Geometric aspects of algebraic cobordism

Scribe notes from a talk by Brian Hwang

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We will work over a field k of characteristic zero, not necessarily algebraically closed. By Sch_k we will denote the category of separable schemes of finite type over k, with a subcategory $\mathrm{Sm}_{/k}$ of smooth quasi-projective schemes.

Problem. $MGL^{*,*}$, the motivic cobordism spectrum, is hard to understand, so we might hope for a geometric interpretation.

Theorem (Levine–Morel)**.** *There is a universal Borel–Moore cohomology theory Ω^* of smooth schemes. Furthermore, $\Omega^*(k) \cong L$, the Lazard ring.*

Remark. We won't define a Borel–Moore cohomology theory in full detail, but it's something like a theory with a projective bundle formula. One of the axioms is

$$c_1(L \otimes M) = F(c_1(L), c_1(M))$$

for some formal group law F. So it's similar to an oriented theory.

Examples (of Borel–Moore cohomology theories)**.** —

- $CH^* \cong \Omega^* \otimes_L \mathbb{Z}$.
- $K^0[\beta, \beta^{-1}] \cong \Omega^* \otimes_L \mathbb{Z}[\beta, \beta^{-1}]$.

Sketch proof of theorem. Use the Quillen "axiomatic" construction of MU^*, and the Chow ring, and the resolution of singularities (so characteristic zero is vital). □

Theorem (Levine)**.** *For $X \in \mathrm{Sm}_{/k}$,*

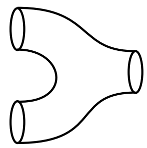
$$\Omega^n(X) \cong MGL^{2n,n}(X).$$

Fact. $CH^n(X) \cong H^{2n,n}_{\mathrm{mot}}(X)$.

Theorem (Levine–Pandharipande)**.** *There is an even more "geometric" theory of algebraic cobordism, $\omega^*(X)$, and a canonical isomorphism $\omega^*(X) \cong \Omega^*(X)$ for $X \in \mathrm{Sm}_{/k}$.*

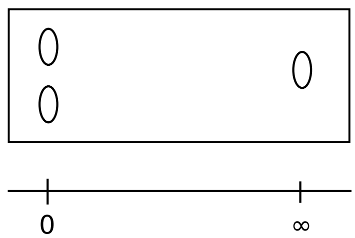
Constructing $\omega(X)$

Geometrically we think of cobordism theory in terms of cobordisms:



Question. How can we algebraize this?

We'll try families of algebraic varieties over $\mathbb{P}^1$:



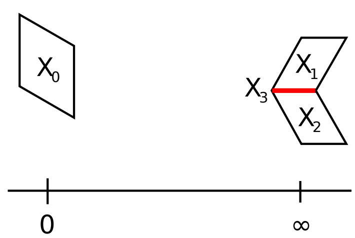
Idea (Naive)**.** For projective morphisms $\pi : Y \to \mathbb{P}^1$, impose the "naive cobordism relation"

$$[\pi^{-1}(0)] = [\pi^{-1}(\infty)].$$

But this resulting ring doesn't look like complex coordism. For instance, even if $X = \operatorname{Spec} k$, with k algebraically closed, and even for dimension 1 objects Y, this relation is not enough (if C_g is a curve of genus $g > 0$, $C_g \not\sim (1-g)\mathbb{P}^1$).

Idea. Impose relations obtained by fibers of π with normal crossing singularities. Surprisingly, if we impose relations given by the simplest such case, namely double point degeneracies, we recover Ω^*.

Definition. Let $X_i \in \mathrm{Sm}_{/k}$.



We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a **double-point relation** if there exists a family X such that

(1) X_0 is a smooth fiber over 0,

(2) the fiber over ∞ has two components X_1 and X_2,

(3) X_1 and X_2 intersect transversely along a smooth divisor D,

(4) $X_3 \cong \mathbb{P}_D(\mathcal{N}_{X/D} \oplus \mathcal{O}_D) \cong \mathbb{P}_D(\mathcal{O}_d \oplus \mathcal{N}_{X_2/D})$.

