ÉTALE CLASSIFYING SPACES AND THE REPRESENTABILITY OF ALGEBRAIC K-THEORY

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1. Introduction

This talk concerns the last section of [MV99], and one of the first applications of motivic homotopy theory. We're aware that algebraic K-theory is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich descent, so that it is well-defined on the motivic homotopy category. What's astounding is that it's actually represented on this category, by a sort of infinite Grassmannian.

This is the sort of talk that has a proof in it, and this proof is also a sort of definition. It's easy to write down an abstract classifying object for algebraic K-theory, essentially using Quillen's plus-construction – the cool part is replacing this object by the more geometric Grassmannian. We'll start with a little intuition about sites, and then quick tour of classifying spaces in the motivic setting. Then, we'll define algebraic K-theory and exhibit its classifying object. Next comes the proof, which splits into three parts. First, a little magic with monoids will make this prettier – up to $\mathbb{A}^1$-homotopy equivalence, it's $BGL_\infty \times \mathbb{Z}$. Second, Hilbert's Theorem 90 will let us replace our Nisnevich classifying space with an étale classifying space. Third, in the technical heart of the proof, we'll show this is $\mathbb{A}^1$-homotopy equivalent to the infinite Grassmannian we crave. Most of these methods are pretty general, and in particular, you can replicate a lot of the pieces for any linear algebraic group. Afterwards, we'll be able to play around with some examples, inside and outside of algebraic K-theory.

Most of the below is phrased in terms of the Nisnevich topology, but ultimately this only matters when algebraic K-theory and GL_n pop up. The general methods, particularly those involving classifying spaces, work in any site. In fact, most of [MV99], which defines $\mathbb{A}^1$-homotopy theory and proves this theorem, actually takes place in the very general setting of a 'site with interval'. At most, this talk will do some switching between the Nisnevich and étale topologies, but this should be fairly explicit.

$(Sm/S)_{\mathrm{Nis}}$ and $(Sm/S)_{\text{ét}}$ are the categories of smooth schemes over S with the Nisnevich and étale topologies respectively. S will always be Noetherian and finite-dimensional. $\mathcal{H}(S)$ is the unstable motivic homotopy category.

2. Prelude: sites and toposes

Some of the topologists I talked to expressed some confusion about sites, so I'll begin with a few minutes of clarification. At some point later on you might want to head over to SGA 4. The following three definitions are basic:

- A **Grothendieck topology** on a category is a notion of when a map $U \to X$ is a covering. For instance, if $\{U_i\}$ is an open covering of a topological space X, then $\coprod U_i \to X$ is a covering map in the category of (disjoint unions of) open subsets of X. There are a few axioms that coverings should satisfy – for instance, they should be closed under composition and base change.
- A **site** is a category with a choice of Grothendieck topology.
- A **topos** is the category of sheaves on a site.

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Usually, when you describe a Grothendieck topology, a morphism $\coprod U_i \to X$ is a covering if it's an epimorphism and satisfies some conditions on the U_i, and we think of the U_i that satisfy these conditions as the 'open subsets' of X. For instance, in the étale topology, the opens are schemes étale over X – note these are not generally subsets of X! In the Nisnevich topology, the opens are the schemes étale over X that split over each fiber, meaning that one of the residue fields in the fiber is the same as the residue field of the point of X. I will shamelessly call these objects 'open sets.' If you're confused, you should always pretend that I'm talking about the category of open subsets of a topological space X and their inclusions.

Okay, so if we're willing to believe these things are open sets, there's a notion of 'local.' For instance, I define a G-torsor as being a locally free G-bundle, which means that it becomes free after pulling back to some open cover. I might even say 'sufficiently small open cover', but by 'small' I really mean 'initial'. For instance, your bundle might not split over $\mathbb{Q}$, but just over $\mathbb{Q}[\sqrt{2}]$, which is an étale cover of $\mathbb{Q}$ – then this bundle is étale-locally trivial.

