Localization and the unstable motivic $(\infty, 1)$ -category

Scribe notes from a talk by Jay Shah

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Goal. Construct the homotopy theory of schemes.

- §1. ∞ -category of presheaves on S.
- §2. Localizations of presentable ∞ -categories.
- §3. Construction of the ∞ -category of motivic spaces.

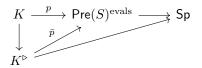
1. The ∞ -category of presheaves on S

Let Sp be the ∞ -category of spaces, i.e. the simplicial nerve of the full subcategory of simplicial sets spanned by the Kan complexes.

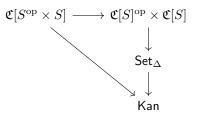
Definition. Let S be a simplicial set. Then $Pre(S) = Fun(S^{op}, Sp)$ is the ∞ -category of presheaves.

Facts. —

• Pre(S) has all limits and colimits, and they are computed pointwise:



• We have the **Yoneda embedding** $S \to \mathsf{Pre}(S)$, defined by

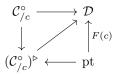


The adjoint to the composite $S^{\text{op}} \to S \to N(\mathsf{Kan}) = \mathsf{Sp}$ is the Yoneda embedding $Y: S \to \operatorname{Fun}(S^{\text{op}}, \mathsf{Sp}) = \mathsf{Pre}(S)$.

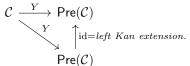
- The Yoneda embedding is fully faithful.
- The Yoneda embedding preserves limits.
- In the following diagram



F is a left Kan extension of F° if the following diagram is a colimit diagram:



Theorem. We can think of $Pre(\mathcal{C})$ as the free cocompletion of \mathcal{C} , in the sense that



This says: any $X \subseteq \mathsf{Pre}(\mathcal{C})$ is canonically a colimit of representables.

Corollary. We have a universal property for Pre(S): for \mathcal{D} a cocomplete category,

$$\operatorname{Fun}^{L}(\operatorname{Pre}(S), \mathcal{D}) \xrightarrow{\simeq} \operatorname{Fun}(S, \mathcal{D}).$$

2. Localization

Let \mathcal{C} and \mathcal{D} be ∞ -categories, and $f: \mathcal{C} \to \mathcal{D}$.

Definition. f is a **localization functor** if f has a fully faithful right adjoint $g: \mathcal{D} \subseteq \mathcal{C}$.

Moreover say that f is **accessible** if f preserves filtered colimits.

Definition. An ∞ -category C is **presentable** if C arises as an accessible localization of an ∞ -category of presheaves.

We think of C as being obtained by means of generators and relations. Let C be an ∞ -category and S a set of morphisms.

Idea. We want to find $S^{-1}\mathcal{C}$ as the full subcategory in \mathcal{C} of "S-local objects", and produce a localization functor $\mathcal{C} \to S^{-1}\mathcal{C}$.

Definition. —

• An S-local object Z is an object such that for all morphisms $f: X \to Y$ in S,

 $\operatorname{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}}(X, Z).$

• An S-equivalence is a morphism $f: X \to Y$ in \mathcal{C} such that for all S-local objects Z,

 $\operatorname{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}}(X, Z).$

Theorem. Let C be a presentable ∞ -category and S a small set of morphisms. Let $C' \subset C$ be the full subcategory of S-local objects.

- (1) The inclusion $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathcal{C}'$, which is an accessible localization.
- (2) f is an S-equivalence in C iff L(f) is an equivalence.
- (3) C' is presentable.

Idea. We find $X \to LX$ as a "terminal S-equivalence", i.e. a final object in some slice category.

Theorem. Let $L: \mathcal{C} \to S^{-1}\mathcal{C}$ as before, and let \mathcal{D} be any ∞ -category. Let

$$\eta: \operatorname{Fun}^{L}(S^{-1}\mathcal{C}, \mathcal{D}) \xrightarrow{L^{*}} \operatorname{Fun}^{L}(\mathcal{C}, \mathcal{D}).$$

Then η is fully faithful, and its essential image consists of $F : \mathcal{C} \to \mathcal{D}$ such that F(s) is an equivalence for all $s \in S$.

