Introduction

Scribe notes from a talk by Brad Drew and Marc Levine

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1 Brad Drew

We'll discuss the basic motivation and idea of infinity-categories.

Infinity-categories are preceded by **triangulated categories**, which are clearly useful but have technical difficulties, centering around a failure of functoriality. For example:

- For $\mathcal{T}$ a triangulated category, $\mathrm{Fun}(\mathcal{C}, \mathcal{T})$ is no longer a triangulated category; so this makes taking nice homotopy (co)limits difficult;
- While we care a lot about ring spectra, there's no well-behaved theory of ring objects or commutative algebra in a triangulated category;

We can talk instead about **model categories**, where we have notions of cofibrations, fibrations, and weak equivalences; then we can take (co)fibrant replacements to compute derived functors, and we can essentially do everything we do in a triangulated setting, in a more functorial way. But here we still have a few issues:

- There's a lot of data that doesn't seem crucial to the homotopy theory;
- For $\mathcal{M}$ a model category, $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ carries several different model structures; it's possible to work with homotopy limits and colimits but you would use different model structures to work with each. Then it's difficult to relate the colimits with the limits.
- It's still technically complicated to work with ring objects or commutative algebra. (Re a question: for instance, algebras over an operad in a model category will only form a semi-model category.)

So we can pass to an in-between solution, namely **quasi-categories**; these are simplicial sets generalizing the nerve $N(\mathcal{C})$ of a category. Here we have a sort of composition that is associative only up to homotopy. This solves many of the above problems:

- $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a quasi-category; this isn't so surprising as maps between simplicial sets form a simplicial set;

- There are nice universal properties;
- There's a clean theory of commutative algebra.

Non-example. We will construct the stable motivic homotopy category $\mathcal{SH}(S) = \Delta^{\mathrm{op}}\mathsf{PSh}(\mathrm{Sm}_{/S})[(\mathbb{P}^1_S, \infty)^{-1}]/($ relations generated by $\mathbb{A}^1$ and the Nisnevich topology $)$. Then we should have a universal property:

$$\begin{array}{ccc} \mathcal{SH}(S)^{\wedge} & \dashrightarrow & D^{\otimes}(\text{stable quasicat}) \\ \uparrow & \nearrow & \\ (\mathrm{Sm}/S)^{\times} & & \end{array}$$

Example. Let $\mathcal{D}^{\otimes} = \mathcal{D}(S(\mathbb{Q}), \mathbb{Q})^{\otimes}$, the derived category of analytic sheaves of $\mathbb{Q}$ vector spaces. Then a functor

$$\mathcal{SH}(S)^{\wedge} \xrightarrow{\rho_B} \mathcal{D}(S(\mathbb{Q}), \mathbb{Q})^{\otimes}$$

arises from a morphism of sites

$$(\mathrm{Sm}/S)_{\mathrm{Nis}} \to \mathrm{AnSm}/S(\mathbb{C}).$$

(The latter is the site of analytic smooth schemes over the complex points of S.)

Example. Let $\mathcal{D}^{\otimes} = \mathcal{D}(\mathcal{D}_S)$, the derived category of $\mathcal{D}$-modules (coefficients for de Rham cohomology). We sholud end up with a functor from the stable homotopy category to derived $\mathcal{D}$-modules, but we don't seem to know how to build this with the language of model categories and Quillen functors. Quasi-categories are an adequate language, giving a functor

$$\mathcal{SH}(S)^{\wedge} \to \mathcal{D}(\mathcal{D}(S)).$$

2 Marc Levine

This part will be on some of the historical background for $\mathbb{A}^1$ homotopy theory, and perhaps Marc's particular historical background with it.

We begin with the diagram:

$$\begin{array}{ccccccc} \mathrm{Sm}/k & \longrightarrow & \mathrm{Spc}_{\bullet}(k) & \longrightarrow & \mathcal{H}_{\bullet}(k) & \longrightarrow & \mathcal{SH}(S) \\ \uparrow & & \| & & & & \\ \mathrm{SmProj}/k & & \text{presheaves of simplicial sets on } \mathrm{Sm}/k & & & & \end{array}$$

with $S = \mathrm{Spec}(k)$ the spectrum of a perfect field.