Now I'll talk about toposes (not 'topoi'!) briefly. If you're, say, Grothendieck, everything you want to do with a site is cohomological, which means it only depends on the topos. This is worth saying because different sites can have equivalent toposes. We think of the topos as being constructed from the site, but we also think of it as containing the site, as follows. There's a **Yoneda embedding**

$$h : \mathcal{C} \to \mathrm{Psh}(\mathcal{C}), \qquad U \mapsto h_U = \mathrm{Hom}_{\mathcal{C}}(\cdot, U).$$

In most of the sites we end up working with, these **representable** presheaves are actually sheaves, so the site embeds into the topos. In this talk, I'll mostly stay within the sites I want – I find it hard to think of everything as a sheaf.

Other ideas from sheaf theory over topological spaces make sense in the topos-theoretic setting. For instance, there's the notion of a 'point' of a site or topos. For a topos, this is defined in terms of its stalk functor, which is an adjunction

$$x^* : \mathcal{T} \leftrightarrows \mathrm{Set} : x_*$$

such that x^* also preserves finite limits. If you're willing to stay at the site level, you can always recover x^* as the limit of a cofiltered diagram of open sets of your topology. You can explicitly state what these points are for the Zariski, étale, and Nisnevich topologies. For the Zariski topology,

$$\mathcal{F}_x = \mathrm{colim}_{U \ni x} \mathcal{F}(U) = \mathcal{F}(\mathrm{Spec}\, \mathcal{O}_{x,X}),$$

which is to say that points are local rings, almost by definition.

For the étale topology, there's now a larger diagram of opens containing a Zariski point: even if your scheme X is Spec of a field k, any finite separable extension of k gives an étale cover of X. So in this case, the point is actually $\mathrm{Spec}\, k^s$. In general, your local ring corresponds to a ring containing $\mathcal{O}_{x,X}$, whose residue field is $k(x)^s$, and with a root for every étale polynomial with coefficients in $\mathcal{O}_{x,X}$. This is called the **strict Henselization** of $\mathcal{O}_{x.X}$, written $\mathcal{O}^{sh}_{x,X}$. The points in the étale topology are the strictly Henselian local rings – those with separably closed residue field and such that any factorization mod $\mathfrak{m}_{x,X}$ of a polynomial in $\mathcal{O}_{x,X}[t]$ lifts to a factorization in $\mathcal{O}_{x,X}[t]$.

In the Nisnevich topology, things are much the same, except that the residue field isn't allowed to change. The Nisnevich points are **Henselian local rings** $\mathcal{O}^h_{x,X}$.

Last is the notion of **morphism of sites**. This is an adjunction

$$f^* : \mathrm{Shv}(\mathcal{S}_1) \leftrightarrows \mathrm{Shv}(\mathcal{S}_2) : f_*$$

such that f^* also preserves finite limits (and is thus exact). This is usually written $f : \mathcal{S}_2 \to \mathcal{S}_1$, but note that the left adjoint is really the pullback. An example is the stalk functors mentioned earlier, which are morphisms to sets – that is, sheaves on a point!

3. Classifying spaces and bundles

If G is just a group, there's a simplicial group EG with $EG_n = G^{n+1}$, faces given by projections, degeneracies given by diagonals, and the group law defined componentwise. G acts on this by acting on each factor of everything. EG is contractible, a contracting homotopy given by inclusions $G^{n+1} \cong G^{n+1} \times \{e\} \subseteq G^{n+2}$, and the G-action is free, so the quotient EG/G is the sort of thing we like to call BG. We can also describe BG as the simplicial set with $BG_n = G^n$, degeneracies given by diagonals, the face d_i forgetting the ith factor if $i = 0$ or n and multiplying the ith and $(i+1)$th if $0 < i < n$. The map $EG \to BG$ is a G-bundle; it's a **principal** G-bundle, or **G-torsor**, since the fibers are isomorphic to G, as G-sets.

Now say G is a sheaf of groups on, let's say, the Nisnevich site. Then defining EG as above on each representable gives a presheaf of groups, which is easily seen to be a sheaf. Again this is the total space of a principal G-bundle $EG \to BG$. That is, $G(U)$ acts freely on $EG(U)$ for each representable U, and the sheafification of the quotient $U \mapsto EG(U)/G(U)$ is $BG(U)$.

