Endomorphisms of the sphere spectrum

Scribe notes from a talk by David Yang

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We will work over a perfect field k of characteristic $\neq 2$.

Goal. Compute $\pi_0^s(S_\bullet)$, which will be the Grothendieck–Witt ring. To rephrase this, we will compute $[S_\bullet, \mathbb{G}_m^{\wedge i}]$.

Transfer maps

Let C be a curve over k, and S a set of points in C. Let U be the complement of S; then we have a distinguished triangle

$$U \to C \to \coprod_{s \in S} \mathbb{P}^1 \wedge k(s).$$

If $\mathcal{F}$ is a strictly $\mathbb{A}^1$-invariant sheaf, we get a long exact sequence

$$\ldots \to \mathrm{Hom}(\coprod \mathbb{P}^1 \wedge k(s), H\mathcal{F}) \to \mathrm{Hom}(C, H\mathcal{F}) \to \ldots$$

This left term is $\mathrm{Hom}(\coprod S^1 \wedge \mathbb{G}_m \wedge k(s), H\mathcal{F})$. Taking a colimit over all U, this becomes

$$0 \to H^0(C, \mathcal{F}) \to H^0(k(C), \mathcal{F}) \to \coprod_{s \in C} H^0(k(s), \mathcal{F}_{-1}) \to H^1(C, \mathcal{F}) \to 0,$$

the last term being zero as fields have no nontrivial Nisnevich covers. The second nontrivial map in this sequence is the sum of all of the **transfer maps**

$$H^0(k(C), \mathcal{F}) \to H^0(k(s), \mathcal{F}_{-1}).$$

This goes forward for an arbitrary smooth scheme X:

$$0 \to H^0(X, \mathcal{F}) \to H^0(k(X), \mathcal{F}) \xrightarrow{\oplus \partial} \coprod_{x \in X^{(1)}} H^0(k(x), \mathcal{F})$$

where $X^{(1)}$ denotes the codimension-1 subschemes of X.

Definition. Define the **Milnor K-theory** of a field F by $K_*^M(F) = \mathrm{Tens}_{\mathbb{Z}}(F^\times)/I$, where I is generated by the elements $a \otimes (1-a)$.

We define transfer maps in Milnor K-theory as follows. For $x \in X^{(1)}$, write $(\pi) = \mathfrak{m}_x \in \mathcal{O}_{X,x}$; then we have

$$\partial_x(\{\pi, a_2, \dots, a_n\}) = \{\bar{a}_2, \dots, \bar{a}_n\} \in K^M_{n-1}(k(x)),$$

with $a_i \in \mathcal{O}^x_{X,x}$ yielding $\bar{a}_i \in k(x)$. For $n = 1$ we have $K_1(k(X)) = k(X)^\times$.

$$\mathbb{Z} \ni \partial_x(f) = \mathrm{div}_x(f) = \mathrm{ord}_x(f).$$

Quadratic forms

Write $\langle a_1, a_2, \dots, a_n \rangle$ for a diagonal quadratic form.

Definition.

- The **Grothendieck–Witt group** $GW(k)$ is the Grothendieck group of isomorphism classes of quadratic forms over k; this is a ring under tensor product.
- The **Witt group** $W(k)$ is the quotient of the Grothendieck group by the "hyperbolic plane" $\langle 1, -1 \rangle$.

The Witt ring has an ideal I of quadratic forms of even rank. We now motivate the Steinberg relation. Define $\langle\langle a \rangle\rangle = \langle 1, -a \rangle$. Then

$$\begin{aligned} \langle\langle a \rangle\rangle \langle\langle 1-a \rangle\rangle &= \langle\langle 1, -a \rangle\rangle \langle\langle 1, a-1 \rangle\rangle \\ &= \langle\langle 1, -a, a-1, -a(a-1) \rangle\rangle \\ &= h \oplus h = 0. \end{aligned}$$

Theorem (Milnor conjecture)**.**

$$\frac{I^n}{I^{n+1}} = K^M_n(F)/2.$$

K^M_n and W are known to be strictly $\mathbb{A}^1$-invariant sheaves. The ideal I^n is also strictly $\mathbb{A}^1$-invariant, as we have this exact sequence from the Milnor conjecture

$$0 \to I^{n+1} \to I^n \to K^M_n \to 0,$$

and by induction we deduce this from the $\mathbb{A}^1$-invariance of K^M_n and W.

