Symmetric spectra

Scribe notes from a talk by Irina Bobkova

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There are two things we like:

1) spectra;

2) multiplication.

Spectra

Let $\mathcal{C}$ be a category and G a left Quillen endofunctor.

Definition. We define the category $\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ of **spectra** as follows:

- Objects are sequences $X_1, \ldots, X_n, \ldots$, with $X_i \in \operatorname{ob}\mathcal{C}$, together with structure maps $\sigma : GX_n \to X_{n+1}$;
- Morphisms are collections of maps $f_n : X_n \to Y_n$ commuting with σ.

Definition. We have the evaluation map

$$\mathrm{Ev}_n : \mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G) \to \mathcal{C}, \qquad X \mapsto X_n,$$

which has a left adjoint

$$(F_n X)_m = \begin{cases} G^{m-n}X & m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let U denote the right adjoint of G.

Lemma. *Suppose we have $F : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\tau : GF \to FG$. Then there is an induced map $F : \mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G) \to \mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$.*

Remark. U and G prolong to adjoint functors on spectra.

Model structures on spectra

Suppose that $\mathcal{C}$ is cofibrantly generated, by generating cofibrations I and generating trivial cofibrations J.

Theorem. *The projective cofibrations (LLP with respect to level trivial fibrations), level fibrations, and level weak equivalences give a model structure, called the* **projective model structure***, on* $\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$*. This is cofibrantly generated, with generating cofibrations* $I_G = \cup_n F_n I$*, and generating trivial cofibrations* $J_G = \cup_n F_n J$*.*

Plan. We would like G to be a Quillen equivalence. We will then Bousfield localize to some appropriate objects.

We define the analogue of Ω-spectra:

Definition. A spectrum X is called a U**-spectrum** if X is levelwise fibrant and the adjoint map $X_n \xrightarrow{\tilde{\sigma}} UX_{n+1}$ is a weak equivalence.

We would like a localizing set S such that the S-local spectra are the U-spectra.

Proposition. *Let* $\mathcal{C}$ *be a cofibrantly generated model category with generating cofibrations* I*, and* $f : X \to Y$*. Then* f *is a weak equivalence iff* $\mathrm{Map}(C, X) \to \mathrm{Map}(C, Y)$ *is a weak equivalence, for* C *running over all domains and codomains of maps in* I*.*

So we need to choose some set S of maps to make the map

$$\begin{array}{ccc} \mathrm{Map}(C, X_n) & \longrightarrow & \mathrm{Map}(C, UX_{n+1}) \\ \| & & \| \\ \mathrm{Map}(F_n QC, X) & \longrightarrow & \mathrm{Map}(F_{n+1} GQC, X) \end{array}$$

a weak equivalence.

Definition. Define $S = \{F_{n+1} GQC \xrightarrow{s_n^{QC}} F_n QC\}$, where C runs over the domains and codomains of maps in I.

Theorem. *Now* $S^{-1}G : S^{-1}\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G) \to S^{-1}\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ *is a Quillen equivalence.*

Theorem. *This general construction of localization coincides with the Bousfield–Friedlander model structure on spectra.*

Properties of this construction are as follows.

- If $G : \mathcal{C} \to \mathcal{C}$ is a Quillen equivalence, then $\mathcal{C} \to \mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ is a Quillen equivalence.
- $\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ is functorial in $(\mathcal{C}, G)$.
- If $\mathcal{C}$ is symmetric monoidal, $\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ is almost never symmetric monoidal.

Compatible model structure and symmetric monoidal structure

Definition. Let $\mathcal{C}$ be a symmetric monoidal model category, and let $f : A \to B$, $g : C \to D$. Define the **pushout product**

$$f \,\square\, g : (A \otimes D) \coprod_{A \otimes C} (B \otimes C) \to B \otimes D$$

induced by

$$\begin{array}{ccc} A \otimes C & \longrightarrow & B \otimes C \\ \downarrow & & \downarrow \\ A \otimes D & \longrightarrow & B \otimes D \end{array}$$

Definition. We say that the model structure and symmetric monoidal structure on $\mathcal{C}$ are **compatible** if the following hold:

- if f and g are cofibrations, then $f \,\square\, g$ is a cofibration;
- if f or g is a trivial cofibration, then $f \,\square\, g$ is a trivial cofibration.