Now say G is a sheaf of simplicial groups. We can apply E levelwise to G, creating a *bisimplicial* group sheaf $EG_{\bullet,\bullet} = G_{\bullet}^{\bullet+1}$. The diagonal of this is what we'll call EG. So EG is the sheaf of simplicial groups

$$EG_n = G_n^{n+1}.$$

To take a face map d_k, we lose the kth factor and apply $d_k : G_n \to G_{n-1}$ on all the remaining ones; degeneracies are similar. Everything above still works as long as we phrase it 'categorically' enough. For example, the action of G_n on EG_n defines an action of G on EG, by which I mean a map of simplicial sheaves

$$\alpha : G \times EG \to EG;$$

to say this is free means that the associated map

$$G \times EG \to EG \times EG, \qquad (g, x) \mapsto (\alpha(g, x), x),$$

is a monomorphism of simplicial sheaves; the quotient $BG = EG/G$ is the obvious coequalizer. We could also get BG by a similar diagonal construction, with

$$BG_n = G_n^n.$$

The importance of BG is that it's the classifying space for homotopy types of G-torsors. Since we're working with simplicial sets, we have to be a little careful.

Proposition 3.1. *Let G be a simplicial sheaf of groups and let $\mathcal{E} \to \mathcal{X}$ be a G-torsor. Then there is a (stalkwise) acyclic fibration $f : \mathcal{Y} \to \mathcal{X}$ and a map $p : \mathcal{Y} \to BG$ such that $f^*\mathcal{E} \cong p^*EG$.*

Proof. Take $\mathcal{Y} = (\mathcal{E} \times EG)/G$; this maps to $\mathcal{X}$ with fibers the contractible fibrant sheaf EG. □

In the case we're interested in, G will be some GL_n, and in particular, just a discrete sheaf of groups. In this case, it's true that the set of isomorphism classes of G-torsors over a simplical sheaf is homotopy invariant, and in particular, in the above replacement $\mathcal{Y} \overset{\sim}{\twoheadrightarrow} \mathcal{X}$, the G-torsors over $\mathcal{Y}$ are the same as those over $\mathcal{X}$. Choosing a good (i.e. fibrant) model for BG, one obtains the following.

Proposition 3.2. *For any discrete sheaf of groups G on $(Sm/S)_{\mathrm{Nis}}$, there is a simplicial sheaf BG such that $\mathrm{Hom}_{\mathcal{H}(S)}(\mathcal{X}, BG)$ is the set of isomorphism classes of G-torsors over $\mathcal{X}$.*

There's a natural map $G \to \Omega BG$, given levelwise by

$$(s_0, \ldots, s_n) : G_n \to G_{n+1}^{n+1}.$$

When G is an ordinary simplicial group and we've chosen a fibrant model for BG, this map is a weak equivalence, which is to say that $G \simeq \mathbb{R}\Omega BG$ in the homotopy category. This is also true for simplicial sheaves of groups because we can check it at each point.

Last, note that the definition $BG_n = G_n^n$ only depended on the multiplication in G. In particular, we could do the same for a simplicial sheaf of *monoids* M.

4. Algebraic K-theory

Now let's switch gears and talk about algebraic K-theory of schemes. If you don't know what K-theory is, I don't know that there's much I can do for you here. I can only guarantee that you'll see at least one talk in your life on the subject of "what is K-theory". Also, someone writes a totally new paper about "what is K-theory" every seven to ten years, so you could just wait to catch up until the next one comes out.

The basics are as follows. First, K-theory says something about vector bundles over a scheme – specifically, $K_0(X)$ is the Grothendieck group of vector bundles on X (equivalently, if X is regular, Noetherian, and separated, the Grothendieck group of coherent sheaves on X). Second, the graded group we call K-theory is naturally the homotopy groups of a certain connective spectrum, also called K-theory. Third, the assignment $U \mapsto K(U)$ is a presheaf of spectra on the Zariski topology of X, which satisfies descent in the sense of Jardine [Jar87] for the Zariski and Nisnevich topologies (though not the étale topology).

