisms of $\widehat{F}_2$.
- Actions of the profinite G-T group on: (towers of) braid groups, dessins d'enfants, mapping class groups, braided tensor categories, etc.
- The proof that the absolute Galois group of $\mathbb{Q}$ injects into the profinite G-T group.

Possible references: [31] (the original source), [74]

5.2. Little 2-disks and the profinite Grothendieck-Teichmüller group (K).

- Pro-categories.
- Model category structures on profinite groupoids and profinite spaces.
- Profinite completion of groupoids and spaces.
- An $\mathcal{E}_2$-operad in groupoids: $\mathcal{P}a\mathcal{B}$.
- Weak operads via dendroidal objects.
- Identification of the profinite G-T monoid with (different flavors of) endomorphism monoids of the profinite completion of $\mathcal{E}_2$.

Possible references: [7], [54], [65], [69]

5.3. Little disks and the prounipotent Grothendieck-Teichmüller group (K).

- Definition of the prounipotent G-T group via parenthesized braids and parenthesized chord diagrams, Drinfeld associators.
- The rational homotopy theory of operads.
- The main steps in the proof of the identification of the prounipotent G-T group and the group of automorphisms of the rationalization of the little 2-disks operad.

Possible references: [9], [31], [44]

5.4. Future directions (Kathryn and Dev).

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