Future directions and discussion

Scribe notes from a talk by Marc Levine and Brad Drew

21 Mar 2014

Geometric model for $MGL^{*,*}$?

We have $H^{*,*}(-,\mathbb{Z}) = CH^*(-,*)$, defined geometrically as follows. Let $Z^q(X,m)$ to be the free abelian group on irreducible subschemes $W \subseteq X \times \Delta^m$ that are integral and such that for all faces F of Δ^m, $\mathrm{codim}_{X \times F} W \cap X \times F = m$. Then we have a map $Z^q(X,m) \to Z^q(X,m-1) \to Z^q(X,*)$. The homology $H_m(Z^q(X,*))$ defines $CH^q(-,m)$, and this equals $H^{2q-m,q}(-,\mathbb{Z})$.

So is there an analogous geometric model for $MGL^{*,*}$?

$\mathcal{SH} \leftrightarrow$ positive part of $\mathcal{SH}(k)$

Something about τ acting on $\mathbb{S} = (S^0, \mathbb{P}^1, \mathbb{P}^1 \wedge \mathbb{P}^1, \ldots)$.

In $\mathcal{SH}(k)[\frac{1}{2}]$, we have $[\mathbb{S}, \mathbb{S}] = GW(k) \to \mathbb{Z}$, with kernel an augmentation ideal I. We have $\mathbb{S}[\frac{1}{2}] = \mathbf{1}^+ \oplus \mathbf{1}^-$, and $\mathbf{1}^-_{\mathbb{Q}} \neq 0$.

Conjecture. $\mathbf{1}^-_{\mathbb{Q}}$ is concentrated in degree (n,n), and $\pi_{n,n}(\mathbf{1}^-_{\mathbb{Q}}) = I^n_{\mathbb{Q}}$.

A program for this is as follows. Find a minus version of Voevodsky's slice tower, and use it to understand $s_0^-(\mathbf{1})$.

Stability

We have
$$K_q^{MW}(F) = [S^n \wedge \mathbb{G}_m^r \wedge \mathrm{Spec}(F)_+, S^n \wedge \mathbb{G}_m^{\wedge q+r}],$$
but this is true unstably for $n \geq 2$, $r \geq 1$, $q \geq 0$.

So what about other stability results? Little is known at the moment.

The Freudenthal theorem compares a fiber of a cofiber to the original space. Take a diagram of $\mathbb{A}^1$-local spaces, and take the pushout Nisnevich-locally:

$$\begin{array}{ccc}
\mathbb{G}_m \;=\!=\; A & \longrightarrow & * \\
\downarrow & & \downarrow \\
* \longrightarrow \Sigma^A & \longrightarrow & L_{\mathbb{A}}\Sigma A
\end{array}$$

(narrative lost)

Recognizing $\mathbb{P}^1$ loop spaces

Is there a theory analogous to May's recognition principle for loop spaces? Is there an "operad" whose monad is $\Omega^n_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1}$? This was Dustin Clausen's thesis project for a while; it might not be easy. David White may explore soon.

The obvious thing to do is to try to produce an E_∞ operad as a cofibrant replacement of the initial object in some suitable category. Much discussion.

Obstruction theory

The usual obstruction theory goes forward to the extent of producing maps in the homotopy category, but it doesn't give maps of varieties.

Theorem (Asok–Morel). *Let X be smooth projective over k, and F a finitely generated field over k. Then $\pi_0^{\mathbb{A}^1}(X)(F) = X(F)/R \sim$, where R-equivalence is the relation of being connected by a rational curve $f : \mathbb{P}^1 \to X$.*

Suppose that $\operatorname{char} k = 0$ and k is algebraically closed. Then a smooth projective scheme X over k is **rationally connected** if for x, y two generic points of X, there is a rational curve $C \subseteq X$ containing x and y. (In this case, there turns out to be a rational curve connecting any two points.)

Now X is rationally connected iff $\pi_0^{\mathbb{A}^1}(X)(\bar{k}) = *$.

Theorem (Harris–Graber–Starr). *Let C/k be a smooth curve, X smooth over k, and $f : X \to C$ a flat surjective projective morphism. Suppose that $X_{\overline{k(C)}}$ is rationally connected. Then f admits a section. (Equivalently, $X(k(C)) \neq \emptyset$.)*

Theorem (deJong–Starr?). *The same theorem holds on replacing C with a surface S, and asking that S is rationally simply connected, and assuming also that the Brauer obstruction $H^2_{\mathrm{et}}(k(S), \operatorname{Hom}(\operatorname{Pic}(X_{\overline{k(S)}}), \mathbb{G}_m))$ vanishes.*

Warning. This term "rationally simply connected" has technical conditions in it, and is defined in terms of rational connectivity of some sort of path space, rather than being defined e.g. through homotopy groups.

So it would be nice to see theorems along these lines stated with the "right" homotopy-theoretic assumptions.

Note. If X is an $\mathbb{A}^1$-connected smooth projective variety over k, and $x \in X$, then $\pi_1^{\mathbb{A}^1}(X, x)$ is never trivial. If $\mathbb{G}_m$ is $\mathbb{A}^1$ discrete, then a $\mathbb{G}_m$-bundle is a covering space.

Various diagrams and sketches to do with étale cohomology and obstructions. Narrative lost.