Definition 4.1. Let $\mathsf{Vect}(X)$ be the category of vector bundles on X – that is, locally free sheaves, equivalently in the Zariski or Nisnevich topologies. Then the **K-theory** of X is

$$K(X) = K(\mathsf{Vect}(X)) = \Omega BQ(\mathsf{Vect}(X)),$$

where $Q(\mathsf{Vect}(X))$ is the monoidal category of vector bundles where the morphisms are inclusions of summands, and the monoid structure is given by direct sum.

The following is the beginning of our investigation:

Proposition 4.2. *In the motivic stable homotopy category $\mathcal{H}(S)$, algebraic K-theory is represented by $\mathbb{R}\Omega B\left(\coprod_{n\geq 0} BGL_n\right)$.*

Here $\coprod BGL_n$ is viewed as a monoid, with the multiplication

$$BGL_n \times BGL_m \simeq B(GL_n \times GL_m) \to BGL_{n+m}$$

induced by the group homomorphism $GL_n \times GL_m \to GL_{n+m}$ corresponding to the direct sum of vector spaces.

Proof. The category $\mathsf{Vect}(X)$ is one where all exact sequences split, so by a theorem of Grayson and/or Quillen [Gra76], we can make the replacement

$$\Omega BQ(\mathsf{Vect}(X)) \simeq \Omega BN(\mathrm{iso}\,\mathsf{Vect}(X)).$$

Locally on X, all vector bundles are trivial, so the right-hand side gets replaced by $\Omega B\left(\coprod BGL_n(X)\right)$, and the rest follows. □

5. Monoids

The following shows that classifying spaces of monoids aren't really much different than classifying spaces of groups.

Proposition 5.1. *If M is a simplicial sheaf of monoids such that each level M_n is a free monoid on a sheaf of sets, then the natural map $BM \to BM^{gp}$ is a weak equivalence, and thus $M^{gp} \simeq \mathbb{R}\Omega BM$.*

By the standard model category arguments, thinking of levelwise free monoids as the cofibrant ones, we can find for any monoid M an acyclic fibration $\widetilde{M} \overset{\sim}{\twoheadrightarrow} M$ with $\widetilde{M}$ levelwise free. In this case, $\mathbb{R}\Omega BM \simeq \widetilde{M}^{gp}$.

In our case of interest, we can simplify this even further.

Definition 5.2. An **augmentation** on a simplicial sheaf of monoids M is a map $f : M \to \underline{\mathbb{N}}$ with a section $\alpha : \underline{N} \to M$.

Note that an augmentation induces a grading on M – take degree n to be $M_n = f^{-1}(n)$. Multiplying by $\alpha(1)$ induces $M_n \to M_{n+1}$, and we define M_∞ to be the colimit. Now, if M is levelwise free, there's a map $M_\infty \times \mathbb{Z} \to M^{gp}$, namely $(x, m) \mapsto \alpha(1)^{m-n} x_n$ if x is represented by $x \in M_n$. For general M, we can take a levelwise free replacement $\widetilde{M} \to M$, inducing weak equivalences $\widetilde{M}_n \to M_n$ and thus $\widetilde{M}_\infty \to M_\infty$. We can thus fill in the bottom map in

$$\begin{array}{ccccc} \widetilde{M}_\infty \times \mathbb{Z} & \longrightarrow & \widetilde{M}^{gp} & \xrightarrow{\sim} & \mathbb{R}\Omega B\widetilde{M} \\ \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\ M_\infty \times \mathbb{Z} & \dashrightarrow & & & \mathbb{R}\Omega BM. \end{array}$$

Proposition 5.3. *Let M be an augmented simplicial sheaf of monoids, and suppose that M is homotopy commutative and the map $\pi_0(M)^a \to \underline{\mathbb{N}}$ induced by the augmentation is an isomorphism of sheaves. Then $M_\infty \times \mathbb{Z} \to \mathbb{R}\Omega BM$ is a weak equivalence.*