Idea. The hard part is to prove that η is fully faithful. We know that

$$\operatorname{Fun}(S^{-1}\mathcal{C},\mathcal{D}) \xrightarrow{L^*} \operatorname{Fun}(\mathcal{C},\mathcal{D})$$

is fully faithful by dualizing our previous observation on left Kan extensions; this functor is the inclusion of those functors $\mathcal{C} \to \mathcal{D}$ that are right Kan extensions of their restriction to $S^{-1}\mathcal{C}$.

The remaining key idea is that $(S^{-1}\mathcal{C})_{X/}$ has an initial object $X \to LX$.

3. Construction of the ∞ -category of motivic spaces

Let S be a noetherian scheme of finite dimension. Let Sm/S denote the category of smooth schemes of finite type over S. We equip this category with the Nisnevich topology.

There are squares that we call **elementary** or **distinguished squares**:

$$p^{-1}U \longrightarrow V$$

$$\downarrow \qquad \qquad \downarrow^p$$

$$U \xleftarrow{i} X$$

where p is étale, i is an open immersion, and with $Z = X \setminus U$, $p: p^{-1}Z \to Z$ is an isomorphism. These squares form a "basis" for the Nisnevich topology: to check that $F: \operatorname{Sm}_{/S}^{\operatorname{op}} \to \operatorname{Set}$ is a sheaf for the Nisnevich topology, it suffices to check that on the elementary distinguish squares,

$$F(X) \longrightarrow (F(U) \times F(V) \Longrightarrow F(p^{-1}(U))$$

is an equalizer diagram.

We want to localize Pre(Sm/S) so that distinguished squares pass to cocartesian schemes. So we take the collection of morphisms

$$S = \{ p^{-1}U \cup_U V \to X, \quad \mathbb{A}^1 \times X \to X, \quad \emptyset_{\text{initial}} \to \emptyset_{\text{scheme}} \}$$

Then we define the ∞ -category of motivic spaces

$$\mathcal{H}(S) = S^{-1} \mathsf{Pre}(\mathrm{Sm}\,/S).$$

This has the following universal property:

Theorem. The functor

$$\operatorname{Fun}^{L}(\mathcal{H}(S), \mathcal{D}) \to \operatorname{Fun}(\operatorname{Sm}/S, \mathcal{D})$$

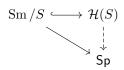
is fully faithful, and its essential image consists of functors $F : \operatorname{Sm} / S \to \mathcal{D}$ which satisfy "Nisnevich descent" and invert \mathbb{A}^1 .

We now turn to applications of this theory.

Change of site functors. A map $f : S \to T$ of base schemes induces $\operatorname{Sm}/T \xrightarrow{f^*} \operatorname{Sm}/S$, and we want this to pass to a map $\mathcal{H}(T) \to \mathcal{H}(S)$ of the homotopy categories. We obtain this from the universal property:

$$\begin{array}{ccc} \operatorname{Sm}/T & \longrightarrow & \mathcal{H}(T) \\ & & & & \downarrow^{f^*} & & \downarrow^{\exists !} \\ \operatorname{Sm}/S & \longrightarrow & \mathcal{H}(S) \end{array}$$

Complex realization. Let $S = \operatorname{Spec} k$, with $k \hookrightarrow \mathbb{C}$. We apply the universal property to produce a realization:



where the left arrow takes a k-scheme X to the complex points $X(\mathbb{C})$.

Going further, let $p: E \to X$ be an étale cover, so that $E(\mathbb{C}) \to X(\mathbb{C})$ is a surjective local homeomorphism. Define the diagram

$$E(\mathbb{C})_{\bullet} = \left(\ldots \Longrightarrow E(\mathbb{C}) \times_{X(\mathbb{C})} E(\mathbb{C}) \Longrightarrow E(\mathbb{C}) \right)$$

Then $\operatorname{hocolim}(E(\mathbb{C})_{\bullet}) \xrightarrow{\sim} X(\mathbb{C})$, a generalization of the nerve theorem.