Note that the case $X_2 = \emptyset$ recovers the naive cobordism relation.

Theorem.

$$\omega_*(X) = \langle \text{ proj. } f : Y \to X \mid Y \text{ irred } \rangle / DPR$$

This is exciting because it means all $X \in \mathrm{Sm}_{/k}$ can be linked to products of $\mathbb{P}^n$s by the following results.

Theorem. *There exists a canonical isomorphism*

$$\omega_*(X) \cong \Omega_*(X).$$

Proof. —

(⇒) This is easy, just show that the DPRs hold in $\Omega_*(X)$.

(⇐) Trickier. We have to check axioms that ω_* is a "B–M functor of geometric type".

□

Corollary.

$$\omega_*(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{\text{partitions } \lambda} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times \ldots \times \mathbb{P}^{\lambda_{\ell(\lambda)}}]$$

Example. If $[Y] \in \omega_3(\mathbb{Q})_{\mathbb{Q}}$, for any $r \in \mathbb{Z}$,

$$r[Y] = s_3[\mathbb{P}^3] + s_{21}[\mathbb{P}^2 \times \mathbb{P}^1] + s_{111}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1] \in \omega_*(\mathbb{C}),$$

for some $s_\lambda \in \mathbb{Z}$.

If Z is the partition function of degree 0 Donaldson–Thomas invariants (this is a rational power series), we have

$$Z(Y,q)^r = \prod_{|\lambda|=3} Z(\mathbb{P}^\lambda, q)^{s_\lambda}.$$

Example. Gromov–Witten theory "equals" Donaldson–Thomas theory, in the sense that their partition functions agree up to change in coordinates. This holds for all toric 3-folds over $\mathbb{C}$.

Question. Gromov–Witten theory is usually phrased in a symplectic context. What kind of symplectic cobordism theory corresponds to ω_* and the DPRs?

Remark. —

- $\Omega^*(X)$ is easy for cellular varieties.
- Additive structure is given by the free $\coprod$-module on cells, so additively $\Omega^*(X)$ is like a Chow group.
- We have a ring homomorphism $\Omega^*(X) \to MU^{2*}(X(\mathbb{C})^{\mathrm{an}})$. If X is cellular then this is an isomorphism.

Problem. Computations in $\Omega^*(X)$ currently only seem to apply well if they were done by algebro-geometric methods (e.g. blow-ups). Using the structure of $MU^*(X(\mathbb{C})^{\mathrm{an}})$ seems difficult, especially if the structural results were gained using topological methods with no algebraic analogue.

Theorem. $\Omega^n(X) \cong MGL^{2n,n}(X)$.

Sketch proof. —

1. Prove the result for function fields of smooth varieties.
2. Induct on the dimension of X, using localization sequences for Ω_* and the description of their connecting homomorphisms.

If $X = \mathrm{Spec}(F)$, where F is a finitely generated field over k, then

$$E_2^{p,q}(n) = H^{p-q,n-q}(F) \otimes \coprod q \Rightarrow MGL^{p+q,n}.$$

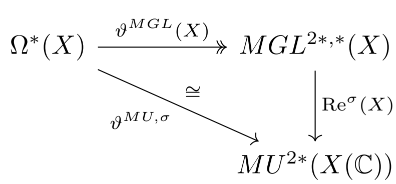
Use this to show that

$$\vartheta^{MGL}(X) : \Omega^*(X) \to MGL^{2*,*}(X)$$

is surjective, and then given $\sigma : F \hookrightarrow \mathbb{C}$, we have a natural transformation

$$\vartheta^{MU,\sigma}(X) : \Omega^*(X) \to MU^{2*}(X(\mathbb{C}))$$

and we have a commutative diagram



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