Universality of MGL

Scribe notes from a talk by Ben Knudsen

20 Mar 2014

We will discuss work of Panin, Pimenov, Röndigs, and Smirnov.

Orientability

One major feature of orientability (for manifolds) is the notion of a tangent bundle. This gives us an Euler class and a Thom class, related by the formula $e(M) = i^*(\mathrm{Th}(M))$.

Orientability also gives us Poincaré duality, which gives us interesting wrong-way maps, proper pushforwards:

$$f_* : H^*(M) \cong \tilde{H}_{m-*}(M^+) \to \tilde{H}_{m-*}(N^+) \cong H^{*+n-m}(N).$$

Similarly, we have a notion of E-orientability.

Example. A manifold is KO-orientable iff it is spin.

Question. Which theories orient all manifolds?

Answer (Thom)**.** Basically just $H\mathbb{F}_2$. There is a universal such cohomology theory MO, and

$$MO \simeq \vee_\alpha \Sigma^{i_\alpha} H\mathbb{F}_2.$$

So instead we just ask about certain classes of manifolds:

Example. The universal theory MG orients all manifolds with G-structure. For instance we have MU, the start of chromatic homotopy theory.

So now let k be a field, and E a (homotopy) commutative $\mathbb{P}^1$-spectrum. Write $\mathbb{P}^\infty = \operatorname{colim} \mathbb{P}^n$.

Definition. An **orientation** for E is a class $c \in E^{2,1}(\mathbb{P}^\infty)$ such that

$$c\big|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1).$$

Remark. This is the same as a natural choice of Chern class for line bundles. It's a theorem of Voevodsky, which we haven't proven, that $\mathbb{P}^\infty$ represents the Picard group.

Examples. —

- Betti cohomology (ordinary cohomolgy is complex orientable).
- Motivic cohomology. We have
$$H^{2,1}_{\mathrm{mot}}(X,\mathbb{Z}) \cong \mathrm{CH}^1(X), \qquad L \mapsto \operatorname{div} L.$$
- Algebraic K-theory,
$$L \mapsto \Sigma_{\mathbb{P}^1}([1]-[L]).$$
- Algebraic cobordism $MGL_n = \mathrm{Th}(\gamma_n)$, where $\gamma_n \to Gr_n$ is the tautological bundle. Then we have
$$\Sigma^{\infty-1}_{\mathbb{P}^1}\mathbb{P}^\infty \to \Sigma^{\infty-1}_{\mathbb{P}^1}\mathrm{Th}(\gamma_1) \to MGL.$$

Theorem (Projective bundle theorem)**.** *Let X be a smooth variety, $V \to X$ a rank n vector bundle, and E oriented. Then*
$$E^{*,*}(\mathbb{P}(V)) \cong E^{*,*}(X)[t]/(t^n)$$
as $E^{*,*}(X)$-modules, where $t = c_1(\mathcal{O}_V(-1))$.

Proof. —

- Using Mayer–Vietoris, reduce to $\mathbb{P}^n$.
- The definition of orientation is the case $\mathbb{P}^1$.

□

Remark. —

- We get Chern classes for all bundles: there exist $c_i(V) \in E^{2i,i}(X)$ such that
$$t^n - c_1(V)t^{n-1} + \ldots + (-1)^n c_n(V) = 0.$$
- We get Thom classes as follows.

Construction. Consider the square
$$\begin{array}{ccc} V \smallsetminus X & \longrightarrow & \mathbb{P}(V\oplus 1)\smallsetminus \mathbb{P}(1) \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathbb{P}(V\oplus 1) \end{array}$$

By excision on this square, and applying the purity theorem,
$$E^{*,*}(\mathrm{Th}(V)) \cong E^{*,*}(\mathbb{P}(V\oplus 1), \mathbb{P}(V\oplus 1)\smallsetminus \mathbb{P}(1)),$$
$$\mathrm{Th}(V) \leftarrow\!\shortmid c_n(\mathcal{O}_{V\oplus 1}(1)\otimes \pi^*(V)).$$

Applying the projective bundle theorem, we get a diagram

$$\begin{array}{ccccccc}
 & & E^{*,*}(\mathrm{Th}(V)) & & & & \\
 & & \Vert & & & & \\
0 & \longrightarrow & E^{*,*}(\mathbb{P}(V \oplus 1), \mathbb{P}(V \oplus 1) \smallsetminus \mathbb{P}(1)) & \longrightarrow & E^{*,*}(\mathbb{P}(V \oplus 1))E^{*,*}(\mathbb{P}(V)) & \longrightarrow & 0 \\
 & & {\scriptstyle -\cup c_n(\mathcal{O}(1)\otimes\pi^*(V))}\uparrow & & \uparrow & & {\scriptstyle \sim}\uparrow \\
0 & \longrightarrow & E^{*,*}(X) & \hookrightarrow & E^{*,*}(X) \oplus E^{*,*}(X)^n & \longrightarrow & E^{*,*}(X)^n \longrightarrow 0
\end{array}$$

By Mayer–Vietoris, it suffices to check the case where V is trivial. Then

$$c_n(\mathcal{O}(1) \otimes \pi^*(V)) = c_n(\mathcal{O}(1)^n) = \bar{t}^n.$$

So we get a Thom isomorphism!

