Étale classifying spaces and the representability of algebraic K-theory

Scribe notes from a talk by Paul VanKoughnett

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Plan.

1. Analysis situs
2. Classifying spaces
3. K-theory
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Definition. —

- A **Grothendieck topology** is a notion of which objects cover which other objects.
- A **site** is a category with a Grothendieck topology.
- A **topos** is the category of sheaves on some site.

If $\{U_i\}$ is an open cover of some open set U, then the map $\coprod U_i \to U$ is a cover.

Example. A topological space has an associated site of open subsets.

The above definitions are loose; there are certainly axioms involved.

Definition. Given a cover $U \to V$, we can form

$$U \twoheadleftarrow V \leftleftarrows V \times_U V$$

The **sheaf condition** on a presheaf F is that

$$U \longrightarrow V \rightrightarrows F(V \times_U V).$$

Examples. —

- We could take the site $\mathrm{Sm}_{/S}$ with the Nisnevich topology. The covers are the surjective étale maps that induce isomorphisms on residue fields.
- We could take the site $\mathrm{Sm}_{/S}$ with the Zariski topology. The covers are the surjective maps consisting of a disjoint union of open immersions.
- We could take the site $\mathrm{Sm}_{/S}$ with the étale topology. The covers are the surjective étale maps.

So every Zariski cover is a Nisnevich cover, and every Nisnevich cover is an étale cover. Relating these sites will be important.

Definition. For a site $\mathcal{S}$, a **point** of $\mathcal{S}$ is an adjunction

$$X^* : \mathsf{Sh}(\mathcal{S}) \rightleftarrows \mathsf{Set} : X_*.$$

We consider X^* to be a "stalk functor", and consider X_* to be a "skyscraper sheaf functor".

Remark. We could also describe this as a morphism of topoi

$$\begin{array}{ccc} \mathsf{Set} & \longrightarrow & \mathsf{Sh}(\mathcal{S}) \\ \| & & \\ \mathsf{Sh}(*) & & \end{array}$$

if we knew what a morphism of topoi was.

We obtain morphisms of topoi

$$(\mathrm{Sm}_{/S})_{\mathrm{Zar}} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} (\mathrm{Sm}_{/S})_{\mathrm{Nis}} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} (\mathrm{Sm}_{/S})_{\mathrm{et}}$$

We would like to identify the points of these topoi.

- A Zariski point is $\operatorname{Spec} \mathcal{O}_{X,x}$.
- An étale point is the **strict henselian local ring** $\operatorname{Spec} \mathcal{O}^{sh}_{X,x}$, formed from $\mathcal{O}_{X,x}$ by adding all roots of polynomials p such that $p'(0) \neq \mathfrak{m}_{X,x}$.
- A Nisnevich point is the **henselian local ring** $\operatorname{Spec} \mathcal{O}^{h}_{X,x}$.

Definition. Let $\mathcal{G}$ be a simplicial sheaf of groups. We define the **classifying space** $B\mathcal{G}$, a simplicial sheaf of sets, as the sheafification of the following presheaf:

$$(U, n) \mapsto \mathcal{G}(U)^n_n,$$

where U is an object of the site, and n is the simplicial degree.

Definition. A **$\mathcal{G}$-torsor** on X is a simplicial sheaf $E \to X$ with locally free $\mathcal{G}$-action such that $E/G \simeq X$.

Theorem. *Let $\mathcal{G}$ be a sheaf of groups. Then*

$$\{\mathcal{G} - \textit{torsors over } X\}/iso \cong [X, R(B\mathcal{G})]$$

Definition. We define **algebraic K-theory**

$$K(X) = \Omega BQ(\mathsf{Vect}_X).$$

Here $Q(\mathsf{Vect}_X)$ is the category of vector bundles over X, where maps are inclusions of subquotients.

