Quasicategories

Scribe notes from a talk by Meng Guo

17 Mar 2014

Definition. A **Kan complex** is a simplicial set K such that the following lift always exists:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Here Λ^n_i is the ith horn in Δ^n, the simplicial set formed by deleting the ith face.

Definition. For $\mathcal{C}$ a category, we define $\mathrm{N}(\mathcal{C})$, the simplicial set given by strings of morphisms

$$\mathrm{N}(\mathcal{C})_n = \{C_0 \to C_1 \to \ldots \to C_n\},$$

with face maps given by composition, and degeneracy maps given by insertion of identity morphisms.

We will define ∞-categories by generalizing the properties of the nerve. Given 1-simplices $C_0 \to C_1 \to C_2$ in $N(\mathcal{C})$, forming a map $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$, we can find a lift of this to a map $\Delta^2 \to \mathrm{N}(\mathcal{C})$; for instance the 1-face of this Δ^2 will map to the composite $C_0 \to C_2$. In fact this lift is unique.

Definition. An ∞-category is a simplicial set $\mathcal{C}$ such that the following lift always exists:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array} \qquad \text{for } 0 < i < n.$$

Proposition. *Let X be a simplicial set. The following are equivalent:*

1. *X is the nerve of a category $\mathcal{C}$;*
2. *X has the horn-filling condition that defines an ∞-category, with the further condition that the lift is unique.*

Definition. A **simplicial category** is a category enriched over the category Set_Δ of simplicial sets. We write Cat_Δ for the category of simplicial categories.

Definition. A functor $\mathcal{C} \to \mathcal{C}'$ between simplicial categories is an equivalence if the induced map $hF : h\mathcal{C} \to h\mathcal{C}'$ on homotopy categories is an equivalence. By $h\mathcal{C}$ we mean $\mathcal{C}$ with the Hom-sets replaced by their homotopy types; so $h\mathcal{C}$ is a category enriched over $\mathrm{Ho}(\mathsf{Set}_\Delta)$.

Definition. We define a functor $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$ as follows. For a finite non-empty linearly ordered set J,

- The objects of $\mathfrak{C}[J]$ are the elements of J;
- The morphisms of $\mathfrak{C}[J]$ are given by:

  $$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) = \begin{cases} \emptyset & \text{if } i > j \\ \mathrm{N}(P_{i,j}) & \text{if } i \le j, \end{cases}$$

  where $P_{i,j}$ is the poset defined by

  $$P_{i,j} = \{I \subseteq J \mid i, j \in I, i \le k \le j, k \in I\}.$$

  For example, $P_{0,2}$ is the poset $\{\{0,2\}, \{0,1,2\}\}$.
- For $i_0 < \ldots < i_n$, we have a composition map

  $$\mathrm{Map}(i_0, i_1) \times \mathrm{Map}(i_1, i_2) \times \ldots \times \mathrm{Map}(i_{n-1}, i_n),$$

  given as the nerve of

  $$P_{i_0,i_1} \times P_{i_1,i_2} \times \ldots \times P_{i_{n-1},i_n}, \qquad (I_1, \ldots, I_n) \mapsto I_1 \cup \ldots \cup I_n.$$

This construction is functorial: given $f : J \to J'$, we obtain a map $\mathfrak{C}[\Delta^J] \to \mathfrak{C}[\Delta^{J'}]$ taking i to $f(i)$, and taking $P_{i,j}$ to $P_{f(i),f(j)}$. We extend this definition of $\mathfrak{C}$ from just the sets Δ^J to all simplicial sets by taking colimits in Set_Δ to colimits in Cat_Δ.

Definition. We now define the **simplicial nerve** $\mathcal{N}(\mathcal{C})$ of a simplicial category $\mathcal{C}$ by

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, \mathcal{N}(\mathcal{C})) = \mathrm{Hom}_{\mathsf{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Remark. By construction, $\mathfrak{C}$ preserves colimits. In fact it is a left adjoint, with right adjoint given by the simplicial nerve.

Remark. Giving an ordinary category $\mathcal{C}$ a discrete simplicial enrichment, the ordinary nerve agrees with the simplicial nerve. But in general, the simplicial nerve of a simplicial category does not agree with the ordinary nerve of the underlying category.

Proposition. *Let $\mathcal{C}$ be a simplicial category. If for any pair x, $Y \in \mathrm{ob}\,\mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(x,y)$ is a Kan complex, then the simplicial nerve $\mathcal{N}(C)$ is an ∞-category. In fact, we can equip Set_Δ and Cat_Δ with appropriate model structures so as to obtain this result as a Quillen adjunction.*

Definition. The **Bergner model structure** on Cat_Δ is given as follows:

(W) The weak equivalences are the maps $f : \mathcal{C} \to \mathcal{D}$ satisfying:

  (W1) For a, $b \in \mathrm{ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{D}}(fa, fb)$ is a weak equivalence of simplicial sets (equivalently, the geometric realization is a weak equivalence of spaces).

  (W2) The induced functor $hf : h\mathcal{C} \to h\mathcal{D}$ is an equivalence of categories.

(F) The fibrations are the maps $f : \mathcal{C} \to \mathcal{D}$ satisfying:

  (F1) For a_1, $a_2 \in \mathrm{ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(a_1, a_2) \to \mathrm{Hom}_{\mathcal{C}}(fa_1, fa_2)$ is a fibration of simplicial sets (i.e. a Kan fibration).

  (F2) For $a_1 \in \mathrm{ob}\,\mathcal{C}$, $b \in \mathrm{ob}\,\mathcal{D}$, and a homotopy equivalence $e : fa_1 \to b$, there exists an object $a_2 \in \mathrm{ob}\,\mathcal{D}$ and a homotopy equivalence $a_1 \to a_2$ in $\mathcal{C}$ such that $fd = e$.

(C) The cofibrations are defined by the left lifting property with respect to trivial fibrations.

Proposition. *Let $\mathcal{C}$ be a Bergner fibrant simplicial category $\mathcal{C}$ (so each* Hom*-space is a Kan complex). The counit map*

$$\mathrm{Map}_{\mathfrak{C}[N(\mathcal{C})]}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, y)$$

is a weak equivalence of simplicial sets.

Proposition. *There exists a model category structure on Set_Δ with properties*

(1) A map $p : S \to S'$ is a cofibration iff it is a monomorphism.

(2) A map $p : S \to S'$ is a weak equivalence, and called a **categorical equivalence***, iff the induced simplicial category $\mathfrak{C}[S] \to \mathfrak{C}[S']$ is an equivalence of simplicial categories.*

With this model structure on Set_Δ, and the Bergner model structure on Cat_Δ, the functors $\mathfrak{C}$ and $\mathcal{N}$ form a Quillen adjunction.

Proposition. *In fact this adjunction is a Quillen equivalence.*

Corollary. *If $\mathcal{C}$ is a locally fibrant simplicial category (i.e. each* Hom*-set is Kan fibrant), then the simplicial nerve $\mathcal{N}(\mathcal{C})$ is an ∞-category.*

Example. Let $\mathcal{M}$ be a simplicial model category. Then $\mathcal{M}_{cf}$ is locally fibrant, and its simplicial nerve is an ∞-category.