Landweber exactness

Scribe notes from a talk by Marc Levine

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Last lecture we saw that the theory of oriented spectra carries over to motivic homotopy. In ordinary topology, we have a formal group law $F_{MU}(u,v) \in MU_*[\![u,v]\!]$, with

$$\begin{array}{ccccc} MU^*(\mathbb{CP}^\infty) & = & MU_*[\![t]\!] & & \ni t \\ \downarrow & & & & \downarrow \\ MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) & = & MU_*[\![u,v]\!] & & \ni F_{MU} \end{array}$$

Goal. We want to use formal group laws to construct new objects in $\mathcal{SH}(S)$.

Motivic (co)homology theories

Definition. A **(co)homology theory** on some $J \subseteq \mathcal{SH}(S)$, localizing and triangulated, is a (co)homological functor (taking distinguished triangles to exact sequences), and into bigraded groups, which (for homology) is notated $F \mapsto E_{*,*}(F)$, and satisfies:

- compatibility with $\oplus$,
- $E_{*,*}(S^{a,b} \wedge F) = E_{*-a,*-b}(F)$.

For cohomolgy we have instead $E^{*,*}(S^{a,b} \wedge F) = E^{*-a,*-b}(F)$.

Theorem (Quillen)**.** *(F_{MU}, MU_{2*}) is the universal formal group law.*

Definition. We say (F, R_*) is a **graded formal group law** if $F(u,v) \in R_*[\![u,v]\!]$ has total degree -1, where $\deg u = \deg v = -1$.

Theorem. *Let (F, R_*) be a graded formal group law classified by $\phi : MU_{2*} \to R_*$. Assume that (F, R_*) is Landweber exact (to be defined). Then*

1. *The presheaf $F \mapsto (MGL \wedge F)_{*,*} \otimes_{MU_*} R_*$ (for $F \in \mathcal{SH}(S)$) defines a bi-graded homology theory on $\mathcal{SH}(S)$.*
2. *This theory is represented by an object in the Tate subcategory (to be defined), $\mathcal{SH}(S)_{\mathcal{T}}$.*

Definition. A formal group law represented by ϕ is **Landweber exact** if R_* is flat over the quotient stack $[MU_*/MU_*MU]$.

The quotient stacks $[MU_{2*}/MU_{2*}MU]$ and $[MGL_*/MGL_*MGL]$

We won't define quotient stacks, but it won't be so necessary.

Proposition. *MU_* is "flat": for all $\mathcal{F} \in \mathcal{SH}(S)$,*

$$MU_*MU \otimes_{MU_*} MU_*\mathcal{F} \to (MU \wedge MU \wedge \mathcal{F})_*$$

is an isomorphism.

$MGL_*(= MGL_{2*,*})$ is "flat" also: the maps

$$MGL_{*,*}MGL \otimes_{MGL_{*,*}} MGL_{*,*}(\mathcal{F}) \to (MGL \wedge MGL \wedge \mathcal{F})_{*,*},$$

$$MGL_*MGL \otimes_{MGL_*} MGL_*(\mathcal{F}) \to (MGL \wedge MGL \wedge \mathcal{F})_*$$

are isomorphisms.

Out of this we get a Hopf algebroid

$$MU \underset{\mathrm{id}\wedge 1}{\overset{1\wedge\mathrm{id}}{\rightrightarrows}} MU \wedge MU,$$

and out of the "1 in the middle" map $MU \wedge MU \to MU \wedge MU \wedge MU$, we get a map

$$MU_*MU \to MU_*MU \otimes_{MU_*} MU_*MU.$$

The same holds for MGL.

We have a notion of MU_*MU**-comodules**, which are MU_*-modules with an action of $\mathrm{Spec}(MU_*MU)$. So we have an action functor

$$\mathrm{Spec}(MU_*MU) \to \mathsf{Mod}\text{-}[MU_*/MU_*MU].$$

Now if N_* is an MU_*-module, we can say that N **is flat over** $[MU_*/MU_*MU]$ if $- \otimes_{MU_*} N_*$ is exact on MU_*MU-comodules.

Definition. For $\phi : MU_* \to R_*$, corresponding to the formal group law (F, R_*), we say this formal group law is **Landweber exact** if R_* is flat over $[MU_*/MU_*MU]$.