Recall that we're trying to compute $[S_0, \mathbb{G}_m^{\wedge i}]$. Of course, $S_0 = (\mathrm{Spec}\, k)_+$. For every element $a \in k^\times$, we get a map $[a] : \mathrm{Spec}\, k \to \mathbb{G}_m$. We have the Hopf map

$$\eta : \mathbb{A}^2 \smallsetminus \{0\} \to \mathbb{P}^1, \qquad (x, y) \mapsto (x : y),$$

and $\mathbb{A}^2 \smallsetminus \{0\} \simeq S^1 \wedge \mathbb{G}_m \wedge \mathbb{G}_m$, as we see from the elementary Nisnevich square

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^2 \smallsetminus \{0\} \end{array}$$

Let $< a >= 1 + \eta[a]$. Then

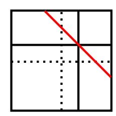
(1) $[a][1-a] = 0$,

(2) $< a >< b >=< ab >$,

(3) $\eta < a >=< a > \eta$,

(4) $\eta h = 0$.

An ingredient here is the map $\mathbb{A}^1 \smallsetminus \{0,1\} \to \mathbb{G}_m \wedge \mathbb{G}_m$ given as follows. We view $\mathbb{G}_m \wedge \mathbb{G}_m$ as the affine plane with the coordinate axes removed and the lines $x = 1$, $y = 1$ collapsed to the basepoint. We now include $\mathbb{A}^1 \smallsetminus \{0,1\}$ as the line $x + y = 1$.



Definition. Define **Milnor–Witt K-theory** $K_\bullet^{MW}$ as the ring generated by $[a]$ and η, quotiented by relations (1) to (4).

Theorem.

$$K_*^{MW} \cong \oplus_{i\in\mathbb{Z}}[S_0, \mathbb{G}_m^{\wedge i}].$$

Note that $K_0^{MW} \cong GW(k)$, and $K_{-n}^{MW} \cong W(k)$.
Define the homotopy module $J^n = I^n \times_{I^n/I^{n+1}} K_n^M$.
Note that when we quotient Milnor–Witt K-theory by the relation $\eta = 0$, we recover Milnor K-theory.

Definition. Define **Witt K-theory** K_n^W as the quotient of Milnor–Witt K-theory by the relation $h = 0$.

Theorem.

$$K_n^W = \begin{cases} W & n < 0 \\ I^n & n \geq 1 \end{cases}$$

We now have a commutative diagram

$$\begin{array}{ccccccc} & & K_n^W & \longrightarrow & K_n^M W & \longrightarrow & K_n^M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{n+1} & \longrightarrow & J^n & \longrightarrow & K_n^M & & \end{array}$$

Theorem. *We have*

$$K_n^W \cong I \otimes_W \ldots \otimes_W I / \text{ ideal generated by } a \otimes (1-a).$$

We effectively have to show the relation $\langle\langle a^2 \rangle\rangle = 0$ using only the Steinberg relation; equivalently, we show $\langle a \rangle = 1$. We do this by rewriting this as

$$0 = 1 - \langle a \rangle^2 = (1 - \langle a \rangle)(1 - \langle -a \rangle)$$

and noting that

$$-a = \frac{1-a}{1-a^{-1}}.$$

We have also

$$I \otimes_W \ldots \otimes_W I/\text{ideal} \cong I^n.$$

This is harder, requiring the Milnor conjecture. We can show

(0) If $\langle\langle a_1, \ldots, a_n \rangle\rangle = 0$ in W, then $\langle\langle a_1, \ldots, a_n \rangle\rangle = 0$.

(1) $\langle\langle a_1, \ldots, a_n \rangle\rangle + \langle\langle b_1, \ldots, b_n \rangle\rangle = \langle\langle ab, \ldots \rangle\rangle + \langle\langle ab(a+b), \ldots \rangle\rangle$.

(2) $\langle\langle ab, c, \ldots \rangle\rangle + \langle\langle a, b, \ldots \rangle\rangle = \langle\langle ac, b, \ldots \rangle\rangle + \langle\langle a, b, \ldots \rangle\rangle$.

Definition (Pfister Chain Equivalence)**.** We take two elements $\langle\langle a_1, \ldots, a_n \rangle\rangle$ and $\langle\langle a_1', \ldots, a_n' \rangle\rangle$ to be **simply equivalent** if they're the same in all but two terms, and ???.

[The note-taker was unable to follow the last few minutes of the lecture.]