We get pushforwards for projective maps. Recall that $f : X \to Y$ is **projective** if it factors as:

$$\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
 & \searrow \quad \nearrow^{p} & \\
 & Y \times \mathbb{P}^N &
\end{array}$$

with i a closed immersion. We will construct pushforwards for i and p, and then show that choices don't matter.

For $i : X \hookrightarrow Y$, we define the pushforward by

$$\begin{array}{ccc}
E^{*,*}(X) & \overset{\mathrm{Thom}}{=\!=\!=} & E^{*+2k,*+k}(\mathrm{Th}(N_{X/Y})) \\
 & & \Vert\,{\scriptstyle \mathrm{purity}} \\
 & & E^{*+2k,*+k}(Y, Y \smallsetminus X) \\
 & {\scriptstyle i_*}\searrow & \downarrow \\
 & & E^{*+2k,*+k}(Y)
\end{array}$$

Theorem (Panin–Pimenov–Röndigs)**.**

$$\{\textit{orientations of } E\} \cong [MGL, E]_{ring}.$$

Proof. To go one way, we assign

$$\phi \circ \mathrm{th}^{MGL} \leftarrow\!\shortmid \phi.$$

We have $MGL_n = \mathrm{Th}(\gamma_n)$.

Note that $\operatorname{hocolim} \mathrm{Th}(\gamma_n) = MGL$. So we have a short exact sequence

$$0 \to \lim{}^1 E^{*+,n-1,*+n}(\mathrm{Th}(\gamma_n)) \to E^{*,*}(MGL) \to \lim E^{*+2n,*+n}(\mathrm{Th}(\gamma_n)) \to 0$$

Claim. This $\lim^1$ term vanishes.

Assuming this:

$$[MGL, E] = E^{0,0}(MGL) = \lim E^{2n,n}(\operatorname{Th}(\gamma_n)).$$

So $\{\operatorname{th}(\gamma_n)\}$ defines a map $\phi : MGL \to E$.

$$\begin{array}{ccc}
\Sigma^{\infty-n}_{\mathbb{P}^1}\operatorname{Th}(\gamma_n) \wedge \Sigma^{\infty-n}_{\mathbb{P}^1}\operatorname{Th}(\gamma_n) & \longrightarrow & \Sigma^{\infty-2n}_{\mathbb{P}^1}\operatorname{Th}(\gamma_{2n}) \\
\downarrow & & \\
MGL \wedge MGL & \longrightarrow & E \wedge E \\
\downarrow & & \downarrow \\
MGL & \longrightarrow & E
\end{array}$$

So these classes are multiplicative.

Theorem.
$$E^{*,*}(\operatorname{Gr}_n) \cong E^{*,*}(k)[\![c_1, \ldots, c_n]\!],$$
where $c_i = c_i(\gamma_n)$.

Proof of claim. It suffices to show that

$$E^{*+2n,*+n}(\operatorname{Th}(\gamma_n)) \to E^{*+2n-2,*+n-1}$$

is surjective. By the Thom isomorphism, it suffices that $\operatorname{Gr}_{n-1} \hookrightarrow \operatorname{Gr}_n$ induces a surjection. □

□

Proof of theorem. Let $\operatorname{Fl}_n(m)$ be the flag variety of flags

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n, \qquad \dim V_i = i,$$

and let $\operatorname{Fl}_n = \operatorname{colim} \operatorname{Fl}_n(m)$. We have a map $\pi : \operatorname{Fl}_n \to \operatorname{Gr}_n$. There is a filtration

$$0 = \gamma_n^0 \subset \gamma_n^1 \subset \ldots \subset \gamma_n^n = \pi^*(\gamma_n),$$

with associated graded $L_n^i = \gamma_n^i / \gamma_n^{i-1}$.

Now,
$$E^{*,*}(\operatorname{Fl}_n) \cong E^{*,*}(k)[\![t_1, \ldots, t_n]\!], \qquad t_i = c_1(L_n^i).$$

We proceed by induction. For $n = 1$, we have $\operatorname{Fl}_1 = \mathbb{P}^\infty$. Apply the projective bundle theorem. For the inductive step, there's a map $\operatorname{Fl}_n \to \operatorname{Fl}_{n-1}$ which is a projective bundle with tautological bundle L_n^n. Now:

- π^* is injective, as $\operatorname{Fl}_n \to \operatorname{Gr}_n$ decomposes as a sequence of projective bundles;
- $\pi^*(c_i) = \sigma_i(t_1, \ldots, t_n)$, by the Whitney sum formula;

- $\operatorname{im}\pi^* = E^{*,*}(\mathrm{Fl}_n)^{\Sigma_n}$, by replacing Fl_n by M_n, which parameterizes flags together with a splitting. Then $M_n \to \mathrm{Gr}_n$ is Σ_n-equivariant and factors through Fl_n, and this map $M_n \to \mathrm{Fl}_n$ is an $\mathbb{A}^1$-homotopy equivalence because it is a sequence of projections of vector bundles.

□

Note that we have

$$E^{*,*}(\mathbb{P}^\infty) \cong E^{*,*}(k)[\![t]\!],$$

$$E^{*,*}(\mathbb{P}^\infty) \cong E^{*,*}(k)[\![x,y]\!].$$

Now the classifying map $\mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$ for the tensor product of line bundles induces

$$F_E(x,y) \hookleftarrow t,$$

with $F_E(x,y)$ a formal group law.