Stable homotopy groups of spheres and their analogue over Spec $\mathbb{C}$

Scribe notes from a talk by Zhouli Xu

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The idea here is to apply motivic homotopy to ordinary homotopy theory, as opposed to more common applications to algebraic geometry.

We first recall what we know about the ordinary homotopy groups $\pi_{n+k}(S^n)$.

Theorem (Serre). *These abelian groups are finitely generated, and we can compute them each prime at a time.*

Theorem (Freudenthal). *$\pi_{n+k}(S^n)$ depends only on k so long as $n \geq k+2$; we call this $\pi_k(S^0)$, the k-**stem**.*

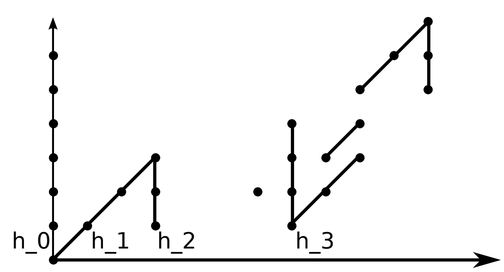
For this talk we'll mainly compute at $p = 2$ and work stably.

- The 0-stem is $\pi_0(S^0) = \mathbb{Z}/2$.
- The 1-stem is $\mathbb{Z}/2$, generated by the Hopf map $\eta : S^3 \to S^2$.
- The 2-stem is $\mathbb{Z}/2$, generated by η^2.
- Serre computed inductively with the Leray–Serre spectral sequence to determine through the 8-stem.
- Toda computed through the 14-stem with the EHP spectral sequence.

Theorem (Adams). *There is a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow \pi_{t-s}(\hat{S_2})$$

where $\mathcal{A}$ is the Steenrod algebra.



For any spectral sequence, there are three problems:

- How do we compute E_2? (E_1?)
- How do we compute differentials?
- How do we compute extensions?

There are some strategies for problem 1:

- Cobar resolution,
- Minimal resolutions,
- Λ-algebra,
- May spectral sequence.

The May spectral sequence is a trigraded spectral sequence based on a filtration $E^\bullet\mathcal{A}$ of the Steenrod algebra:

$$E_2^{u,v,t} = \mathrm{Ext}^{u,v,t}_{E^\bullet\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow \mathrm{Ext}^{n+v,t}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2).$$

The input is very computable.

Further progress:

- May used his spectral sequence to compute $\pi_* S$ through degree 29.
- Tangora used the May spectral sequence to compute the Adams E_2 through degree 45.
- Barratt–Mahowald–Tangora computed $\pi_*(S^0)$ through the Adams spectral sequence through degree 45. (Correction by Bruner: one missing differential originating in $\mathrm{Ext}^{4,4+38}$.
- Kochman computed through degree 64 using the Atiyah–Hirzebruch spectral sequence on BP. (Corrections by Kochman–Mahowald and Isenksen.)

A little bit of chromatic theory: there is a spectrum BP with

$$BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots], \qquad |v_1| = 2, |v_2| = 6, |v_3| = 14, \ldots.$$

These elements relate to periodicity behavior and vanishing lines in the Adams spectral sequence. For instance, the v_1 case gives Bott periodicity.

Theorem (Adams–Novikov)**.** *There is a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_{t-s}(\hat{S_2}).$$

This spectral sequence is sparser, and we know $\mathrm{d}_{2n} = 0$. For $p = 2$ the best computations with this are to degree around 30 or 40 (due to Ravenel). Results at odd primes are much better.

Theorem (Miller)**.** *There is a diagram of spectral sequences*

$$\begin{tikzcd} & \square \arrow[dl, "\text{Mahowald SS}"'] \arrow[dr, "\text{May SS}"] & \\ \text{ASS} \arrow[dr] & & \text{ANSS} \arrow[dl] \\ & \pi_*(S^0) & \end{tikzcd}$$

For odd primes, the Mahowald SS degenerates; so the ASS remains complicated, whereas the May SS still makes the ANSS simpler at odd primes. At $p = 2$ this is not true, and the ASS still has some advantage.

Motivic spheres

Recall the various spheres in play:

- $S^{1,0} = S^1$,
- $S^{1,1} = \mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$,
- $\mathbb{P}^1 = S^{2,1}$,
- $\mathbb{P}^n/\mathbb{P}^{n-1} = S^{2n,n}$,
- $\mathbb{A}^n \smallsetminus \{0\} = S^{2n-1,n}$.

