Localization and the unstable motivic $(\infty, 1)$-category

Scribe notes from a talk by Jay Shah

17 Mar 2014

Goal. Construct the homotopy theory of schemes.

§1. ∞-category of presheaves on S.

§2. Localizations of presentable ∞-categories.

§3. Construction of the ∞-category of motivic spaces.

1. The ∞-category of presheaves on S

Let Sp be the ∞-category of spaces, i.e. the simplicial nerve of the full subcategory of simplicial sets spanned by the Kan complexes.

Definition. Let S be a simplicial set. Then $\mathsf{Pre}(S) = \mathrm{Fun}(S^{\mathrm{op}}, \mathsf{Sp})$ is the **∞-category of presheaves**.

Facts. —

- $\mathsf{Pre}(S)$ has all limits and colimits, and they are computed pointwise:

$$\begin{array}{ccccc} K & \xrightarrow{p} & \mathsf{Pre}(S)^{\mathrm{evals}} & \longrightarrow & \mathsf{Sp} \\ \downarrow & \overset{\bar{p}}{\nearrow} & & \nearrow & \\ K^{\triangleright} & & & & \end{array}$$

- We have the **Yoneda embedding** $S \to \mathsf{Pre}(S)$, defined by

$$\begin{array}{ccc} \mathfrak{C}[S^{\mathrm{op}} \times S] & \longrightarrow & \mathfrak{C}[S]^{\mathrm{op}} \times \mathfrak{C}[S] \\ & \searrow & \downarrow \\ & & \mathsf{Set}_{\Delta} \\ & & \downarrow \\ & & \mathsf{Kan} \end{array}$$

  The adjoint to the composite $S^{\mathrm{op}} \to S \to \mathrm{N}(\mathsf{Kan}) = \mathsf{Sp}$ is the Yoneda embedding $Y : S \to \mathrm{Fun}(S^{\mathrm{op}}, \mathsf{Sp}) = \mathsf{Pre}(S)$.

- The Yoneda embedding is fully faithful.
- The Yoneda embedding preserves limits.
- In the following diagram

$$\begin{array}{ccc} \mathcal{C}^\circ & \xrightarrow{F^\circ} & \mathcal{D} \\ \big\downarrow & \nearrow_{F} & \\ \mathcal{C} & & \end{array}$$

  F is a left Kan extension of F° if the following diagram is a colimit diagram:

$$\begin{array}{ccc} \mathcal{C}^\circ_{/c} & \longrightarrow & \mathcal{D} \\ \big\downarrow & \nearrow & \big\uparrow{\scriptstyle F(c)} \\ (\mathcal{C}^\circ_{/c})^{\triangleright} & \longleftarrow & \mathrm{pt} \end{array}$$

Theorem. *We can think of* $\mathsf{Pre}(\mathcal{C})$ *as the free cocompletion of* $\mathcal{C}$*, in the sense that*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Y} & \mathsf{Pre}(\mathcal{C}) \\ & {\scriptstyle Y}\searrow & \big\uparrow{\scriptstyle \mathrm{id}=\textit{left Kan extension.}} \\ & & \mathsf{Pre}(\mathcal{C}) \end{array}$$

This says: any $X \subseteq \mathsf{Pre}(\mathcal{C})$ *is canonically a colimit of representables.*

Corollary. *We have a universal property for* $\mathsf{Pre}(S)$*: for* $\mathcal{D}$ *a cocomplete category,*

$$\mathrm{Fun}^L(\mathsf{Pre}(S), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(S, \mathcal{D}).$$

2. Localization

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and $f : \mathcal{C} \to \mathcal{D}$.

Definition. f is a **localization functor** if f has a fully faithful right adjoint $g : \mathcal{D} \subseteq \mathcal{C}$.

Moreover say that f is **accessible** if f preserves filtered colimits.

Definition. An ∞-category $\mathcal{C}$ is **presentable** if $\mathcal{C}$ arises as an accessible localization of an ∞-category of presheaves.

We think of $\mathcal{C}$ as being obtained by means of generators and relations.

Let $\mathcal{C}$ be an ∞-category and S a set of morphisms.

Idea. We want to find $S^{-1}\mathcal{C}$ as the full subcategory in $\mathcal{C}$ of "S-local objects", and produce a localization functor $\mathcal{C} \to S^{-1}\mathcal{C}$.

Definition. —

- An S-**local object** Z is an object such that for all morphisms $f : X \to Y$ in S,
$$\operatorname{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}}(X, Z).$$

- An S-equivalence is a morphism $f : X \to Y$ in $\mathcal{C}$ such that for all S-local objects Z,
$$\operatorname{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}}(X, Z).$$

Theorem. *Let $\mathcal{C}$ be a presentable ∞-category and S a small set of morphisms. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of S-local objects.*

(1) The inclusion $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathcal{C}'$, which is an accessible localization.