Let $\mathcal{D}$ be a bicomplete, closed symmetric monoidal category with unit S. Let $K \in \operatorname{ob} \mathcal{D}$. Let $\mathcal{C}$ be bicomplete, enriched, tensored, and cotensored over $\mathcal{C}$. We will consider functors of the form $G(X) = X \otimes K$, where $K = GS$.

Definition. Let $\Sigma = \coprod_{n \geq 0} \Sigma_n$, the category with

- objects as sets $\bar{n} = \{1, \ldots, n\}$,
- morphisms as automorphisms of the set $\bar{n}$.

Definition. A **symmetric sequence** is a sequence $X_0, \ldots, X_n$ of objects of $\mathcal{C}$ with an action of Σ_n on X_n. Denote the category of symmetric sequences as $\mathcal{C}^\Sigma$.

Fact. If $\mathcal{C}$ is a symmetric monoidal category, then $\mathcal{C}^\Sigma$ is a symmetric monoidal category as follows. If $x,\, y \in \mathcal{C}^\Sigma$,

(A)

$$(X \otimes Y)(\mathcal{C}) = \coprod_{\substack{A \cup B = C \\ A \cap B = 0}} X(A) \otimes Y(B),$$

or equivalently,

(B)

$$(X \otimes Y)_n = \coprod_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} (X_p \otimes Y_q).$$

The unit in $\mathcal{C}^\Sigma$ is $(S, 0, 0, \ldots)$. We have the commutative monoid

$$\operatorname{Sym}(K) = (S, K, K \otimes K, \ldots).$$

Definition. Let $\mathcal{D}$ be a symmetric monoidal model category, $\mathcal{C}$ a $\mathcal{D}$-model category, and $K \in \operatorname{ob} \mathcal{D}$. Define $\mathsf{Sp}^\Sigma(\mathcal{C}, K)$ to be the category of modules over $\operatorname{Sym}(K)$ in $\mathcal{C}^\Sigma$.

Model structure on $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$

There exists a model structure on Sp^{Σ}, defined analogously to the model structure on $\mathsf{Sp}^{\mathbb{N}}$.

Theorem. —

- *With this projective model structure,* $\mathsf{Sp}^{\Sigma}(\mathcal{D}, K)$ *is a symmetric monoidal model category.*
- $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$*, with its projective model structure, is a* $\mathsf{Sp}^{\Sigma}(\mathcal{D}, K)$*-model category.*

Definition. A symmetric spectrum $X \in \mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$ is an **Ω-spectrum** if X is levelwise fibrant and the adjoint of the structure map $X_n \to X_{n+1}^K$ is a weak equivalence for all n.

We want to localize $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$ the same way as in the non-symmetric case, plus symmetric conditions. The localization is called the **stable model structure** on $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$.

Theorem. *If* $GX = X \otimes K$*, then* $\mathsf{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ *and* $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$*, with stable model structures, are related by a sequence of Quillen equivalences, so long as the cyclic permutation*

$$K \otimes K \otimes K \to K \otimes K \otimes K$$

is the identity in $\mathrm{Ho}(\mathcal{D})$*.*

Example. We can take $K = S^1$ in the topological context to obtain the ordinary notion of symmetric spectra in stable homotopy theory.

Example. In the motivic context, taking $K = \mathbb{P}^1$, we obtain a category of motivic symmetric spectra

$$\mathsf{Sp}^{\Sigma}(\mathrm{spc}_*(S), \mathbb{P}^1).$$

Here $\mathrm{spc}_*(S)$ is the model category of simplicial presheaves on $\mathrm{Sm}_{/S}$, with elementary Nisnevich squares and $\mathbb{A}^1$ localized.

Remark. Last time we constructed the ∞-category $\mathcal{SH}(S)$. Let us notationally identify the model category $\mathsf{Sp}^{\Sigma}(\mathrm{spc}_*(S), \mathbb{P}^1)$ with its underlying ∞-category, the simplicial nerve of its fibrant-cofibrant subcategory. By the universal property of $\mathcal{SH}(S)$, we have a map

$$\mathcal{SH}(S) \to \mathsf{Sp}^{\Sigma}(\mathrm{spc}_*(S), \mathbb{P}^1).$$

One can show that this is an equivalence, essentially by the universal property of the latter model category. We build objects on both sides out of smooth schemes; showing that this map takes smooth schemes to smooth schemes will show essential surjectivity. Keeping track of how the localizations involved match up will yield full faithfulness.

The upshot is that we have a symmetric monoidal model category that presents the stable motivic ∞-category.