# PROGRAM FOR THE TALBOT WORKSHOP ON MOTIVIC HOMOTOPY THEORY

Mentors: Brad Drew and Marc Levine

Part I: **Foundations**

Lecture 1: Model categories. Recall the basic definitions: model structures, homotopy categories, Quillen adjunctions and derived functors. Discuss the equivalence between the homotopy category of a model category and the full subcategory of bifibrant objects modulo homotopy equivalence ([Hov99, 1.2.10]). As an example, introduce the standard model structure on the category of simplicial sets ([Hov99, §3.2]), emphasizing the role of the small object argument ([Hov99, 2.1.14]) in the construction of functorial factorizations rather than the characterization of the fibrations as Kan fibrations.

Further sources: [DS95, Hir03, Hov99, Qui67].

Lecture 2: Quasi-categories. Define quasi-categories (henceforth referred to as $(\infty, 1)$-categories) and the fundamental data associated thereto: objects, morphisms, mapping spaces, homotopy categories, equivalences and functors ([Lur09, §§1.1, 1.2], [Gro10, §1]. Discuss the Quillen equivalence between J. Bergner's model structure on the category of categories enriched in simplicial sets and A. Joyal's model structure on the category of simplicial sets ([Lur09, 2.2.5.1], [Gro10, 1.27]). In particular, define the homotopy coherent nerve functor and its left adjoint $\mathfrak{C}$, which encodes associativity of composition "up to coherent homotopy" ([Lur09, 1.1.5.2]). As examples, explain how ordinary categories and simplicial model categories give rise to $(\infty, 1)$-categories ([Gro10, 1.2, 1.30]).

Further sources: [BV73, Joy08, 3]

Lecture 3: Basic constructions. Define joins, slices and limits in $(\infty, 1)$-categories ([Lur09, §1.2], [Gro10, §2]). Define stable $(\infty, 1)$-categories ([Lur12, §1.1], [Gro10, §5.1]) and remark that the homotopy category of a stable $(\infty, 1)$-category admits a natural triangulated category structure. Define Cartesian fibrations ([Lur09, §2.4], [Gro10, §3.2]) and the $(\infty, 1)$-categorical Grothendieck construction ([Lur09, 3.2.0.1]). As an illustration, introduce adjunctions of $(\infty, 1)$-categories as Cartesian bifibrations ([Lur09, 5.2.2.1]) and also via unit transformations ([Lur09, 5.2.2.8]). Time permitting, explain how Quillen adjunctions induce adjunctions of $(\infty, 1)$-categories ([Lur12, 1.3.4.21]).

Lecture 4: Localization and the unstable motivic $(\infty, 1)$-category. Define presentable $(\infty, 1)$-categories ([Lur09, §5.5], [Gro10, §2.6]) and discuss the associated theory of localizations ([Lur09, 5.5.4.15, 5.5.4.20]). Construct the unstable motivic $(\infty, 1)$-category $\mathcal{H}(S)$ of a noetherian scheme $S$ of finite dimension and explain its universal property ([Rob13, Thm. 5.2]). Time permitting, talk about the connections between presentable $(\infty,1)$-categories and combinatorial model categories ([Lur09, A.3.7.6]) and Jeff Smith's theorem ([Bar10, 2.2]).

Further sources: [Bek00, MV99, DLØ07].

Lecture 5: Symmetric monoidal $(\infty,1)$-categories and the stable motivic $(\infty,1)$-category. Define symmetric monoidal $(\infty,1)$-categories ([Lur12, 2.0.0.7], [Gro10, §4.1], [Rob13, §3.1.3]). By only considering the symmetric case, one might avoid a long digression on general $\infty$-operads. Interpret symmetric monoidal $(\infty,1)$-categories as commutative algebra objects in the symmetric monoidal $(\infty,1)$-category of $(\infty,1)$-categories ([Lur12, 2.4.2.6], [Rob13, §§3.2.2, 3.6.1]). As examples, consider the $(\infty,1)$-category $\mathcal{H}(S)$ of the previous talk as well as its pointed version $\mathcal{H}_\bullet(S)$ ([Rob13, §5.2]).

Explain formal inversion of objects in presentable symmetric monoidal $(\infty,1)$-categories ([Rob13, Def. 4.7]). Use this to construct the $(\mathbf{P}^1,\infty)$-stable and $S^1$-stable motivic $(\infty,1)$-categories $\mathcal{SH}(S)$ and $\mathcal{SH}_{S^1}(S)$ of a noetherian scheme $S$ of finite dimension ([Rob13, Def. 5.10]) by formally inverting the objects $(\mathbf{P}^1,\infty)$ and $S^1 := \Delta^1/\partial\Delta^1$ of $\mathcal{H}_\bullet(S)$. State the universal property of these objects ([Rob13, Cor. 5.11]).

Further sources: [Ayo07, CD09, DLØ07].

