The purity theorem and consequences

Scribe notes from a talk by Marc Hoyois

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The bulk of the talk will prove the following Purity Theorem, due to Morel–Voevodsky, and at the end we will give an application.

Theorem. *Let S be a base scheme, $X \in \mathrm{Sm}_{/S}$. Let $Z \subset X$ be a closed subscheme. Then there is an equivalence*

$$\frac{X}{X \smallsetminus Z} \simeq \frac{N_{X,Z}}{N_{X,Z} \smallsetminus Z}$$

in $\mathcal{H}_*(Z)$, where $N_{X,Z}$ is the normal bundle of Z in X.

Let's assume first that X and Z are not smooth schemes but rather smooth manifolds. Then we have the tubular neighborhood theorem, saying that $N_{X,Z}$ can be embedded as an open subset of X.

$$\begin{array}{ccccc}
N_{X,Z} \smallsetminus Z & \hookrightarrow & N_{X,Z} & \longrightarrow & \frac{N_{X,Z}}{N_{X,Z} \smallsetminus Z} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \wr \\
Z & \hookrightarrow & X & \longrightarrow & \frac{X}{X \smallsetminus Z}
\end{array}$$

From this we get the Gysin long exact sequences.

So we can interpret the purity theorem as an analogue in the motivic homotopy category. It will use more or less everything in the definition of the motivic category.

Deformation to the normal bundle

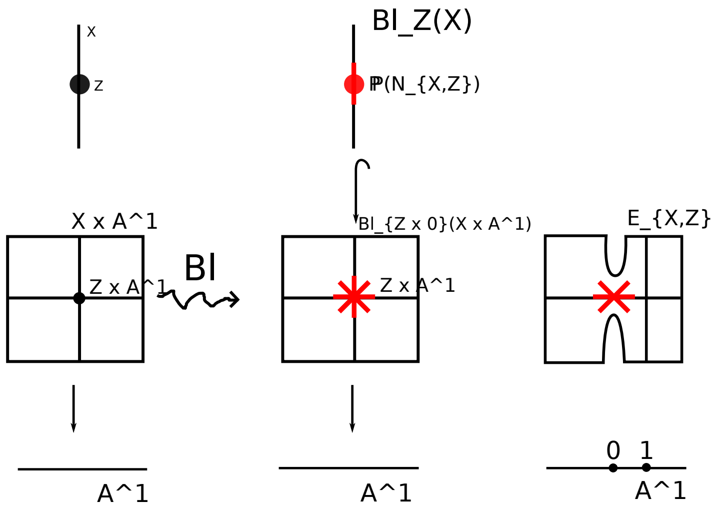
This geometric ingredient will be a substitute for the tubular neighborhood theorem. Given a closed immersion $Z \hookrightarrow X$, we will construct another smooth closed pair $Z \times \mathbb{A}^1 \hookrightarrow E_{X,Z}$ over $S \times \mathbb{A}^1$, such that the fiber over 0 is $Z \hookrightarrow N_{X,Z}$ and the fiber everywhere else is the inclusion $Z \hookrightarrow X$.

Definition. We let

$$E_{X,Z} = \mathrm{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1) \smallsetminus \mathrm{Bl}_{Z \times \{0\}}(X \times \{0\}),$$

where Bl denotes the blow-up.

This is illustrated below:



$$
\begin{array}{ccc}
Z & \hookrightarrow & X \\
& \searrow & \downarrow \\
& & S
\end{array}
\quad \overset{i_1}{\hookrightarrow} \quad
\begin{array}{ccc}
Z \times \mathbb{A}^1 & \hookrightarrow & E_{X,Z} \\
& \searrow & \downarrow \\
& & S \times \mathbb{A}^1
\end{array}
\quad \overset{i_0}{\hookleftarrow} \quad
\begin{array}{ccc}
Z & \hookrightarrow & N_{X,Z} \\
& \searrow & \downarrow \\
& & S
\end{array}
$$

So we have maps

$$\frac{X}{X \smallsetminus Z} \xrightarrow{i_1} \frac{E_{X,Z}}{E_{X,Z} \smallsetminus (Z \times \mathbb{A}^1)} \xleftarrow{i_0} \frac{N_{X,Z}}{N_{X,Z} \smallsetminus Z}$$

We will show that i_0 and i_1 are equivalences in $\mathcal{H}_*(S)$.

The vector bundle case

$Z \hookrightarrow X = V$ is the zero section of a vector bundle. We have $V = N_{V,Z}$ and $V \times \mathbb{A}^1 = N_{V \times \mathbb{A}^1, Z}$.

Then

- $\mathrm{Bl}_{Z \times \{0\}}$ is the tautological bundle on $\mathbb{P}(V \times \mathbb{A}^1)$,
- $E_{V,Z}$ is the restriction of that tautological bundle to $\mathbb{P}(V \times \mathbb{A}^1) \smallsetminus \mathbb{P}(V) \cong V$.
- $i_0 : N_{V,Z} = V \hookrightarrow E_{V,Z}$ is the zero section,
- $i_1 : V \hookrightarrow E_{V,Z}$ is the 1-section.
- These latter two are $\mathbb{A}^1$ homotopy equivalences.

