Morel's $\mathbb{A}^1$-connectivity theorem and homotopy t-structures

Scribe notes from a talk by Florian Strunk

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Starting with a triangulated category $\mathcal{D}$, we can form an abelian category, so long as we have equipped $\mathcal{D}$ also with a t-structure. This process is called taking the "heart". A good source of triangulated categories are the pointed simplicial stable model categories, by taking the homotopy category.

Definition. —

- A (homological) **t-structure** is a pair of strictly full subcategories $\mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq -1}$, such that
  - (T1) For all $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq -1}$, $[X, Y] = 0$;
  - (T2) $\mathcal{D}_{\geq 0}$ is closed under the shift [1], and $\mathcal{D}_{\leq -1}$ is closed under the shift $[-1]$;
    (now set $\mathcal{D}_{\geq n} = \mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n-1} = \mathcal{D}_{\leq -1}[n]$);
  - (T3)
- The subcategory $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$ is called the **heart**, and is an abelian category.
- The t-structure is **non-degenerate** if $\cap_{n\geq 0}\mathcal{D}_{\geq m} = \{0\}$ and $\cap_{m\geq 0}\mathcal{D}_{\leq -m} = \{0\}$.

Example. We can take $\mathcal{D} = \mathrm{Ho}(\mathsf{Sp})$. Take

$$\mathcal{D}_{\geq n} = \{X \mid \pi_i^s(X) = 0, i < m\}, \qquad \mathcal{D}_{\leq n} = \{X \mid \pi_i^s X = 0, i > m\}.$$

There exists an adjunction

$$\mathcal{D}_{\geq m} \rightleftarrows \mathcal{D} : \tau_{\geq m}, \ , \qquad X_{\geq m} \mapsto X,$$

$$\tau_{\leq m} : \mathcal{D} \rightleftarrows \mathcal{D}_{\leq m}. \ , \qquad X \leftarrow\!\shortmid X_{\leq m}.$$

For $m \geq n$, $X_{geqn\geq m} \cong X_{\geq m}$, and $X_{\leq n \leq m} \cong X_{\leq n}$.

Let $\mathcal{D} = \operatorname{Ho}\mathcal{C}$. Then

$$\begin{array}{ccccccc}
\operatorname{holim}_{n\to\infty} X_{\geq n} & \cdots & X_{\geq n+1} & \longrightarrow & X_{\geq n} & \cdots & \operatorname{hocolim} X_{\geq n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \cdots & X & = & X & \cdots & X \\
\downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{holim}_{n\to\infty} X_{\leq n} & \cdots & X_{\leq n} & \longrightarrow & X_{\leq n-1} & \cdots & \operatorname{hocolim} X_{\leq n}
\end{array}$$

The equivalence in the lower left expresses that Postnikov towers converge, the following lemma:

Lemma. *If the t-structure is non-degenerate and the sequential homotopy limit of objects in $\mathcal{D}_{\geq 0}$ is in $\mathcal{D}_{\geq -1}$ (e.g. $\mathcal{D}_{\geq 0}$ is stable under $\prod$), then*

$$\tau_{\leq m} X \xrightarrow{\sim} \tau_{\leq n} \operatorname{holim} X_{\leq m}.$$

How to get a t-structure

Let $\mathcal{C}$ be an (I, J)-cofibrantly generated, pointed, simplicial, stable, and finitely generated model category. According to Hovey,

$$\cup_{n\geq 0} \operatorname{cofib}(I)[n]$$

detects weak equivalences, and each object of it is small ($[-, \coprod A_i] \cong \coprod[-, A_i]$).

Lemma. *Let J consist of small objects. Then taking*

$$\mathcal{D}_{\leq -1} = \{Y \in \mathcal{D} \mid [S[n], Y] = 0 \textit{ for all } n \geq 0, S \in J\},$$

$$J = \mathcal{D}_{\geq 0} = \{X \in \mathcal{D} \mid [X, Y] = 0 \textit{ for all } Y \in \mathcal{D}_{\leq -1}\}$$

is a t-structure.

Proof. (T3) Consider

$$X \to Y \to Z,$$

with $X \in \mathcal{D}_{\geq 0}$ and $Z \in \mathcal{D}_{\leq -1}$.

$$\coprod_{\substack{n\geq 0 \\ S\in J \\ [S[n],Y]}} S[n] \to Y \xrightarrow{d_1} \Phi^1 Y$$

Inducting, we produce

$$\begin{array}{ccccc}
F_0 = 0 & \longrightarrow & Y & = & Y \\
\downarrow & & \| & & \downarrow \\
F_1 & \longrightarrow & Y & \longrightarrow & \Phi^1 Y \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{hocolim} F_k = X & \longrightarrow & Y & \longrightarrow & \operatorname{hocolim} \Phi^k Y
\end{array}$$

□

We now apply this to motivic homotopy. Let $\mathcal{SH}^s_{S^1}$ be the stable category with only the Nisnevich topology localized, not $\mathbb{A}^1$, and with only S^1 inverted.