Lecture 6: Symmetric spectra. Define the stable model structures on the categories of spectra and symmetric spectra associated to a cofibrant object of a symmetric monoidal model category ([Hov01]). Compare the associated $(\infty,1)$-categories with the $(\infty,1)$-category obtained by the process of formal inversion introduced in the previous talk ([Rob13, §4.3]). Time permitting, discuss higher algebra in the context of symmetric monoidal model categories, including but not limited to such topic as ring spectra, $E_\infty$-ring spectra and modules over such ([SS00, Spi01]).

Lecture 7: Étale classifying spaces and representability of algebraic $K$-theory. This is essentially Chap. 4 of [19]. Give a brief outline of the results of §4.1 (classifying spaces, $G$-torsors, etc.). Discuss the geometric construction of étale classifying spaces §4.2 and go through the arguments in §4.3.2 showing that algebraic $K$-theory is represented in $\mathcal{H}_\bullet(S)$ by the infinite Grassmannian.

Part II: **Purity, connectivity, $t$-structures and the endomorphisms of the sphere spectrum**

Lecture 1. The purity theorem and consequenses. Prove the purity theorem

**Theorem 1.** *Let $i : Z \to X$ be a closed immersion in $\mathbf{Sm}/S$. There is a canonical isomorphism $Th(N_{Z/X}) \cong X/(X \setminus Z)$ in $\mathcal{H}_\bullet(S)$.*

Use this to show

**Theorem 2** (lemma 3.3.6 of [18])**.** *Let $k$ be a perfect field, $\mathcal{X} \in \mathbf{Spc}(k)$ be an $\mathbb{A}^1$-local space. Then $\mathcal{X}$ is $n$-connected if and only if $\mathcal{X}$ is weakly $n$-connected, i.e., $\pi_m^{\mathbb{A}^1}(\mathcal{X}) = 0$ for all $m \leq n$ iff for all fields $F$ finitely generated over $k$, $\pi_m(\mathcal{X}(F)) = 0$ for all $m \leq n$.*

Sources: [19, §3.2], [15], [18, §3].

Lecture 2. Morel's $\mathbb{A}^1$-connectivity theorem and homotopy $t$-structures Topics to be covered: First prove the $\mathbb{A}^1$-connectedness theorem:

**Theorem 3.** *Let $k$ be a perfect field. Let $\mathcal{X}$ be a presheaf of spectra on $\mathbf{Sm}/k$ such that each Nisnevich stalk of $\mathcal{X}$ is $n$-connected. Then $\pi_m^{\mathbb{A}^1}(\mathcal{X}) = 0$ for $m \leq n$.*

Then discuss the homotopy $t$-structures for $S^1$-spectra and $T$-spectra and the heart of the homotopy $t$-structure (homotopy modules). Construct examples: Milnor $K$-theory, Witt sheaves. Source: [16], for the Witt sheaves, cite results from [6], [25] and [4] without proof.

Lecture 3. Endomorphisms of the sphere spectrum. Give a construction of Milnor-Witt $K$-theory, the homotopy module $J^*$ and show

**Theorem 4.** *For a field $F$ (of char. not 2), there is a canonical isomorphism $J^n(F) \cong K_n^{MW}(F)$. In particular, $K_0^{MW}(F) \cong GW(F)$, the Grothendieck-Witt group of quadratic forms over $F$.*

Then sketch Morel's arguments that show

**Theorem 5.** *Let $k$ be a perfect field of charactersitc $\neq 2$. There is a natural isomorphism of sheaves $J^n \cong \pi_0^{\mathbb{A}^1}(\Sigma^n_{\mathbb{G}_m}\mathbb{S}_k)$, where $\mathbb{S}_k$ is the motivic sphere spectrum in $\mathcal{SH}(k)$.*

Sources: The main source is [16], supplemented by [17].

Part III: **Algebraic cobordism and oriented theories**

Lecture 1. Introduction: Constructions of H$\mathbb{Z}$, $K$-theory, MGL as objects in $\mathcal{SH}(S)$ (for H$\mathbb{Z}$, take $S = \operatorname{Spec} k$, $k$ a characteristic zero field) plus the conjectures in Voevodskys papers "Open problems. and "A possible new approach...". Include in this a construction of the slice tower and the functors $f_n, s_n$. Sources: [32] [33], [34]

Lecture 2. The universality of MGL. This is the main thereom in [27]. Discuss oriented commutative ring spectra in $\mathcal{SH}(S)$, and also Panin's results from [26] that relate Chern classes, Thom classes and projective pushforward.

Lecture 3. Landweber Exactness. Show how to construct a (co)homology theory from MGL and a Landweber exact formal group law. This is the main result in [22], relying on results from [23] (which go back to Landweber [9, 10]) and [20] (going back to [24]).

Lecture 4. The computation of the slices of a Landweber exact theory. These are the main results from [29, 30], computing first the sllices of MGL, then extending to a compuation of the slices of a Landweber exact theory.. In addition, use [21] to show that $K$-theory is a Landweber exact theory, giving as an application a computation of the slices of $K$-theory. The lecturer should explain the computation of the slices of MGL assuming the main result of Hoyois (discussed in lectures 5-7 below) as well as Voevodsky's theorem on the 0th slice of the sphere spectrum [31].