Proof. Since these properties are inherited by $\widetilde{M}$, it suffices to prove this for the top map in the above square. Weak equivalences are local, so we can check this at each point, thus reducing to the case where M is a levelwise free simplicial monoid. Since M is commutative, its homology is a commutative ring, and $H_*(M_\infty \times \mathbb{Z})$ is easily seen to be $H_*(M_\infty)[\pi_0(M)^{-1}]$. By a theorem of Friedlander and Mazur [FM94], $M_\infty \times \mathbb{Z} \to M^{gp}$ is a homology isomorphism. Since $\pi_0(M) = \mathbb{N}$ given by the grading, each M_n and thus M_∞ is connected. Since M is commutative, we obtain a multiplication $M_m \times M_n \to M_{n+m}$ and thus $M_\infty \times M_\infty \to M_\infty$. Thus, by the Eckmann-Hilton argument, $\pi_1 M_\infty$ is abelian. Thus, by the Whitehead theorem, the map is a weak equivalence. □

Corollary 5.4. *Algebraic K-theory is represented in the motivic homotopy category by $BGL_\infty \times \mathbb{Z}$.*

Proof. The monoid $\coprod_{n\geq 0} BGL_n$ is easily seen to be augmented in the above sense: the grading is already there and the section is given by sending n to the point of BGL_n classifying a trivial n-dimensional vector bundle. One should check that BGL_∞ in this augmented monoid sense is what we'd ordinarily mean by BGL_∞, namely, the classifying space of $GL_\infty = \operatorname{colim} GL_n$. Now let's check the two conditions of the proposition. The condition on π_0 comes from the fact that each BGL_n is connected: $\pi_0 BGL_n$ classifies n-dimensional bundles over points, of which there's only one. The commutativity condition is where we need to pass to the motivic homotopy category, and [MV99] only proves this after introducing the Grassmannian. But in fact, the maps $GL_n \times GL_m \to GL_{n+m}$ are $\mathbb{A}^1$-homotopy commutative: they correspond to block sum of matrices, and we can use $\mathbb{A}^1$-homotopies to slide the blocks around like Sokoban. □

6. The étale classifying space

Just as we can talk about Nisnevich G-torsors, we can talk about étale G-torsors, and it's convenient/'fun' to compare the two. There's a morphism of sites

$$\pi : (Sm/S)_{\text{ét}} \to (Sm/S)_{\text{Nis}},$$

whichi is to say a pair of adjoint functors

$$\pi^* : \operatorname{Shv}((Sm/S)_{\text{Nis}}) \leftrightarrows \operatorname{Shv}((Sm/S)_{\text{ét}}) : \pi_*$$

such that π^* preserves finite limits (note that it's a left adjoint, so it automatically preserves all colimits). The pushforward π_* is a forgetful functor: an étale sheaf is *a fortiori* a Nisnevich sheaf. The pullback π^*, meanwhile, is étale sheafification. The differences between the étale and Nisnevich topologies are measured by the cohomological properties of these functors.

This adjunction extends to simplicial sheaves, and on homotopy categories, we get an adjunction of derived functors:

$$\pi^* : \operatorname{Ho}(\operatorname{sShv}(Sm/S)_{\text{Nis}}) \leftrightarrows \operatorname{Ho}(\operatorname{sShv}(Sm/S)_{\text{ét}}) : \mathbb{R}\pi_*.$$

Note that π^* is exact, so equal to $\mathbb{L}\pi^*$.

Definition 6.1. The **étale classifying space** of G is

$$B_{\text{ét}}G = \mathbb{R}\pi_*\pi^*BG.$$

By adjointness, if G is a discrete sheaf of groups, this classifies *étale* π^*G-torsors over X. This is a little awkward, and the awkwardness propagates to parts of the geometric construction below. The counit of the derived adjunction is a map $BG \to \mathbb{R}\pi_*\pi^*BG$. Still assuming that G is discrete, we can calculate the homotopy groups of the two classifying spaces. We know that maps in the motivic homotopy category from a scheme U to BG correspond to G-torsors over U, which correspond to $H^1_{\text{Nis}}(U,G)$. Likewise,

$$[U, B_{\text{ét}}G] = H^1_{\text{ét}}(U,G).$$

Moreover,

$$[\Sigma U, BG] = [U, \mathbb{R}\Omega BG] = [U, G] = G(U)$$

and likewise

$$[\Sigma U, B_{\text{ét}}G] = G_{\text{ét}}(U).$$

The higher homotopy groups are zero, since G is discrete. We can conclude as follows.

