Basic constructions with quasicategories

Scribe notes from a talk by Michael Catanzaro

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Join

The **join** of two categories $\mathcal{C}$ and $\mathcal{D}$ is a category $\mathcal{C} * \mathcal{D}$ with objects as the disjoint union, and with exactly one morphism $c \to d$ for $c \in \mathcal{C}$ and $d \in \mathcal{D}$, but no morphisms $d \to c$. We generalize this to ∞-categories:

Definition. We define the **join** of ∞-categories (or even simplicial sets) M and N by

$$(M * N)_k = M_k \cup N_k \cup \bigcup_{i+j+1=k} M_i \times N_j.$$

Remark. This is characterized by two properties:

- $M * -$ and $- * N$ preserve colimits;
- $\Delta^i * \Delta^j \cong \Delta^{i+j+1}$.

Notation. We have the **left cone** on K, $K^{\triangleleft} = \Delta^0 * K$, and the **right cone** on K, $K^{\triangleright} = K * \Delta^0$.

Overcategory

Classically we have the idea of an **overcategory** $\mathcal{C}_{/c}$, the category of arrows into c and morphisms of such arrows. This is characterized by the property:

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}_{/p}) \cong \mathrm{Fun}_p(\mathcal{D} * I, \mathcal{C}).$$

This generalizes readily to the higher context.

Definition. Let $\mathcal{C}$ be an ∞-category, $M \in \mathsf{Set}_\Delta$, and $p : M \to \mathcal{C}$. Then there exists an ∞-category $\mathcal{C}_{/p}$ such that for all $K \in \mathsf{Set}_\Delta$,

$$\hom_{\mathsf{Set}_\Delta}(K, \mathcal{C}_{/p}) \simeq \hom_{\mathsf{Set}_\Delta, p}(K * M, \mathcal{C}).$$

Final

Definition. An object $c \in \mathcal{C} \in \mathsf{Cat}_\infty$ is **final** if $\mathcal{C}_{/c} \to \mathcal{C}$ is a trivial Kan fibration.

Remark. $c \in \mathcal{C}$ is final iff $\mathrm{Map}(c', c)$ is contractible for all $c' \in \mathcal{C}$.

Proposition. *If $\mathcal{D} \subseteq \mathcal{C}$ is the full subcategory spanned on final objects, then $\mathcal{D}$ is either empty or is a contractible Kan complex.*

(Co)limits

Definition. Let K be a simplicial set, $\mathcal{C}$ an ∞-category, and $p : K \to \mathcal{C}$ a diagram in $\mathcal{C}$. A **limit** of p is a final cone, i.e. a final object in $\mathcal{C}_{/p}$.

Likewise, a **colimit** of p is an initial cocone, i.e. an initial object in $\mathcal{C}_{p/}$.

If $\mathcal{C}$ has all limits, we say $\mathcal{C}$ is **complete**. Likewise, if $\mathcal{C}$ has all colimits, we say it is **cocomplete**.

Proposition. *For a fixed diagram p, the space of all limits or colimits is either empty or a contractible Kan complex.*

Stability

Definition. We say $\mathcal{C}$ is **pointed** if there exists an object that is both initial and final. In this case, there exists a zero morphism between any two objects.

Definition. A **triangle** in $\mathcal{C}$ is a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

Note that this definition implicitly encodes a homotopy, not an equality, between $g \circ f$ and 0.

Remark. We can see this as a diagram from any of the following isomorphic simplicial sets:

$$(\Delta^1)^2 \cong (\Lambda^2_0)^{\triangleright} \cong (\Lambda^2_2)^{\triangleleft}.$$

Definition. —

- A triangle τ is **exact** if $\tau : (\Lambda^2_2)^{\triangleleft} \to \mathcal{C}$ is a limit diagram.
- A triangle τ is **coexact** if $\tau : (\Lambda^2_0)^{\triangleright} \to \mathcal{C}$ is a colimit diagram.

Definition. An ∞-category $\mathcal{C}$ is **stable** if it is bicomplete, pointed, and if every exact triangle is exact iff it is coexact.

Question. Is $\mathrm{Ho}(\mathcal{C})$ triangulated? (Yes!)