- On $\mathcal{SH}^s_{S^1}$, we obtain the **simplicial t-structure**, giving the heart $\mathsf{Shv}(\mathrm{Sm}_{/k})$.
- On $\mathcal{SH}_{S^1}$, we obtain the S^1 **homotopy t-structure**, giving the heart $\mathsf{Shv}_{st}(\mathrm{Sm}_{/k})$.
- On $\mathcal{SH}$, we obtain the $\mathbb{P}^1$ **homotopy t-structure**, giving as heart the π_* homotopy modules.

For the first, the weak equivalences are maps that induce isomorphisms on all homotopy sheaves

$$\pi_n(E) = \operatorname*{colim}_{k\to\infty} a_{\mathrm{Nis}} \pi_{n+k}(E_k(-)),$$

where a_{Nis} is the sheafification. Our suspension is $\Sigma(-) = S^1 \wedge -$. We take $J = \{\Sigma^\infty_{S^1} U_+ \mid U \in \mathrm{Sm}_{/k}\}$.

Lemma.

$$\mathcal{D}_{\leq -1} = \{\textit{abelian no-positive-homotopy sheaves}\}, \qquad \mathcal{D}_{\geq 0} = \{\textit{abelian no-negative-homotopy sheaves}\}.$$

Theorem (Kato, Saito)**.** *Let $\mathcal{G} \in \mathsf{Shv}$, X a noetherian scheme of Krull dimension d. Then*

$$0 = H^n_{\mathrm{Nis}}(X; \mathcal{G}) = [\Sigma^\infty_{S^1} X_+, H\mathcal{G}[n]], \qquad n > d.$$

Here $H\mathcal{G}$ is the Eilenberg–Mac Lane spectrum. We then have an equivalence

$$\mathrm{Heart} \underset{\pi_0}{\overset{H}{\rightleftarrows}} \mathsf{Shv}(\mathrm{Sm}_{/k}).$$

$$\begin{aligned}\mathrm{Pic}(X) &= H^1_{\mathrm{Zar}}(X;\mathbb{G}_m)\\ &= H^1_{\mathrm{Nis}}(X;\mathbb{G}_m)\\ &= [X_+, H\mathbb{G}_m[+1]]\\ &= [S^{-1}\wedge X_+, H\mathbb{G}_m]\end{aligned}$$

Let $J = \{\Sigma^\infty_{S^1} U_+ \mid U \in \mathrm{Sm}_{/k}\}$.

Theorem (Morel's stable connectivity theorem)**.** *If $E \in \mathcal{D}_{\geq 0}$, then E is $\mathbb{A}^1$-connective:*

$$a_{\mathrm{Nis}}[S^n \wedge (-)_+, L^\infty E]_{\mathcal{SH}^s_{S^1}} = 0$$

for $n < 0$.

Fact. Let $E \in \mathsf{Sp}^s_{S^1}$ and $X \in \mathrm{Sm}_{/k}$ of Krull dimension d. Then $[X_+, E_{\geq n}] = 0$ for $n > d$.

The construction is roughly as follows. We have the evaluation map

$$\begin{array}{ccccc} \hom_\bullet(\mathbb{A}^1, E) & \xrightarrow{ev_1} & E & & \\ \downarrow & & \downarrow & & \\ \bullet & \longrightarrow & \mathrm{cof} & \xrightarrow{R(-)} & L^1 E \end{array}$$

and the upper-left is $\mathbb{A}^1$-contractible.

So

$$\mathcal{D}_{\geq 0} = \{E \mid \phi_{n,m}E = 0, n < 0, \forall m\},$$
$$\mathcal{D}_{\leq -1} = \{\ldots\},$$

where we follow the convention

$$\pi_{n,m} = \operatorname*{colim}_k a_{\mathrm{Nis}}[\underbrace{S^{n+k}\wedge \mathbb{G}_m^{k+m}}_{S^{n+2k+m,k+m}}, E_k]_{\mathbb{A}^1}.$$

Definition. A **homotopy module** $(M_\bullet, \mu_\bullet)$ is a $\mathbb{Z}$-graded strictly $\mathbb{A}^1$-invariant sheaf, together with

$$\begin{aligned}\mu_m : M_m &\to (M_{m+1})_{-}1\\ &= \ker(M_{m+1}(\mathbb{G}_m \times -) \to M_{m+1}(-))\\ &= \hom_\bullet(\mathbb{G}_m, M_{m+1})\end{aligned}$$

By saying that $\mathcal{G}$ is **strictly $\mathbb{A}^1$-invariant**, we mean that

$$H^n_{\mathrm{Nis}}(X\times \mathbb{A}^1, \mathcal{G}) \cong H^n_{\mathrm{Nis}}(X;\mathcal{G})$$

for all n.