(2) f is an S-equivalence in $\mathcal{C}$ iff $L(f)$ is an equivalence.

(3) $\mathcal{C}'$ is presentable.

Idea. We find $X \to LX$ as a "terminal S-equivalence", i.e. a final object in some slice category.

Theorem. *Let $L : \mathcal{C} \to S^{-1}\mathcal{C}$ as before, and let $\mathcal{D}$ be any ∞-category. Let*
$$\eta : \operatorname{Fun}^L(S^{-1}\mathcal{C}, \mathcal{D}) \xrightarrow{L^*} \operatorname{Fun}^L(\mathcal{C}, \mathcal{D}).$$
Then η is fully faithful, and its essential image consists of $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an equivalence for all $s \in S$.

Idea. The hard part is to prove that η is fully faithful. We know that
$$\operatorname{Fun}(S^{-1}\mathcal{C}, \mathcal{D}) \xrightarrow{L^*} \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$
is fully faithful by dualizing our previous observation on left Kan extensions; this functor is the inclusion of those functors $\mathcal{C} \to \mathcal{D}$ that are right Kan extensions of their restriction to $S^{-1}\mathcal{C}$.

The remaining key idea is that $(S^{-1}\mathcal{C})_{X/}$ has an initial object $X \to LX$.

3. Construction of the ∞-category of motivic spaces

Let S be a noetherian scheme of finite dimension. Let Sm/S denote the category of smooth schemes of finite type over S. We equip this category with the Nisnevich topology.

There are squares that we call **elementary** or **distinguished squares**:

$$\begin{array}{ccc} p^{-1}U & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xhookrightarrow{i} & X \end{array}$$

where p is étale, i is an open immersion, and with $Z = X \setminus U$, $p : p^{-1}Z \to Z$ is an isomorphism. These squares form a "basis" for the Nisnevich topology: to check that $F : \mathrm{Sm}_{/S}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf for the Nisnevich topology, it suffices to check that on the elementary distinguish squares,

$$F(X) \longrightarrow (F(U) \times F(V) \rightrightarrows F(p^{-1}(U))$$

is an equalizer diagram.

We want to localize $\mathsf{Pre}(\mathrm{Sm}/S)$ so that distinguished squares pass to cocartesian schemes. So we take the collection of morphisms

$$S = \{p^{-1}U \cup_U V \to X, \quad \mathbb{A}^1 \times X \to X, \quad \emptyset_{\mathrm{initial}} \to \emptyset_{\mathrm{scheme}}\}$$

Then we define the ∞**-category of motivic spaces**

$$\mathcal{H}(S) = S^{-1}\mathsf{Pre}(\mathrm{Sm}/S).$$

This has the following universal property:

Theorem. *The functor*

$$\mathrm{Fun}^L(\mathcal{H}(S), \mathcal{D}) \to \mathrm{Fun}(\mathrm{Sm}/S, \mathcal{D})$$

is fully faithful, and its essential image consists of functors $F : \mathrm{Sm}/S \to \mathcal{D}$ which satisfy "Nisnevich descent" and invert $\mathbb{A}^1$.

We now turn to applications of this theory.

Change of site functors. A map $f : S \to T$ of base schemes induces $\mathrm{Sm}/T \xrightarrow{f^*} \mathrm{Sm}/S$, and we want this to pass to a map $\mathcal{H}(T) \to \mathcal{H}(S)$ of the homotopy categories. We obtain this from the universal property:

$$\begin{array}{ccc} \mathrm{Sm}/T & \longrightarrow & \mathcal{H}(T) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \exists!} \\ \mathrm{Sm}/S & \longrightarrow & \mathcal{H}(S) \end{array}$$

Complex realization. Let $S = \operatorname{Spec} k$, with $k \hookrightarrow \mathbb{C}$. We apply the universal property to produce a realization:

$$\begin{array}{ccc} \mathrm{Sm}/S & \hookrightarrow & \mathcal{H}(S) \\ & \searrow & \downarrow \\ & & \mathsf{Sp} \end{array}$$

where the left arrow takes a k-scheme X to the complex points $X(\mathbb{C})$.

Going further, let $p : E \to X$ be an étale cover, so that $E(\mathbb{C}) \to X(\mathbb{C})$ is a surjective local homeomorphism. Define the diagram

$$E(\mathbb{C})_\bullet = \Big(\ldots \Rrightarrow E(\mathbb{C}) \times_{X(\mathbb{C})} E(\mathbb{C}) \rightrightarrows E(\mathbb{C}) \Big)$$

Then $\mathrm{hocolim}(E(\mathbb{C})_\bullet) \xrightarrow{\sim} X(\mathbb{C})$, a generalization of the nerve theorem.