We study SmProj/k by means of "cohomological invariants", such as:

- $k = \mathbb{C}$, $X \mapsto H^*_{\text{sing}}(X(\mathbb{C}), \mathbb{Z})$,
- k of char. 0, $X \mapsto H^*_{\text{dR}}(X/k) = \mathbb{H}^*(X_{\text{Zar}}, \Omega^\bullet_{X/k})$,
- for ℓ, $X \mapsto H^*_{\text{et}}(\bar{X}/\bar{k}, \mathbb{Q}_\ell)$.

These all fall into the family of **Weil cohomology theories** on SmProj/k.

We also have the "cycle classes"

$$\begin{array}{ccc} Z^n(X) & = & \mathbb{Z}\{W \subseteq X \mid W \text{ irred. of codim } n\} \\ \downarrow \mathrm{cl}^n_X & & \\ H^{2n}(X) & \longleftarrow & \text{Weil cohomology} \end{array}$$

We can construct motives as a quotient of the category of correspondences:

$$\mathsf{Mot}_{k,H} \leftarrow \mathsf{Cor}_{k,H}[X],$$

where $\operatorname{Hom}_{\mathsf{Cor}_{k,H}}([X],[Y]) = Z^{\dim X}(X \times Y)/\sim_H$. In particular, if $\alpha \in \operatorname{End}_{\mathsf{Cor}}([X])$, then $\alpha^2 = \alpha$; α is idempotent.

Conjecture. We can lift the Künneth components

$$\alpha^i_H \in H^{2\dim X}(X \times X)$$

to idempotents $\alpha^i \in \operatorname{Hom}_{\mathsf{Cor}}(X, X)$, and $H^i(X) = ([X], \alpha^i)$.

Assuming this conjecture, we will get a map $\mathsf{Mot}_{k,\mathrm{CH}} \to \mathsf{Mot}_{k,\mathrm{hom}}$, where the former are the motives formed by the rational equivalence relation.

We should define the **Chow group**:

$$\mathrm{CH}^n(X) = Z^n(X)/R^n(X),$$

where

$$R^n(X) = \{\operatorname{div}(f) \mid f \in k(W), W \subseteq X \text{ of codimension } n-1\}.$$

We have Weil cohomology for SmProj/k, but the extension of this to Sm/k is called **Bloch–Ogers cohomology**. For $X/\mathbb{R}$, this looks like $H^*_{\text{et}}(X, \mathbb{Q}_\ell(*))$, and for $X/\mathbb{C}$, this is $H^*_{\text{sing}}(X, \mathbb{Z}$ with mixed Hodge structure.

We have the **mixed motives** MM_k, due to Beilinson (at least). Two constructions have been given:

1. Bloch's higher Chow groups

$$X \mapsto Z^q(X, *),$$

$$\mathrm{CH}^q(X, n) = H_n(Z^q(X, *)) = H^{2q-n}(X, \mathbb{Z}/Q),$$

2. Voevodsky's triangulated category of motives, $\mathrm{DM}(k)$, where $M(X)$ is the motive associated to X, with motivic cohomology given by

$$H^p(X, \mathbb{Z}(Q)) = \mathrm{Hom}_{DM(k)}(M(X), \mathbb{Z}(q)[p]).$$

The connection with K-theory is as follows: as a definition due to Beilinson, or a theorem due to others,

$$H^p(X, \mathbb{Q}(q)) = K_{2q-p}^{(q)}(X).$$

And now we come to the homotopy theory. In ordinary stable homotopy theory, we see $\mathcal{D}(\mathsf{Ab})$ as the category of $H\mathbb{Z}$-modules in the stable homotopy category. Here we have $DM(k)$ as the homotopy category of $H_{\mathrm{mot}}\mathbb{Z}$-modules in the stable motivic homotopy category. When k is of characteristic zero, we have $\mathsf{Mot}_{k,\mathrm{CH}} \hookrightarrow \mathrm{DM}(k)$, so that this relates to our stable homotopy category. We also have an adjunction between an Eilenberg–Maclane functor $\mathrm{Ho}(H_{\mathrm{mot}}\mathbb{Z}) \to \mathcal{SH}(k)$ and the functor $- \wedge H_{\mathrm{mot}}\mathbb{Z} : \mathcal{SH}(k) \to \mathrm{Ho}(H_{\mathrm{mot}}\mathbb{Z})$.