Proposition 6.2. *For a discrete sheaf of groups G on the Nisnevich site, $BG \simeq B_{\text{ét}}G$ if and only if*

(1) G satisfies étale descent, and
(2) $H^1_{\text{ét}}(U,G) \cong H^1_{\text{Nis}}(U,G)$ for all smooth schemes U over S.

For $G = GL_n$, these conditions are satisfied by Hilbert's theorem 90:

Theorem 6.3 (Hilbert's Theorem 90)**.**

$$H^1_{\text{Zar}}(U, GL_n) \cong H^1_{\text{Nis}}(U, GL_n) \cong H^1_{\text{ét}}(U, GL_n).$$

This is just saying that vector bundles in the Zariski topology already satisfy étale descent.

Remark 6.4. There's a nice pointwise characterization of the property $H^1_{\text{ét}}(U,G) \cong H^1_{\text{Nis}}(U,G)$. This says that all Nisnevich G-torsors satisfy étale descent. Equivalently, a free Nisnevich G-torsor on a sufficiently small Nisnevich open becomes free when restricted to a sufficiently small étale open. Taking the colimit over Nisnevich opens, it's sufficient to check that there are no étale G-torsors on Nisnevich points, i. e. that for all points x of smooth schemes X over S,

$$H^1_{\text{ét}}(\mathcal{O}^h_{x,X}, G) = 0.$$

Conversely, if $H^1_{\text{ét}}(U,G) \cong H^1_{\text{Nis}}(U,G)$ for all smooth schemes U over S, then this is in particular true for Henselian local rings, so that

$$H^1_{\text{ét}}(\mathcal{O}^h_{x,X}, G) = H^1_{\text{Nis}}(\mathcal{O}^h_{x,X}, G) = 0.$$

Corollary 6.5. *$BGL_n \simeq B_{\text{ét}}GL_n$, and $BGL_\infty \simeq B_{\text{ét}}GL_\infty$, giving another representing object for algebraic K-theory.*

Though the objects are the same, the argument below extends to étale classifying spaces of linear algebraic groups, which are not necessarily equivalent to the Nisnevich classifying spaces.

7. The Grassmannian

Now we come to the real meat of the proof, using geometric properties of GL_n to create a geometric model for $B_{\text{ét}}GL_n$. In fact, this will work for any closed subgroup of GL_n, though I'll restrict attention to GL_n.

Consider the affine space over S, $\mathbb{A}^{nm}_S \cong \text{Hom}_S(\mathbb{A}^n, \mathbb{A}^m)$. GL_n acts on this by acting on the source, and on the open subscheme U_m of *monomorphisms* $\mathbb{A}^n \hookrightarrow \mathbb{A}^m$, this action is free. Let Z_m denote the complement of this open subscheme. There are GL_n-equivariant monomorphisms $U_m \hookrightarrow U_{m+1}$ induced by $\mathbb{A}^m \to \mathbb{A}^{m+1}$, $x \mapsto (x, 0)$. The sequence of objects $(\mathbb{A}^{nm}, U_m, i_{m,m+1} : U_m \to U_{m+1})$ satisfies the following key properties:

(1) At every (scheme-theoretic) point $s \in S$ with residue field $k(s)$, the fiber $U_m \times_S s$ has a $k(s)$-rational point for some m.
(2) For each m, there is an $m' > m$ such that $i_{m,m'} : U_m \to U_{m'}$ factors through $\mathbb{A}^{nm} - Z_m \to (\mathbb{A}^{nm})^2 - Z_m^2$.
(3) For every smooth scheme X over S and étale GL_n-torsor $E \to X$, there is an m such that $((U_m \times_S X) \times E)/GL_n \to X$ is an epimorphism of sheaves in the Nisnevich topology.

In the terrible lingo of [MV99], this sequence is called an **admissible gadget**.

Let's check the conditions. Condition 1 is saying that there exists a monomorphism $\mathbb{A}^n \to \mathbb{A}^m$ over a field, for which we can just take $m \geq n$. For condition 2, take $m' = 2m$. Then $U_m \to \mathbb{A}^{2nm}$, $x \mapsto (x, 0)$, lands outside $Z_m \times \mathbb{A}^{nm}$, so certainly outside Z_m^2. For condition 3, take $m = n$ – the map you get will be the vector bundle associated to E.

We define $U_\infty = \operatorname{colim}_m U_m$. Let $V_m = U_m/GL_n$ and $V_\infty = \operatorname{colim}_m V_m = U_\infty/GL_n$. V_∞ is going to be our geometric model for $B_{\text{ét}}GL_n$. To make this work, we need to show that U_∞ is contractible, and that this is all we need. This is the roughest part of this talk – get ready!

Proposition 7.1. *The canonical morphism $U_\infty \to S$ is an $\mathbb{A}^1$-homotopy equivalence.*

Proof. The point is to model the standard simplices and their boundaries as affine schemes. Specifically, we'll model Δ^n with $\mathbb{A}^n_S$, and $\partial\Delta^n$ with

$$\partial\Delta^n_{\mathbb{A}^1} := V\left(x_1 x_2 \cdots x_n \left(1 - \sum x_i\right)\right) \subseteq \mathbb{A}^n.$$

I claim that it suffices to show that the diagonal map exists in every diagram

$$\begin{array}{ccc} \partial\Delta^n_{\mathbb{A}^1} & \longrightarrow & U_\infty \\ \downarrow & \nearrow & \\ \mathbb{A}^n & & \end{array}$$

over S. Basically, there's an $\mathbb{A}^1$-local replacement, written $\operatorname{Sing}_*^{\mathbb{A}^1}(U_\infty)$ and q. v. [MV99], such that this diagram is equivalent to

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \operatorname{Sing}_*^{\mathbb{A}^1}(U_\infty) \\ \downarrow & \nearrow & \\ \mathbb{A}^n. & & \end{array}$$

Moreover, weak equivalences can be checked pointwise, so it suffices to take S to be a Henselian local affine scheme, a 'Nisnevich point.'

For $n = 0$, all this is saying is that U_∞ has a point. By condition 1, it has a point over the residue field, and using smoothness of the U_n, we can extend this to any local ring.

For $n > 0$, suppose given a map $\partial f : \partial\Delta^n_{\mathbb{A}^1} \to U_k \to \mathbb{A}^{km}$. On coordinate rings, this is equivalent to

$$\mathcal{O}_S[x_1, \ldots, x_{km}] \to \mathcal{O}_S[y_1, \ldots, y_n] / \left(y_1 \cdots y_n \left(1 - \sum y_i\right)\right).$$

We can just choose a lift of each x_i to $\mathcal{O}_S[y_1, \ldots, y_n]$, and this gives an extension of the map to $f : \mathbb{A}^n \to \mathbb{A}^{km}$. The problem is now that $\mathbb{A}^n$ might hit the bad set V_k. Let B, for 'bad,' be the preimage of V_k in $\mathbb{A}^n$. This is disjoint from $\partial\Delta^n_{\mathbb{A}^1}$. Thus, we can define another morphism $g : \mathbb{A}^n \to \mathbb{A}^{km}$ which is 0 on $\partial\Delta^n_{\mathbb{A}^1}$ and sends all of B to another point $x \in U_i$. The product $f \times g : \mathbb{A}^n \to \mathbb{A}^{2km}$ lands outside V_k^2, so it restricts to a map $\mathbb{A}^n \to U_{2km}$ by condition 2. Moreover, on $\partial\Delta^n_{\mathbb{A}^1}$, it's clear that this agrees with $i_{m,2m} \circ \partial f$. □

We've shown that U_∞ has contractible, and it has free GL_n-action by definition. The same could be said for $E(U_\infty)$. This gives a *monomorphism*

$$B(U_\infty, GL_n) := E(U_\infty)/(GL_n)_{\text{ét}} \to B_{\text{ét}}GL_n.$$

Using condition 3 above, one can show that this is a weak equivalence in the Nisnevich topology, and that this is also weakly equivalent to U_∞/GL_n. The proof isn't terribly complicated – there are some subtleties in changing between the two topologies at play – but it's tedious enough that I don't feel like giving it here. Of course, U_∞/GL_n is just the Grassmannian of n-planes. Thus, we get the following.

