From Algebraic Cobordism to Motivic Cohomology, Part II

Scribe notes from a talk by David White

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Recall. We have a category Cor_S whose objects are smooth schemes over S, with

$$\mathsf{Cor}_S(X,Y) = \mathbb{Z}[Z \not\hookrightarrow X \times Y \mid \text{finite over } X, Z \text{ is integral}]$$

Example. For any $f : A \to B$ in $\mathrm{Sm}_{/S}$, its graph is in $\mathsf{Cor}_S(X,Y)$.

A **presheaf with transfers** is $F : \mathsf{Cor}_S \to \mathsf{Ab}$. $\mathrm{Spc}_{\mathrm{tr}}$ is the category of simplicial objects F as above. We have $\mathbb{Z}_{\mathrm{tr}} : \mathsf{Cor} \to \mathrm{Spc}_{\mathrm{tr}}$ and $\mathbb{Z}_{\mathrm{tr}} : \mathsf{Spt} \to \mathsf{Spt}_{\mathrm{tr}}$. We use this to define $H\mathbb{Z}$. We could apply $-\otimes R$ to get HR.

We will tackle the following. Let $f : MGL/(a_1, a_2, \dots) \to H\mathbb{Z}$, the map from last talk. We want to show that $H\mathbb{Q} \wedge f$ and $H\mathbb{Z}/\ell \wedge f$ are weak equivalences, and deduce that $H\mathbb{Z} \wedge f$ is a weak equivalence.

Goal. $H\mathbb{Z}/\ell \wedge H\mathbb{Z}$ is a cellular $H\mathbb{Z}/\ell$-module, whose homotopy grousp are the kernel of the Bockstein.

Theorem. $HR^{p,q} \sim H^{p,q}(-,R)$.

Theorem (Röndigs–Østvaer 2008)**.** *The derived adjunction between $\mathcal{D}(HR)$ and $\mathcal{SH}_{\mathrm{tr}}(S,R)$ restricts to an equivalence on full subcategories of cellular objects. For $\mathrm{char}(k) = 0$ this equivalence is even stronger.*

We have cofiber sequences:

$$H\mathbb{Z} \xrightarrow{\ell} H\mathbb{Z} \to H\mathbb{Z}/\ell, \qquad H\mathbb{Z}/\ell \xrightarrow{\ell} H\mathbb{Z}/\ell^2 \to H\mathbb{Z}/\ell.$$

Classical Steenrod algebra

We consider cohomology operations such as

$$H^n(X;R) \to H^{2n}(X;R), x \mapsto x \cup x.$$

(Here e.g. $R = \mathbb{F}_2$). But these are unstable (they don't commute with suspension); "stabilizing", we obtain

$$\mathrm{Sq}^i : H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2),$$

the **Steenrod squares**. These turn out to generate all stable cohomology operations over $\mathbb{Z}/2$. These Sq^n are characterized by:

1. They are additive homomorphisms and natural;
2. $\mathrm{Sq}^0 = \mathrm{id}$;
3. $\mathrm{Sq}^n(x) = x \cup x$ when $|x| = n$;
4. If $|x| < n$ then $\mathrm{Sq}^n x = 0$;
5. $\mathrm{Sq}^n(x \cup y) = \sum_{i+j=n}(\mathrm{Sq}^i(x) \cup \mathrm{Sq}^j(y))$ (the Cartan formula).

From this we can formally deduce the Adem relations between these elements. We obtain the Steenrod algebra $\mathcal{A}^*$.

Given $0 \to A \to B \to C \to 0$, we apply $H^*(X; -)$ to get a long exact sequence; then we define the **Bockstein** to be the connecting homomorphism (given by the Snake Lemma). We denote this

$$\beta : H^i(X; C) \to H^{i+1}(X; A).$$

This is a cohomology operation.

For $p > 2$ we have to factor this in: instead of Sq^i, we have operations $P^i : H^n(X; \mathbb{Z}/p) \to H^{n+2i(p-1)}(X; \mathbb{Z}/p)$ as well as the Bockstein β arising from $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$, which together generate the mod p Steenrod algebra. This is a graded $\mathbb{F}_p$-algebra, and in fact a Hopf algebra with

$$\psi : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$$

given by the Cartan formula.

We can dualize to $\mathcal{A}_*$, which is a commutative, associative algebra. We can write

$$\mathcal{A}^* \cong H\mathbb{F}_p^*(H\mathbb{F}_p), \qquad \mathcal{A}_* \cong H\mathbb{F}_{p*}(H\mathbb{F}_p).$$

By applying the Serre spectral sequence inductively to fiber sequences $K(\mathbb{Z}/2, n) \to * \to K(\mathbb{Z}/2, n+1)$, we get that

$$H\mathbb{F}_{2*}(H\mathbb{Z}) = \ker(\beta) = \mathbb{F}_2[\xi_1^2, \xi_2, \xi_3, \ldots].$$

Define Q_i inductively by $Q_0 = \beta$, and

$$Q_i = q_i\beta - \beta q_i, \qquad q_i = P^{\ell^{i-1}} \ldots P^{\ell} P^1.$$

Theorem (Milnor 1958)**.**

$$\mathcal{A}_* \cong (E(\tau_0, 1) \otimes E(\tau_1, 2p-1) \otimes \ldots) \otimes (P(\xi_1, 2p-2) \otimes P(\xi_2, 2p^2-2) \otimes \ldots),$$

a tensor product of an exterior product on generators τ_i of degree $2p^i - 1$ and a polynomial algebra on generators ξ_i of degree $2p^i - 2$. (If $\ell = 2$ then there is no exterior part.)

Motivic Steenrod algebra

Let S be some base scheme, $\ell \neq \mathrm{char}(S)$. (We may as well take S to be a field.) We define the **motivic Steenrod algebra** $\mathcal{A}^{**}$ as the algebra of all bistable natural transformations

$$\tilde{H}^{**}(-, \mathbb{Z}/\ell) \to H^{**}(-, \mathbb{Z}/\ell).$$

Theorem (Voevodsky 2003)**.** *There are reduced power operations*

$$P^i \in \mathcal{A}^{2i(\ell-1),i(\ell-1)}, \qquad \beta : H\mathbb{Z}/\ell \to \Sigma^{1,0} H\mathbb{Z}/\ell.$$

The $\mathbb{F}_\ell$ basis for the classical $\mathcal{A}^*$ comes from P^i and β:

$$H^{**}_{\mathbb{Z}/\ell} \otimes_{\mathbb{F}_\ell} \mathcal{A}^* \cong \mathcal{A}^{**}.$$

So $\mathcal{A}^{**}$ is generated by P^i, β, and $u \mapsto au$ where $a \in H^{**}(S, \mathbb{Z}/\ell)$.

Fact. $(H\mathbb{Z}/\ell_{**}, \mathcal{A}_{**})$ is a Hopf algebroid.

Milnor basis

Recall τ from the previous talk, in bidegree $(0, 1)$. We will have a new element ρ in bidegree $(1, 1)$.

Consider the Hopf algebroid (A, Γ), with

$$A = \mathbb{Z}/\ell[\tau, \rho],$$

$$\Gamma = A[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_1^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1}),$$

$$\Delta(\rho) = \rho \otimes 1, \quad \Delta(\tau) = \tau \otimes 1,$$

and $\Delta(\tau_i)$ and $\Delta(\xi_i)$ are as in the classical case. The units are

$$\eta_R(\tau) = \tau + \rho\tau_0, \qquad \eta_L(\tau) = \tau.$$

If $k = \mathbb{C}$ or $\mathbb{F}_\ell$ for $\ell \neq 2$, $\rho \neq 0$, and this is a Hopf algebra (left and right units agree).

Now $A \to H\mathbb{Z}/\ell$ makes $H\mathbb{Z}/\ell$ into an A-module, and

$$\mathcal{A}_{**} \cong \Gamma \otimes_A H\mathbb{Z}/\ell_{**}.$$

Punchline. Let M denote the Milnor basis on $\mathcal{A}_{**}$. Then

$$\bigvee_{\zeta \in M} \Sigma^{|\zeta|} H\mathbb{Z}/\ell \xrightarrow{\cong} H\mathbb{Z}/\ell \wedge H\mathbb{Z}/\ell.$$

We have $H\mathbb{Z}/\ell_{**} \cong \ker(\beta)$, and

$$0 \to H\mathbb{Z}/\ell_{**}H\mathbb{Z} \to H\mathbb{Z}/\ell_{**}H\mathbb{Z}/\ell \xrightarrow{\beta_*} H_{**}\Sigma^{1,0}H\mathbb{Z} \to 0.$$

Let M_Z be the sub-basis of M generated by $\tau(E)\xi(R) with \varepsilon_0 = 0$, $E = (\varepsilon_0, \varepsilon_1, ldots)$.

Corollary.

$$\bigvee_{\zeta \in M_Z} \Sigma^{|\zeta|} H\mathbb{Z}/\ell \to H\mathbb{Z} \wedge H\mathbb{Z}/\ell$$

is an equivalence of $H\mathbb{Z}/\ell$-modules.