Theorem. *Algebraic K-theory is represented in the motivic homotopy category $\mathcal{H}_*(S)$ by the object*

$$\mathbb{R}\Omega B(\coprod_{n\geq 0} BGL_n).$$

Remark. The monoid structure on this object is induced from

$$GL_n \times GL_m \to GL_{n+m},$$

the block sum of matrices. Note that the classifying space construction goes forward for monoids, not just groups.

Proposition. *If $\mathcal{M}$ is a simplicial sheaf of monoids that is levelwise free (on a simplicial sheaf of sets), then $B(M^{gp}) \simeq BM$, so that $M^{gp} \simeq \mathbb{R}\Omega BM$. Here the superscript ^{gp} denotes the group completion.*

Note that $\coprod_{n\geq 0} BGL_n$ is an **augmented** monoid, i.e. it is equipped with a map down to $\underline{\mathbb{N}}$ which has a section.

$$\begin{array}{c} \coprod_{n\geq 0} BGL_n \\ \beta \big\downarrow \big\uparrow \alpha \\ \underline{\mathbb{N}} \end{array}$$

So $\coprod BGL_n$ is graded, and we have a map $\alpha(1) : BGL_n \to BGL_{n+1}$. We define BGL_∞ as the colimit.

Corollary.

$$\mathbb{R}\Omega B(\coprod BGL_n) \simeq BGL_\infty \times \mathbb{Z}.$$

1. By Hilbert's Theorem 90, étale GL_n-torsors are Nisnevich GL_n-torsors.

   The fact
   $$H^1_{\text{et}}(X, GL_n) \cong H^1_{\text{Nis}}(X, GL_n) \cong H^1_{\text{Zar}}(X, GL_n)$$
   can be seen to imply the usual statement of Theorem 90. Any G-torsor in $H^1_{\text{et}}(X, GL_n)$ maps to zero in $H^1_{\text{et}}(U, GL_n)$, i.e. in $H^1_{\text{et}}(\mathcal{O}_{X,x}, GL_n) = H^1_{\text{et}}(k(x), GL_n) = 0$. Therefore

   $$BGL_\infty \times \mathbb{Z} \simeq \mathbb{R}\pi_*\pi^* BGL_\infty \times \mathbb{Z},$$

   where the maps π_*, π^* are the maps relating the Nisnevich and étale sites, mentioned earlier.

2.
$$\mathbb{R}\pi_*\pi^* BGL_\infty \simeq \mathrm{Gr}(\infty, \infty).$$

Note that GL_n acts on $\mathbb{A}^{mn} = \mathrm{Hom}(\mathbb{A}^n, \mathbb{A}^m)$. It acts freely on some open subset $\mathrm{Inj}(\mathbb{A}^n, \mathbb{A}^m) = U_m$. We write $EGL_n = \mathrm{colim}_{m\to\infty} U_m$.

Define
$$\partial\Delta^n_{\mathbb{A}^1} = V(x_1 \dots x_n(\sum x_i - 1)) \subseteq \mathbb{A}^n.$$

[Some argument about contractibility. See Paul's notes.]

Theorem. *K-theory is represented in $\mathcal{H}_*(S)$ by*
$$R(G(\infty, \infty) \times \mathbb{Z}).$$

Example. We now compute the bigraded homotopy groups (sheaves) of this representing space. Write $S^{p,q} = (S^1)^{\wedge p-q} \wedge \mathbb{G}_m^{\wedge q}$ for the bigraded spheres. We want to compute the bigraded homotopy group
$$[S^{p,q} \wedge X_+, R(G(\infty, \infty) \times \mathbb{Z})].$$

If $q = 0$, this is $K_p(X)$. Note that
$$\mathbb{G}_m \wedge X_+ = (\mathbb{G}_m \times X)/X,$$
$$K_n(\mathbb{G}_m \times X) = K_n(X) \oplus K_{n-1}(X),$$
$$K_n(\mathbb{G}_m \wedge X_+) = K_{n-1}(X).$$

So we conclude that the bigraded homotopy group above equals $K_{p-2q}(X)$.