Theorem 7.2. *In the motivic homotopy category, there are canonical equivalences*

$$BGL_n \simeq B_{\text{ét}}GL_n \simeq G(n,\infty).$$

Theorem 7.3. *In the motivic homotopy category $\mathcal{H}_*(S)$, where S is a finite-dimensional, regular, Noetherian scheme, algebraic K-theory is represented by $G(\infty,\infty)\times\mathbb{Z}$, where $G(\infty,\infty)$ is an $\mathbb{A}^1$-local model of* $\operatorname{colim} G(n,\infty)$.

If S is not regular, we instead get Weibel's homotopy-invariant K-theory [Wei89].

8. Computations and examples

In this final section, I list homotopy groups of some things. What do I mean by homotopy groups? Well, they're a sheaf on the Nisnevich site, so I should give their values on any smooth S-scheme U; they're also *bigraded*, with

$$S^{p,q} = S_s^{p-q} \wedge S_t^q = S^{p-q} \wedge \mathbb{G}_m^{\wedge q}$$

for $p \geq q \geq 0$. If $p \geq 2q$, this is also equal to $S^{p-2q} \wedge (\mathbb{P}^1)^{\wedge q}$.

Let's start with the homotopy groups of K-theory. I've told you these for suspensions with respect to the simplicial sphere – they're just the K-groups of X. Consider

$$[\mathbb{G}_m \wedge X_+, G(\infty,\infty)\times\mathbb{Z}].$$

The target is $\mathbb{G}_m \wedge X_+ = (X\times\mathbb{G}_m)/X = X[t^{\pm1}]/X$. By the fundamental theorem of algebraic K-theory, the K-groups of $X\times\mathbb{G}_m$ are

$$K_i(X\times\mathbb{G}_m) = K_iX \oplus K_{i-1}X.$$

The quotient by the sheaf X kills the first of these things, and we're able to conclude that, for X a scheme,

$$[S^{p,q}\wedge X_+, G(\infty,\infty)\times\mathbb{Z}] = K_{p-2q}X$$

where this is interpreted as 0 if the degree is negative. In particular, we get the calculation

$$K_i(\mathbb{P}^1) = K_i(S^{2,1}) = K_0(S^{2+i,1}\wedge(\operatorname{Spec} k)_+) = K_i(k).$$

Another thing we could look at is $BGL_1 = B\mathbb{G}_m$. The geometric model is $G(1,\infty) = \mathbb{P}^\infty$, and one can show that it's $\mathbb{A}^1$-local, so that it computes the right homotopy groups. Over a regular scheme, are computed as follows.

$$[S^{p,q}\wedge X_+, \mathbb{P}^\infty] = \begin{cases} \operatorname{Pic}(X) & p=q=0 \\ \mathcal{O}(X)^\times & p=1, q=0 \\ H^0(X,\mathbb{Z}) & p=2, q=1 \\ 0 & \text{otherwise.} \end{cases}$$

For the first line, BGL_1 classifies line bundles, which are the Picard group. For the second, we've suspended X, so we get $GL_1(X)$, which is the units in X. For the third, we've smashed X_+ with $\mathbb{G}_m$ and a simplicial circle, so we get the units in $X\times\mathbb{G}_m$ mod those in X, which is precisely the free abelian group on the connected components of X. If we simplicially suspend any further, we get zero since GL_1 is discrete. Let's see what happens if we smash with $\mathbb{G}_m$ further. We have

$$[\mathbb{G}_m^{\wedge q}\wedge X_+, \mathbb{P}^\infty] = \left[\frac{\mathbb{G}_m^{\times q}\times X}{*\times\mathbb{G}_m^{\times(q-1)}\times X\cup\cdots\cup\mathbb{G}_m^{\times(q-1)}\times X}, \mathbb{P}^\infty\right],$$

but every unit in $\mathbb{G}_m^{\times q}\times X$ is a product of those induced from X and the $\mathbb{G}_m$ factors.

One can use the same theory to find homotopy groups of classifying spaces for other groups. [MV99] does this for étale groups, which are the simplest. On the other hand, the situation is more difficult for the groups GL_n, $n > 1$ – in particular, BGL_n is not known to be $\mathbb{A}^1$-local.

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