The slices of a Landweber exact theory

Scribe notes from a talk by Lorenzo Mantovani

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Notation. —

- $\mathcal{SH}^{\mathrm{eff}}(S)$ is the localizing triangulated subcategory of $SH(S)$ generated by $\Sigma^\infty_T X_+$ for $X \in \mathrm{Sm}_{/S}$.
- $\mathcal{SH}(S)_T$ is the Tate subcategory, the full localizing triangulated subcategory of $SH(S)$ generated by $S^{p,q} = \mathbb{S}^{p-q} \wedge \mathbb{G}_m^q$.
- $\mathcal{SH}(S)_{T\geq 0}$ is like the above, but generated by $S^{p,q}$ for $q \geq 0$.

1 $s_0 MGL$

The main theorem is

Theorem. *The cofiber* $\mathrm{cofib}(\mathbf{1} \to MGL)$ *belongs to* $\Sigma_T \mathcal{SH}(S)_{T\geq 0}$.

Corollary. *$s_0\mathbf{1} \to s_0 MGL$ is an isomorphism in $\mathcal{SH}(S)$. (In particular, if $S = \operatorname{Spec} k$, then $s_0 MGL = H\mathbb{Z}$.)*

2 Description of $s_0 MGL$

Idea. We have seen a map $\mathbb{Z}[x_1, \ldots, x_n] = MU_* \to MGL_{2*,*}$. We want this to define a map "multiplication by x_i" on MGL. We will need to bifibrantly replace MGL and cofibrantly replace $S^{2n,n}$. We can form the object

$$MGL/(x_1, x_2, \ldots)MGL.$$

Definition. A **multi-index** is a function $f : \mathbb{N} \to \mathbb{N}$ almost everywhere zero; these will encode exponents for monomials. In particular, multi-indices are not of fixed length. We let I be the set of such multi-indices.

We define a map $D : I \to \mathsf{Spt}^{\Sigma}_{\mathbb{P}^1}(\mathrm{Sm}_{/S})$ by

$$D_{\underline{a}} = \Sigma^{2,1} MGL^{\wedge_{MGL} a_1} \quad \wedge_{MGL} \quad \ldots \quad \wedge_{MGL} \quad \Sigma^{2m,m} MGL^{\wedge_{MGL} a_m}.$$

Taking two multi-indices $\underline{a}$ and $\underline{b}$ which differ only in that $a_i \geq b_i$ for some i, we have a map $D_{\underline{b}} \to D_{\underline{a}}$ given by $\underbrace{\mathrm{id} \wedge \ldots \wedge \mathrm{id}}_{b_i} \wedge \underbrace{x_i \wedge \ldots \wedge x_i}_{a_i - b_i}$.

Proposition. *If $F : I_{\deg \geq n} \to \mathcal{C}$ is a diagram in a cofibrantly generated model category, and* $\operatorname{hocolim} F\big|_{\deg I \geq \underline{a}} = F(\underline{a})$ *for all a of degree m, then* $\operatorname{hocolim} F = \operatorname{hocolim} F\big|_{\deg I \geq n+1}$.

Proposition. *$f_i : \mathcal{SH}(S) \to \Sigma_T^i \mathcal{SH}(S)^{\mathrm{eff}} \subseteq \mathcal{SH}(S)$ preserves homotopy colimits.*

Theorem. *Suppose that $MGL \to Q := MGL/(x_1, \ldots)MGL$ is the map to the zero slice. Then $s_i MGL \simeq \Sigma_T^i s_0 MGL \otimes MU_{2i}$, and this isomorphism is compatible to $MU_* \to MGL_{*,*}$.*

Sketch proof. —

- The map $D\big|_{\deg I \geq 1} \hookrightarrow D$ induces a map on hocolims, and this is the map $MGL \to Q$. So $\operatorname{hocolim} D\big|_{\deg I \geq 1} \to f_1 MGL$ is an isomorphism in $\mathcal{SH}(S)$. Here f_1 is the slice truncation functor introduced in a previous talk.
- Next show that $f_2 \operatorname{hocolim} D\big|_{\deg I \geq 1} \simeq \operatorname{hocolim} F_2 D\big|_{\deg i \geq 2}$, where $F_n : \mathsf{Spt}^{\Sigma}_{\mathbb{P}^1}(-) \to \mathsf{Spt}^{\Sigma}_{\mathbb{P}^1}(-)$ lifts f_n.
- By induction one shows that $f_n MGL \simeq \operatorname{hocolim} D\big|_{\deg I \geq n}$.
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$$\begin{aligned} s_n MGL &\simeq \operatorname{cofib}(f_{n+1} MGL \to f_n MGL) \\ &\simeq \operatorname{hocolim} \operatorname{cofib}(F_{n+1} D\big|_{\deg I \geq n} \to F_n D\big|_{\deg I \geq n}). \end{aligned}$$
- Now $\operatorname{cofib}(\underline{a}) = 0$ if $\deg(\underline{a}) \geq n+1$, and
$$\operatorname{cofib}(\underline{a}) = \Sigma_T s_0 MGL \quad (\deg(\underline{a}) = n),$$
$$\operatorname{hocolim}(-) = \bigoplus_{\text{quotients of } \mathbb{N}_{2n}} \Sigma_T^n s_0 MGL = \Sigma_t^n s_0 MGL \otimes MU_{2n}.$$

We use in this last step that $s_n \Sigma_T E = \Sigma_T s_{n-1} E$. □

3 Computation of the slices of a Landweber exact spectrum

Theorem. *If M_* is a Landweber exact MU_*-module, and $E \in \mathcal{SH}(S)$ is the element representing the (co)homology theory defined by M_*, namely*

$$(MGL \wedge -)_{*,*} \otimes_{MU_*} M_* \quad : \mathcal{SH}(S) \to \mathsf{GrAb},$$

then
$$s_i E \simeq \Sigma_T^i s_0 MGL \otimes M_{2i}.$$

Main idea. We take a projective resolution

$$0 \to A_* \to B_* \to M_* \to 0$$

of M_* by MU_*-modules. (Landweber exactness guarantees homological dimension 1.) □