

# Model categories

Scribe notes from a talk by Vivian Bailey

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## I.

**Definition.** A **model category structure** on a category consists of three distinguished subcategories, the **weak equivalences**, **fibrations**, and **cofibrations**, satisfying axioms:

1. Weak equivalences have the “2 out of 3” property: if two of  $f$ ,  $g$ , and  $gf$  are weak equivalences then so is the third;
2. All three subcategories are closed under retracts;
3. Lifts always exist in the following diagrams:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow \text{---} & \downarrow \\ B & \longrightarrow & Y \end{array} \quad \begin{array}{ccc} A & \longrightarrow & X \\ \downarrow \sim & \nearrow \text{---} & \downarrow \\ B & \longrightarrow & Y \end{array}$$

4. Morphisms  $A \rightarrow B$  factor functorially into the following forms:

$$A \hookrightarrow X \xrightarrow{\sim} B, \quad \forall \xrightarrow{\sim} X \rightarrow B$$

**Definition.** A **model category** is a category with all small limits and colimits, equipped with a model category structure.

**Notation.** —

- An object  $X$  is **fibrant** if  $X \rightarrow *$  is a fibration, and **cofibrant** if  $\emptyset \rightarrow X$  is a cofibration.
- A **trivial fibration** is a fibration that is a weak equivalence. Similarly **trivial cofibration**.

**Note.** Some of this data determines others: if we only know the trivial fibrations and trivial cofibrations, we can for instance determine the fibrations as those maps with right lifting with respect to the trivial cofibrations.

Our functorial factorizations give us a functorial **cofibrant replacement**  $Q$  and **fibrant replacement**  $R$ , so that we have

$$\hookrightarrow QX \xrightarrow{\sim} X, \quad Y \xrightarrow{\sim} RY \twoheadrightarrow *.$$

**Definition.** We define the **homotopy category**  $\text{Ho } \mathcal{C} = F(\mathcal{C}, W^{-1}) / \sim$ , where the morphisms are composites of morphisms in  $\mathcal{C}$  and reversals of arrows in  $W$ , with relations

$$1_A = (1_A), \quad (f, g) = (g \circ f), \quad 1_{\text{dom } w} = (w, w^{-1}).$$

**Lemma.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor that sends maps of  $W$  to isomorphisms in  $\mathcal{D}$ . Then there exists a functor

$$\text{Ho } F : \text{Ho } \mathcal{C} \rightarrow \mathcal{D}$$

such that  $(\text{Ho } F) \circ \gamma = F$ , where  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$  is the canonical functor.

**Proposition.** Suppose that  $\mathcal{C}$  is a model category. Then the inclusion functors induce equivalences of categories

$$\text{Ho } \mathcal{C}_{cf} \hookrightarrow \text{Ho } \mathcal{C}_c \rightarrow \text{Ho } \mathcal{C},$$

where  $\mathcal{C}_{cf}$  is the full subcategory of  $\mathcal{C}$  on the cofibrant-fibrant objects,  $\mathcal{C}_c$  the cofibrant objects.

**Definition.** Let  $\mathcal{C}$  be a model category, and  $f, g : B \rightarrow X$  two maps in  $\mathcal{C}$ .

1. A **cylinder object** for  $B$  is a factorization of the fold map  $\nabla : B \amalg B \rightarrow B$  into

$$B \amalg B \hookrightarrow B' \xrightarrow{\sim} B.$$

2. A **left homotopy** from  $f$  to  $g$  is a map  $H : B' \rightarrow X$  for some cylinder object  $B'$ , such that

$$Hi_0 = f, \quad Hi_1 = g.$$

3. Dually we have the notion of right homotopy of maps.
4. We say two maps  $f$  and  $g$  are **homotopic**, and we write  $f \sim g$ , if they are both left and right homotopic.
5. A map  $f : B \rightarrow X$  is a homotopy equivalence if there exists a map  $h : X \rightarrow B$  with  $fh \sim 1_X$ ,  $hf \sim 1_B$ .

We have some results on homotopy in model categories:

- If  $f \stackrel{\ell}{\sim} g$ , then  $hf \stackrel{\ell}{\sim} hg$ .
- Left homotopy and right homotopy coincide on  $\mathcal{C}_{cf}$ .