We have our formal group law for MGL, (F_{MGL}, MGL_*), which gives a classifying map $\phi : MU_{2*} \to MGL_*$. This makes $(MGL \wedge \mathcal{F})_{*,*}$ into an MU_{2*}-module.

Note.

$$MU_*MU = MU_*[b_1, b_2, \ldots].$$

MU_{2*} are the coefficients for the universal formal group law $F_{MU}(u, v) = u + v + \sum a_{ij}u^iv^j$. Write $\lambda(t) = t + \sum b_it^i$; we can form $\lambda(F(\lambda^{-1}(u), \lambda^{-1}(v)))$. We have

$$MU_*MU \to MGL_*MGL \to \text{isomorphisms of } F_{MGL}.$$

Consider the diagram

$$\begin{array}{ccc} G_{MGL} \times X_{MGL} & \longrightarrow & G_{MU} \times X_{MU} \\ \downarrow & & \downarrow \\ \operatorname{Spec} MGL_* & \longrightarrow & \operatorname{Spec} MU_* \\ \downarrow & & \downarrow \\ [MGL_*/MGL_*MGL] & \longrightarrow & [MU_*/MU_*MU] \end{array}$$

where e.g. $X_{MGL} = \operatorname{Spec} MGL_*$.

Proposition. *This diagram is cartesian:*

$$MGL_*MGL = MU_*MU \otimes_{MU_*} MGL_*,$$

$$G_{MGL} = \operatorname{Spec}(MGL_*MGL),$$

etc.

Proof of theorem part (1). The point is that for any $\mathcal{F} \in \mathcal{SH}(S)$, $MGL_{*,*}(\mathcal{F})$ is a comodule for $MGL_{*,*}MGL$. By flatness, we have

$$\begin{array}{ccc} (MGL \wedge \mathcal{F})_{*,*} & \longrightarrow & (MGL \wedge MGL \wedge \mathcal{F})_{*,*} \\ \| & & \| \\ MGL_{*,*}(\mathcal{F}) & & MGL_*MGL \otimes_{MGL_*} MGL_{*,*}(\mathcal{F}) \end{array}$$

We want to see that

$$\mathcal{F} \mapsto MGL_{*,*}(\mathcal{F}) \otimes_{MU_*} R_*$$

is homological on $\mathcal{SH}(S)$. By Landweber exactness and the cartesian property above, R_* is flat over $[MGL_*/MGL_*MGL]$. □

Proof of theorem part (2). The **Tate subcategory** $\mathcal{SH}(S)_\mathcal{T}$ is the localizing subcategory generated by $S^{a,b}$. Then $MGL \in \mathcal{SH}(S)_\mathcal{T}$; this comes from the Schubert cell decomposition of Grassmannians. Let $\operatorname{proj}_\mathcal{T}$ denote the right adjoint to inclusion $\mathcal{SH}(S)_\mathcal{T} \hookrightarrow \mathcal{SH}(S)$. Then $\operatorname{proj}_\mathcal{T}(E \wedge F) = E \wedge \operatorname{proj}_\mathcal{T}(F)$.

Proposition. *If $E \in \mathcal{SH}(S)_\mathcal{T}$ represents a homology theory on $\mathcal{SH}(S)$ restricted to $\mathcal{SH}(S)_\mathcal{T}$, then E represents the theory on $\mathcal{SH}(S)$, and*

$$E_{*,*}(\mathcal{F}) = E_{*,*}(\operatorname{proj}_\mathcal{T}(\mathcal{F})), \qquad \mathcal{F} \in \mathcal{SH}(S).$$

Proposition. *$E \in \mathcal{SH}(S)_\mathcal{T}$ behaves well under pullback $S' \to S$.*

Theorem. *For $S = \operatorname{Spec}(\mathbb{Z})$, $\mathcal{SH}(S)_\mathcal{T}$ is a Brown category.*

So given (F, R_*), we can represent the restriction of $(MGL/\mathbb{Z})_{*,*}(-) \otimes_{MU_*} R_*$ to $\mathcal{SH}(\mathbb{Z})_\mathcal{T}$ by some $E_\mathbb{Z}$. □