Theorem (Morel)**.** $\pi_{p,q} = 0$ *if* $p < q$*, and* $\pi_{p,p}$ *is Milnor–Witt K-theory.*

We take the stable setting, $p = 2$, and $\operatorname{Spec}\mathbb{C}$ for base scheme. Let $\mathbb{M}^2 = H^{*,*}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}/2)$. We have the motivic Steenrod algebra $\mathcal{A}_{\mathrm{mot}}$, which acts on $H^{*,*}(X, \mathbb{Z}/2)$.

Theorem (Voevodsky)**.** $\mathbb{M}_2 = \mathbb{F}_2[\tau]$*, with* $|\tau| = (0,1)$*, and the motivic Steenrod algebra is described by*

$$\mathcal{A}_{\mathrm{mot}} = \mathbb{M}_2[\mathrm{Sq}^i \mid i \geq 1]/\textit{motivic Adem relations.}$$

Here Sq^{2k} *has bidegree* $(2k, k)$*, and* Sq^{2k-1} *has bidegree* $(2k-1, k-1)$*.*

Theorem (Dugger–Isaksen, Hu–Kriz–Ormsby)**.** *There is a motivic Adams spectral sequence*

$$E_2^{s,t,w} = \operatorname{Ext}^{s,t,w}_{\mathcal{A}_{\mathrm{mot}}}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \pi_{t-s,w}(\hat{S_2}),$$

with differentials bigraded as $\mathrm{d}_r : E_r^{s,t,w} \to E_r^{s+r,t+r-1,w}$*.*

Theorem (Isaksen)**.** *There is a (quadruply-graded) motivic May spectral sequence, which is computed through degree 70.*

Realization $X \mapsto X(\mathbb{C})$ takes these motivic spectral sequences to their classical analogues. These are closely related; in fact,

$$E_2\text{-MMSS} \cong E_2\text{-MSS} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau],$$

$$E^0\mathcal{A}_{\mathrm{mot}} = E^0\mathcal{A} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau],$$

$$E_2\text{-MASS} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau^{-1}] \cong E_2\text{-ASS} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}].$$

Theorem (Isaksen)**.** *Computation of the MASS through degree 59. The differential* $\mathrm{d}_3(Q_2) = \tau^2 gt$ *in the MASS implies a new differential* $\mathrm{d}_3(Q_2) = gt$ *in the ASS.*

One application is as follows. A differential $\mathrm{d}_r(x) = y$ in the ASS corresponds to a differential $d_r(x) = \tau^n y$ in the MASS. We must have

$$\mathrm{weight}(X) = \mathrm{weight}(\tau^n y) = \mathrm{weight}(y) - n,$$

and in particular $\mathrm{weight}(x) \leq \mathrm{weight}(y)$. This is probably less powerful in practice than it sounds.

Reverse engineering of the ANSS

Theorem (Hur–Kriz–Ormsby)**.** *There is a motivic Adams–Novikov spectral sequence*

$$\mathrm{Ext}^{s,t,w}_{BPGL_* BPGL}(BPGL_*, BPGL_*) \Rightarrow \pi_{t-s,w}(\hat{S_2}).$$

$$E_2\text{-}MANSS = E_2\text{-}ANSS \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\tau],$$

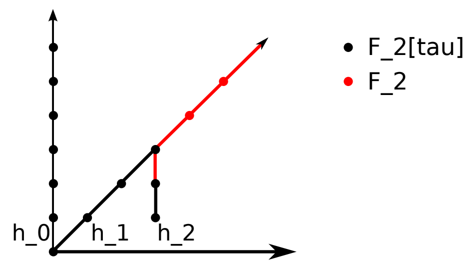
$$\mathrm{Ext}^{s,t,\frac{t}{2}-n} \to \mathrm{Ext}^{s,t}, \qquad \tau^n x \mapsto x.$$

Theorem. *There is a diagram of spectral sequences*

$$\begin{tikzcd}
MANSS \arrow[dd] \arrow[dr] & & MASS \arrow[dl] \arrow[dd] \\
& \pi_{*,*}(S) \arrow[d] & \\
& \pi_*(S) & \\
ANSS \arrow[ur] & & ASS \arrow[ul]
\end{tikzcd}$$

The MASS is computed through degree 59.

Let's look at a basic computation:



Here $\eta^4\tau = 0$ and $\eta^3\tau = 4\nu$.

We can tabulate some filtrations:

$\pi_{*,*}$	$t-s$	w	MANSS:	t	w	s
η	1	1		2	1	1
η^2	2	2		4	2	2
ν	3	2		4	2	1
2ν	3	2		4	2	1
η^3	3	3		6	3	3
η^4	4	4		8	4	4

And we can draw the motivic Adams–Novikov spectral sequence: