Introduction to algebraic cobordism and oriented theories

Scribe notes from a talk by Adeel Khan

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Notation. —

- Write $\mathsf{Spc}(S)$ for the $\mathbb{A}^1$-local Nisnevich-local projective schemes,
- write $\mathsf{Spt}_{S^1}(S)$ for the symmetric S^1-spectra in pointed motivic spaces, with the stable projective model structure induced from the above,
- write T for the Tate sphere $S^1 \wedge \mathbb{G}_m$,
- write $\mathsf{Spt}_T(S)$ for the symmetric T-spectra,
- $H(S) = \mathrm{Ho}(\mathsf{Spc}(S))$,
- $\mathcal{SH}_{S^1}(S) = \mathrm{Ho}(\mathsf{Spt}_{S^1}(S))$,
- $\mathcal{SH}(S) = \mathrm{Ho}(\mathsf{Spt}_T(S))$.

Definition. In $\mathcal{SH}(S)$, define **simplicial spheres**

$$\mathbb{S}^n = \Sigma^\infty_T(S^n),$$

$$\mathbb{S}^{-n} = (\underbrace{*,\ \dots,\ *}_{n \text{ copies}},\ \mathbb{G}_m^{\wedge n},\ \mathbb{G}_m^{\wedge n} \wedge T,\ \mathbb{G}_m^n \wedge T^{\wedge 2},\ \dots),$$

and define **algebraic spheres**

$$\mathbb{G}_m^n = \Sigma^\infty_T(\mathbb{G}_m^{\wedge n}),$$

$$\mathbb{G}_m^{-n} = (\,*,\ \dots,\ *,\ S^n,\ S^n \wedge T,\ \dots).$$

Remark. $\mathbb{S}^n \wedge \mathbb{S}^{-n} \simeq \mathbf{1}$ in $\mathcal{SH}(S)$, and $\mathbb{G}_m^n \wedge \mathbb{G}_m^{-n} \simeq \mathbf{1} = \Sigma^\infty_T(S_+)$.

Definition. We let $\mathbb{S}^{p,q} = \mathbb{S}^{p-q} \wedge \mathbb{G}_m^q$ in $\mathcal{SH}(S)$, and define the functor

$$\Sigma^{p,q} = \mathbb{S}^{p,q} \wedge - \quad : \quad \mathcal{SH}(S) \to \mathcal{SH}(S).$$

Definition. For $E \in \mathsf{Spt}_T(S)$, we define the associated **cohomology theory**, a presheaf:

$$E^{p,q} = H_E^{p,q} : \mathsf{Spc}(S) \to \mathsf{Spc}_\bullet(S) \to \mathsf{Ab},$$
$$X \mapsto X_+ \mapsto \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma_T^\infty(X_+), \Sigma^{p,q}(E)),$$

the **homology theory**

$$E_{p,q} = H_{p,q}^E : X \mapsto \mathrm{Hom}_{\mathcal{SH}(S)}(\mathbb{S}^{p,q}, \Sigma^\infty(X_+) \wedge E),$$

and the **homotopy sheaf** $\pi_{p,q}(E)$ as the Nisnevich sheafification of

$$X \mapsto E^{-p,-q}(X).$$

Proposition (Voevodsky)**.** *The triangulated category $\mathcal{SH}(S)$ is compactly generated by the objects $\Sigma^{\infty-n}(X_+)$, for $X \in \mathrm{Sm}_{/S}$ and $n \geq 0$.*

Definition. We define the category of **effective motivic spectra**

$$\mathcal{SH}^{\mathrm{eff}}(S) = \langle \Sigma_T^{\infty-n}(\Sigma_T^{p,q}(X_+)) \rangle^\oplus,$$

taken over $X \in \mathrm{Sm}_{/S}$, $n \geq 0$, and $p \geq q \geq n$. These angle brackets denote the full triangulated subcategory generated by these objects under coproducts.

Definition. For $k \geq 0$, define

$$\Sigma^k \mathcal{SH}^{\mathrm{eff}}(S) = \langle \Sigma^{\infty-n}(\Sigma_T^{p,q}(X_+)) \rangle^\oplus$$

taken over $X \in \mathrm{Sm}_{/S}$, $n \geq 0$, $p \geq q \geq n + k$. Then the filtration

$$\ldots \subseteq \Sigma^{k+1} \mathcal{SH}^{\mathrm{eff}}(S) \subseteq \Sigma^k \mathcal{SH}^{\mathrm{eff}}(S) \subseteq \ldots \subseteq \mathcal{SH}^{\mathrm{eff}}(S) \subseteq \mathcal{SH}(S)$$

is the **slice filtration** on S.

Proposition (Voevodsky)**.** *Each of these inclusions $i_k : \Sigma^k \mathcal{SH}^{\mathrm{eff}}(S) \hookrightarrow \mathcal{SH}(S)$ is a coreflective subcategory, and admits a right adjoint $r_k : \mathcal{SH}(S) \to \Sigma^k \mathcal{SH}^{\mathrm{eff}}(S)$.*

Definition. The map $f_k = i_k \circ r_k : \mathcal{SH}(S) \to \mathcal{SH}(S)$ is the k**th truncation functor**.

Remark. —

- The unit $\mathrm{id} \xrightarrow{\sim} r_k \circ i_k$ is invertible.
- The counits $f_k \to \mathrm{id}$ induce $f_k = f_k \circ f_{k-1} \to f_{k-1}$.

Definition. For $k \geq 0$, and $E \in \mathsf{Spt}_T(S)$, we define

$$s_k(E) = \mathrm{cofib}(f_{k+1}(E) \to f_k(E)),$$

called the **slices** of the filtration. This induces a triangulated functor $s_k : \mathcal{SH}(S) \to \mathcal{SH}(S)$.

Definition. The cofiber sequences

$$f_{k+1}(E) \to f_k(E) \to s_k(E) \to$$

form the layers of the **slice tower** of E

$$\ldots \to f_{k+1}(E) \to f_k(E) \to f_{k-1}(E) \to \ldots$$

Definition. We define $\mathrm{Sm}^{\mathrm{corr}}(S)$ as the category with

- Objects as the objects of $\mathrm{Sm}_{/S}$,
- $\mathrm{Hom}(X, Y)$ as the free abelian group on closed integral immersions $Z \not\hookrightarrow X \times_S X$ such that $Z \to X$ is finite and equidimensional.
- We define a composition $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ by taking the pullback to $X \times_S Y \times_S Z$ and pushing down to $X \times_S Z$.

By taking graphs, we obtain a functor $\gamma : \mathrm{Sm}_{/S} \to \mathrm{Sm}^{\mathrm{corr}}(S)$.

Definition. Define the **spaces with transfer** $\mathsf{Spc}_{\mathrm{tr}}(S) = \mathsf{PSh}_{\Delta}(\mathrm{Sm}^{\mathrm{corr}}(S))$.

Lemma. *We have an adjunction*

$$\mathbb{Z}_{\mathrm{tr}} : \mathsf{Spc}_*(S) \rightleftarrows \mathsf{Spc}_{\mathrm{tr}}(S) : u_{\mathrm{tr}},$$

where u_{tr} is restriction along 0, and $\mathbb{Z}_{\mathrm{tr}}$ maps X_+ to the presheaf represented by X and preserves colimits.

Motivic cohomology

Definition. Note that we have an induced adjunction

$$\mathbb{Z}_{\mathrm{tr}} : \mathsf{Spt}_T(S) \rightleftarrows \mathsf{Spt}_T^{\mathrm{tr}}(S) : u_{\mathrm{tr}}.$$

We define the **Eilenberg–Mac Lane spectrum**

$$H_{\mathbb{Z}} = \mathbb{R}u_{\mathrm{tr}}(\mathbb{L}\mathbb{Z}_{\mathrm{tr}}(\mathbf{1})) \quad \in \mathsf{Spt}_T(S).$$

Theorem (Voevodsky)**.** *This spectrum represents motivic cohomology:*

$$H_{\mathbb{Z}}^{p,q} \simeq H^{p,q}(-, \mathbb{Z}).$$

Theorem (Levine)**.** *The slices are $s_0(H_{\mathbb{Z}}) = H_{\mathbb{Z}}$ and $s_k(H_{\mathbb{Z}}) = 0$ for $k > 0$.*

Cobordism

Definition. Let X be a scheme, E a vector bundle on X, and $s : X \to E$ the zero section. We define the **Thom space**

$$\mathrm{Th}(E) = \mathrm{cofib}(E\ s(X) \hookrightarrow E).$$

This is functorial on a category of pairs (X, E), where a morphism $(X, E) \to (Y, F)$ consists of morphisms $f : X \to Y$ and $f^*F \twoheadrightarrow E$. In particular, given (E, X) and a morphism $f : Y \to X$, we obtain a map $\mathrm{Th}(f^*(E)) \to \mathrm{Th}(E)$. So from the diagram

$$\begin{array}{ccc} E(m,n) & \longrightarrow & \mathrm{Gr}(m,n) \\ \big\downarrow & & \big\downarrow \\ E(m,n+1) & \longrightarrow & \mathrm{Gr}(m,n+1) \end{array}$$

one gets an induced morphism

$$\mathrm{Th}(E(m,n)) \to \mathrm{Th}(E(m,n+1)).$$

Definition. We define

$$MGL_n = \operatorname*{colim}_{n \geq m} \mathrm{Th}(E(m,n)).$$

The morphism

$$i : \mathrm{Gr}(m,n) \hookrightarrow \mathrm{Gr}(m+1,n+1), \qquad A \mapsto A \times \mathbb{A}^1$$

induces

$$i^*(E(m+1,n+1)) \simeq \mathbf{1} \oplus E(m,n),$$
$$T \wedge \mathrm{Th}(E(m,n)) \to \mathrm{Th}(E(m+1,n+1)),$$
$$T \wedge MGL_m \to MGL_{m+1}.$$

These form the structure maps for a T-spectrum MGL, the **algebraic cobordism spectrum**.

Theorem (Levine)**.** $MGL^{2n,n}(X) \simeq \Omega^n(X)$.

Theorem (Voevodsky)**.** $s_0(MGL) = H_{\mathbb{Z}}$ *in the regular equi-characteristic case.*