From Algebraic Cobordism to Motivic Cohomology, Part III

Scribe notes from a talk by Dylan Wilson

21 Mar 2014

Plan. Let $f : MGL/(a_0, a_1, \ldots) \to H\mathbb{Z}$ be the map of the previous lectures.

1. $MGL_{\leq 0} \simeq HZ_{\leq 0} \simeq (1/\eta)_{\leq 0}$.
2. $H\mathbb{Q} \wedge f$ is an equivalence.
3. $H\mathbb{Z}/\ell \wedge f$ is an equivalence.

Assuming these steps, we can run the following argument.

Proof of thm.

$$\begin{array}{ccccc} F & \xrightarrow{\alpha} & MGL/(a_0, a_1, \ldots) & \xrightarrow{f} & H\mathbb{Z} \\ {\scriptstyle\delta}\uparrow & & & & \\ \Omega^{1,0} H\mathbb{Z} & & & & \end{array}$$

We want to show that $\alpha \sim 0$ and that any section s of δ satisfies $s \sim 0$.

We know (from steps 2 and 3) that $H\mathbb{Z} \wedge F \simeq *$. Note that $MGL/(a_0, a_1, \ldots)$ and $\Omega^{1,0} H\mathbb{Z}$ are homotopy MGL-modules.

Lemma. *Suppose that $H\mathbb{Z} \wedge F \simeq *$, and that X is a homotopy MGL-module. If X is r-connective for $r \in \mathbb{Z}$, then $[F, X] = 0$.*

Proof. By the homotopy t-structure, reduce to $[\Sigma^{p,0} F, \kappa_n X] = 0$ for all p and n.

$$\begin{array}{ccc} \Sigma^{p,0} F & \longrightarrow & \kappa_n X \\ \downarrow & & \uparrow \\ \Sigma^{p,0} F \wedge x_0 MGL & \longrightarrow & \kappa_n X \wedge \kappa_0 MGL \end{array}$$

□

Now

$$F \wedge \kappa_0 MGL \simeq F \wedge \kappa_0 H\mathbb{Z} \simeq F \wedge H\mathbb{Z}_{\leq 0} \simeq 0,$$

as desired. □

Theorem. $H\mathbb{Z}_{\leq 0} \simeq (1/\eta)_{\leq 0}$.

Proof. We want to show that this sequence is exact:

$$\underline{\pi}_{n-1,n-1}1 \xrightarrow{\eta} \underline{\pi}_{n,n}1 \to \underline{\pi}_{n,n}H\mathbb{Z} \to 0.$$

This follows from the diagram0

$$\begin{array}{ccccccc} \pi_{n-1,n-1}1 & \xrightarrow{\eta} & \pi_{n,n}1 & \longrightarrow & K^m_{-n}(L) & \longrightarrow & 0 \\ & & & \searrow & \simeq \big\downarrow \lambda & & \\ & & & & \pi_{n,n}H\mathbb{Z} & & \end{array}$$

□

Computation. We want to compute $H\mathbb{Z}/\ell^{**}MGL/(a_0, a_1, \ldots)$.

1.
$$\begin{array}{ccc} L & \xrightarrow{h_R} & R[b_0, b_1, \ldots] \\ \big\downarrow & & \big\downarrow \\ MGL_{**} & \longrightarrow & HR_{**}MGL \end{array}$$

2. $L \simeq \mathbb{Z}[a_0, a_1, \ldots]$, and
$$h_{\mathbb{Z}}(a_n) = \begin{cases} \ell \cdot b_n & n = \ell^i - 1 \\ b_n & \text{otherwise} \end{cases}$$
modulo decomposables.

Plan. —

I. $H\mathbb{Z}/\ell_{**}$ as an $\mathcal{A}_{**}$ comodule algebra.

II. See what happens inductively as we kill x_i, the a_ns with $n \neq \ell^i - 1$.

III. Kill ℓ-typical elements.

$H\mathbb{Z}/\ell^{**}BGL \simeq H\mathbb{Z}/\ell^{**}[\![c_1, c_2, \ldots]\!]$, a Hopf algebra with

$$\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j.$$

Check that $H\mathbb{Z}/\ell_{**}BGL = H\mathbb{Z}/\ell[\beta_1, \beta_2, \ldots]$ as Hopf algebras, with β_i dual to c_1^2.

By the Thom isomorphism,

$$H\mathbb{Z}/\ell_{**}MGL = H\mathbb{Z}/\ell_{**}[b_1, b_2, \ldots], \qquad |b_n| = (2n-2, n-1).$$

In the action of $\mathcal{A}^{**}$ on $H^{**}\mathbb{P}^{**}$, the Q_i act trivially, and the P^i are defined by the Cartan formula.

Theorem. *We describe the coaction:*

$$\begin{array}{ccc} H\mathbb{Z}/\ell_{**}MGL & \longrightarrow & \mathcal{A}_{**} \otimes_{H\mathbb{Z}/\ell_{**}} H\mathbb{Z}/\ell_{**}MGL \\ & \searrow \quad \nearrow & \\ & P_{**} \otimes_{H\mathbb{Z}/\ell_{**}} H\mathbb{Z}/\ell_{**}MGL & \end{array}$$

$H\mathbb{Z}/\ell_{**}MGL \simeq P_{**}[x_i]$ as comodule algebras and L-modules.

Corollary. *$H\mathbb{Z}/\ell_{**}(MGL/(x_1, x_2, \ldots)) \simeq P_{**}$ as $\mathcal{A}_{**}$-comodule algebras.*

Theorem. *Let I be a set of integers ≥ 0, and $v_i = a_{\ell^i - 1}$. There is an isomorphism of $\mathcal{A}^{**}$-modules*

$$\mathcal{A}^{**}/(Q_i \mid i \notin I) \simeq H\mathbb{Z}/\ell^{**}(MGL/(x, v_i \mid i \in I)),$$

$$[\phi] \mapsto \phi(\theta),$$

with θ the Thom class.

Remark. For $I = \{1, 2, \ldots\}$, we get $H^{**}MGL/(a_0, \ldots) \simeq \mathcal{A}^{**}/Q_0$.

Proof. Suppose inductively that this is true for I; pick $r \notin I$. Let $E = MGL/(x, v_i \mid i \in I)$. Then

$$H^{**}(E/v_r) \simeq H^{**}E \,\square_{H^{**}MGL}\, H^{**}(MGL/v_r).$$

□

We now come to the last step.

$$\Sigma^? MGL \xrightarrow{v_r} MGL \to MGL/v_r.$$

We want to understand the relationship between Q_r, v_r, $\delta = c \circ b$, and θ. In fact

$$\theta \circ \delta = Q_r \circ \theta$$

up to a unit in $\mathbb{Z}/\ell$.