This proves the purity theorem in this case.

Functoriality

Let $\begin{array}{ccc} W & \hookrightarrow & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \hookrightarrow & X \end{array}$ be a cartesian square in $\mathrm{Sm}_{/S}$. Then

$$\begin{array}{ccc} W \times \mathbb{A}^1 & \hookrightarrow & E_{Y,W} \\ \downarrow{\scriptstyle g \times \mathbb{A}^1} & & \downarrow{\scriptstyle \hat{f}} \\ Z \times \mathbb{A}^1 & \hookrightarrow & E_{X,Z} \end{array} \qquad \begin{array}{ccc} W & \hookrightarrow & N_{Y,W} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_0} \\ Z & \hookrightarrow & N_{X,Z} \end{array}$$

Fact. If f is étale, then $\hat{f}$ and f_0 are étale.

Lemma. *If f is étale and g is an isomorphism, then purity holds for $Z \hookrightarrow X$ iff it holds for $W \hookrightarrow Y$.*

Proof. By Nisnevich,

$$\begin{array}{c} \frac{X}{X \smallsetminus Z} \\ \simeq \uparrow \\ \frac{Y}{Y \smallsetminus W} \end{array}$$

□

Lemma. *If $\{U_i \to X\}$ is an open cover of X, and if purity holds for $Z \cap U_i \hookrightarrow U_i$ for all i, then purity holds for $Z \hookrightarrow X$.*

Lemma. *Suppose we have two cartesian squares*

$$\begin{array}{ccc} Z & \hookrightarrow & X_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_1} \\ W & \hookrightarrow & Y \end{array} \qquad \begin{array}{ccc} Z & \hookrightarrow & X_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_2} \\ W & \hookrightarrow & Y \end{array}$$

Then purity holds for $Z \hookrightarrow X_1$ iff it holds for $Z \hookrightarrow X_2$.

Proof. We can see that the square in

$$\begin{array}{ccc} Z & & \\ \downarrow{\scriptstyle \Delta} & & \\ Z \times_W Z & \hookrightarrow & X_1 \times_Y X_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_i} \\ Z & \hookrightarrow & X_i \end{array}$$

is a pullback diagram. We would like to complete this into a pair of pullback squares. So we write:

$$\begin{array}{ccc}
Z & \longrightarrow & X_1 \times_Y X_2 \smallsetminus (Z \times_W Z \smallsetminus \Delta_Z) = T \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow \\
Z \times_W Z & \hookrightarrow & X_1 \times_Y X_2 \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_i} \\
Z & \hookrightarrow & X_i
\end{array}$$

The left-hand map is an isomorphism and the right-hand map is étale. Writing this square twice with $i = 1, 2$, we have

$$\begin{array}{ccc}
Z & \hookrightarrow & X_2 \\
\Big\| & & \Big\uparrow{\scriptstyle \text{étale}} \\
Z & \hookrightarrow & T \\
\Big\| & & \Big\downarrow{\scriptstyle \text{étale}} \\
Z & \hookrightarrow & X_1
\end{array}$$

Applying the previous lemma gives the result. □

Proof of purity. Suppose given a diagram
$$\begin{array}{ccc}
Z & \hookrightarrow & X \\
& \searrow & \Big\downarrow \\
& & S
\end{array}.$$

Recall the structure theorem for smooth morphisms: there exists an open cover $\{U_i \to X\}$ and étale maps $f_i : U_i \to \mathbb{A}^n_S$ such that

$$Z \cap U_i = f_i^{-1}(\mathbb{A}^{n-c}_S \times \{0\}^c).$$

We have a PB square:

$$\begin{array}{ccc}
Z \cap U_i & \longrightarrow & U_i \\
\Big\downarrow & & \Big\downarrow{\scriptstyle f_i} \\
\mathbb{A}^{n-c} & \longrightarrow & \mathbb{A}^n_S
\end{array}$$

We also have a PB square:

$$\begin{array}{ccc}
Z \cap U_i & \hookrightarrow & (Z \cap U_i) \times \mathbb{A}^c \\
\Big\downarrow & & \Big\downarrow \\
\mathbb{A}^{n-c}_S & \hookrightarrow & \mathbb{A}^n_S
\end{array}$$

The result follows. □

Application

Theorem (Morel)**.** *Let k be a perfect field, and let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant Nisnevich sheaf of spaces on $\mathrm{Sm}_{/k}$. Let $n \geq 0$. The following are equivalent:*

(1) $\mathcal{F}$ is n-connected (as a Nisnevich sheaf);

(2) For all field extensions L/k of finite transcendence degree, $\mathcal{F}(\operatorname{Spec} L)$ is n-connected.

($\mathcal{F}(\operatorname{Spec} L) := \operatorname{colim}_\alpha \mathcal{F}(X_\alpha)$, where $\operatorname{Spec} L = \lim_\alpha X_\alpha$ with $X_\alpha = \mathrm{Sm}_{/k}$.)