Definition. The diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is **distinguished** in $\mathrm{Ho}(\mathcal{C})$ if

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0' \\ \downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{\tilde{h}} & U \end{array}$$

in $\mathcal{C}$, with both squares being pushouts, where $U \cong \Sigma X$.

Grothendieck construction

The Grothendieck construction in ordinary category theory is a correspondence between functors $\mathcal{D} \to \mathsf{Cat}$ and categories cofibered over $\mathcal{D}$.

Definition. Let $p : \mathcal{C} \to \mathcal{D}$ be a map of ∞-categories, and $f : c_1 \to c_2$ a morphism in $\mathcal{C}$.

- f is **p-cartesian** if
$$\mathcal{C}_{/f} \to \mathcal{C}_{/c_2} \times_{\mathcal{D}_{/p(c_2)}} \mathcal{D}_{/p(f)}$$
is a trivial Kan fibration.

- Dually, f is **p-cocartesian** if
$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}.$$

- p is an **inner fibration** if it satisfies right lifting with respect to inner horn fillings.

- p is a **cartesian fibration** if it is an inner fibration and for all $c_2 \in \mathcal{C}$, and for all $\alpha : d_1 \to p(c_2)$, there exists a p-cartesian morphism f that lifts α.

- p is a **cocartesian fibration** dually to the above.

Let $S \in \mathsf{Set}_\Delta$, and $\mathfrak{C}[S]$ its associated simplicial category. The ∞-Grothendieck construction relates $(\mathsf{Set}_\Delta)_{/S}$ and $(\mathsf{Set}_\Delta)^{\mathfrak{C}[S]^{\mathrm{op}}}$ by a Quillen equivalence, given by functors St and Un, the "straightening and unstraightening functors".

$$(\mathsf{Set}_\Delta)_{/S} \underset{\mathrm{Un}}{\overset{\mathrm{St}}{\rightleftarrows}} (\mathsf{Set}_\Delta)^{\mathfrak{C}[S]^{\mathrm{op}}}$$

Theorem. *This is a Quillen equivalence, if we take the Joyal model structure on the left, and the projective model structer on the right.*

Remark. There exists a generalization, taking as input a simplicial set S, a simplicial category $\mathcal{C}$, and a map $\mathfrak{C}[S] \to \mathcal{C}^{\mathrm{op}}$. We obtain a Quillen adjunction which is an equivalence iff ϕ is an equivalence. (The categories on both sides of the adjunction depend on the input data.)

Definition. Let $p : M \to \Delta^1$ be a cartesian fibration, together with $h_0 : \mathcal{C} \to p^{-1}\{0\}$, $h_1 : \mathcal{D} \to p^{-1}\{1\}$. Then $g : \mathcal{D} \to \mathcal{C}$ is **associated to** M if if

$$\begin{array}{ccc} \mathcal{D} \times \Delta^1 & \xrightarrow{s} & M \\ & \searrow \quad \swarrow_{p} & \\ & \Delta^1 & \end{array}$$

such that $s\big|_{\mathcal{D}\times\{0\}} = h_1$, $s\big|_{\mathcal{D}\times\{1\}} = h_0 \circ g$, and $s\big|_{\{x\}\times\Delta^1} \times \Delta^1$ is a p-cartesian morphism for all $x \in \mathcal{D}$.

Dually for co-cartesian fibrations; we obtain a map $\mathcal{C} \to \mathcal{D}$.

Definition. An **adjunction** is a map $q : M \to \Delta^1$, which is both a cartesian and a co-cartesian fibration, together with equivalences $\mathcal{C} \to M_{\{0\}}$, $\mathcal{D} \to M_{\{1\}}$. The associated maps $\mathcal{C} \to \mathcal{D}$ and $\mathcal{D} \to \mathcal{C}$ are the left and right adjoints, respectively.

Definition. A **unit transformation** for (f, g) is a morphism $U : \mathrm{id}_C \to g \circ f$ such that for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$,

$$\hom_{\mathcal{D}}(f(c), d) \to \hom_{\mathcal{C}}(gf(c), g(d)) \to \hom_{\mathcal{C}}(c, g(d))$$

is an isomorphism in the homotopy category.