**Proposition.** *If  $\mathcal{C}$  is a model category, then a map in  $\mathcal{C}_{cf}$  is a weak equivalence iff it is a homotopy equivalence.*

**Theorem.** *Let  $\mathcal{C}$  be a model category.*

(i) *We have an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \xrightarrow{\cong} \mathrm{Ho}\mathcal{C}_{cf} \xrightarrow{\cong} \mathrm{Ho}\mathcal{C}.$$

(ii) *The canonical map  $\gamma : \mathcal{C} \rightarrow \mathrm{Ho}\mathcal{C}$  identifies left or right homotopic maps.*

(iii) *If  $f : A \rightarrow B$  is a map in  $\mathcal{C}$  such that  $\gamma f$  is an isomorphism in  $\mathrm{Ho}\mathcal{C}$ , then  $f$  is a weak equivalence.*

## II. Quillen functors and derived functors

**Definition.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a **left Quillen functor** if  $F$  is a left adjoint and preserves cofibrations and trivial cofibrations. Dually for a **right Quillen functor**.

Suppose that  $(F, U, \phi)$  is an adjunction  $\mathcal{C} \rightarrow \mathcal{D}$ . Then  $(F, U, \phi)$  is a **Quillen adjunction** if  $F$  is a left Quillen functor. (Equivalently, if  $U$  is a right Quillen functor.)

[The notation for adjunctions is that  $F$  is the left adjoint,  $U$  is the right adjoint, and  $\phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, UY)$ .

**Definition.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories. If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a left Quillen functor, we can defined the **total left derived functor**

$$LF : \mathrm{Ho}\mathcal{C} \xrightarrow{\mathrm{Ho}Q} \mathrm{Ho}\mathcal{C}_c \xrightarrow{\mathrm{Ho}F} \mathrm{Ho}\mathcal{D},$$

and similarly the **total right derived functor**

$$RF : \mathrm{Ho}\mathcal{D} \xrightarrow{\mathrm{Ho}R} \mathrm{Ho}\mathcal{D}_f \xrightarrow{\mathrm{Ho}U} \mathrm{Ho}\mathcal{C}.$$

(It turns out that left Quillen functors preserve weak equivalences between cofibrant objects, and dually, which justifies this.)

For a Quillen adjunction  $(F, U, \phi)$ , it turns out that we get an adjunction  $(LF, RU, R\phi)$ , called the **derived adjunction**.

**Definition.** A Quillen adjunction  $(F, U, \phi)$  is called a **Quillen equivalence** if for all fibrant  $Y$  in  $\mathcal{D}$  and cofibrant  $X \in \mathcal{C}$ ,  $f : FX \rightarrow Y$  is a weak equivalence in  $\mathcal{D}$  iff  $\phi(f) : X \rightarrow UY$  is a weak equivalence in  $\mathcal{C}$ .

Now we can say that  $(F, U, \phi)$  is a Quillen equivalence iff the derived adjunction  $(LF, RU, R\phi)$  is an adjoint equivalence of categories.

### III. Cofibrantly generated model categories

**Definition.** —

- Let  $\lambda$  be an ordinal. A  $\lambda$ -**sequence** in  $\mathcal{C}$  is an colimit-preserving functor  $X : \lambda \rightarrow \mathcal{C}$ ,

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots,$$

where we identify  $\lambda$  with the poset category of ordinals less than  $\lambda$ .

- We say that  $X_0 \rightarrow \operatorname{colim}_{\beta < \lambda} X_\beta$  is the **composition** of the  $\lambda$ -sequence.
- We say an ordinal  $\gamma$  is  $\lambda$ -**filtered** iff the cofinality of  $\lambda$  is greater than  $\gamma$ . (The **cofinality** of a cardinal  $\lambda$  is the largest cardinal  $\gamma$  that can be embedded cofinally into  $\lambda$ , in the usual category-theoretic sense of the word “cofinal”.)
- Let  $\mathcal{D}$  be a collection of morphisms in  $\mathcal{C}$ , let  $A$  be an object of  $\mathcal{C}$ , and let  $\kappa$  be a cardinal. We say  $A$  is  $\kappa$ -**small relative to**  $\mathcal{D}$  if for all  $\kappa$ -filtered ordinals  $\lambda$  and all  $\lambda$ -sequences

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots$$

with  $X_\beta \rightarrow X_{\beta+1}$  in  $\mathcal{D}$ , the map of sets

$$\operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism.

**Notation.** —

- $I$ -inj is the morphisms that have right lifting with respect to  $I$ .
- $I$ -proj is the morphisms that have left lifting with respect to  $I$ .
- $I$ -fib is the morphisms that have left lifting with respect to  $I$ -inj.
- $I$ -cof is the morphisms that have right lifting with respect to  $I$ -proj.

The small object argument uses this machinery to produce functorial factorizations in the right circumstances.

**Example.** The model category of simplicial sets is formed as follows: the generating cofibrations are the inclusions of boundaries, and the generating trivial cofibrations are the inclusions of horns, and we apply the above to produce the rest of the model structure. We then have a Quillen equivalence with the model category  $\mathbf{Top}$ , given by geometric realization and the singular complex.