Lectures 5,6,7. Prove the main result of Hopkins-Morel (in the case of a field $k$ of char. 0): Take polynomial generators $x_1, x_2, \dots$ for $MU^*$. Then the classifying map $\mathrm{MGL} \to \mathrm{H}\mathbb{Z}$ descends to a isomorphism $\mathrm{MGL}/(x_1, x_2, \dots) \to \mathrm{H}\mathbb{Z}$. This is used in llecture 4 to compute the slices of MGL. Consequence is the Hopkins-Morel spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(n-q) \otimes MU^{-2q} \Longrightarrow \mathrm{MGL}^{p+q,n}(X).$$

Main source: [7]. The lecturers should assume the construction and facts on Voevodsky's motivic Steenrod operations [35], the construction of Voevodsky's triangulated category of motives $\mathcal{DM}(k)$ [36], and the main result of Röndigs-Østvær giving an equivalence of $\mathcal{DM}(k)$ with the homotopy category of $\mathrm{H}\mathbb{Z}$-modules in $\mathcal{SH}(k)$.

A suggested division of labor, sections refer to [7]:

Lecture 5. Introduction, sections 1-3
Lecture 6. Section 4
Lecture 7. Section 5 and the main result section 6.

Lecture 8. Geometric aspects: describing $\mathcal{E}^{2*,*}$ in terms of $\Omega^*$. The lecturer should give an overview of Levine-Morel-Pandharipande algebraic cobordism $\Omega^*$ and prove the main result of [12], that the classifying map $\Omega^*(-) \to \mathrm{MGL}^{2*,*}(-)$ is an isomorphism. If time permits, give the extension to Landweber exact theories and their (slice) connected covers [11]. If even more time permits, discuss the example of $K$-theory and connective $K$-theory [1].

## References


[1] S. Dai and M. Levine, *Connective algebraic $K$-theory*, preprint (2012) 31 pages `arXiv:1212.0228 [math.KT]`, to appear Journal of $K$-Theory.

[2] Dundas, B. I., Levine, M., Østvær, P. A., Röndigs, O. and Voevodsky, V. **Motivic homotopy theory**. Lectures from the Summer School held in Nordfjordeid, August 2002. Universitext. Springer-Verlag, Berlin, 2007

[3] Fresse, B. *Modules over Operads and Functors*, LNM 1967, Springer 2009. Available at `http://math.univ-lille1.fr/~fresse/OperadModuleFunctors.pdf`

[4] Gille, Stefan, *On Witt groups with support.* Math. Ann. 322 (2002), no. 1, 103137.

[5] M. Hopkins, F. Morel, *Slices of MGL.* lecture (Hopkins), Harvard Univ. Dec. 2, 2004. Available as "Week 8" on a webpage of T. Lawson presenting notes from a seminar given by Mike Hopkins, Harvard, fall of 2004. `http://www.math.umn.edu/~tlawson/motivic.html`

[6] Hornbostel, Jens, *$\mathbb{A}^1$-representability of Hermitian $K$-theory and Witt groups.* Topology 44 (2005), no. 3, 661–687

[7] M. Hoyois, *From algebraic cobordism to motivic cohomology.* J. reine u. ang. Math. (online-June 2013) `http://dx.doi.org/10.1515/crelle-2013-0038`

[8] Jardine, J.F., *Motivic symmetric spectra*, *Doc. Math.* 5 (2000), 445–553.

[9] Landweber, Peter S., *Associated prime ideals and Hopf algebras.* J. Pure Appl. Algebra 3 (1973), 43–58.

[10] Landweber, Peter S., *Annihilator ideals and primitive elements in complex bordism.* Illinois J. Math. 17 (1973), 273–284.

[11] M. Levine, *Motivic Landweber exact theories and their connective covers* Preprint 2014 `arXiv:1401.0284 [math.AG]`

[12] Levine, Marc, *Comparison of cobordism theories.* J. Algebra 322 (2009), no. 9, 3291–3317

[13] Levine, M.; Morel, F., **Algebraic cobordism**. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[14] Levine, M.; Pandharipande, R. *Algebraic cobordism revisited.* Invent. Math. 176 (2009), no. 1, 63–130.
[15] Morel, Fabien, *The stable* $\mathbb{A}^1$*-connectivity theorems.* K-Theory 35 (2005), no. 1-2, 1–68.
[16] Morel, Fabien, *On the motivic* $\pi_0$ *of the sphere spectrum.* Axiomatic, enriched and motivic homotopy theory, 219–260, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004.
[17] Morel, Fabien, *Sur les puissances de l'idéal fondamental de l'anneau de Witt.* Comment. Math. Helv. 79 (2004), no. 4, 689–703.
[18] Morel, Fabien, *An introduction to* $\mathbb{A}^1$*-homotopy theory.* Contemporary developments in algebraic K-theory, 357–441 (electronic), ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.
[19] Morel, F. and Voevodsky, V., $\mathbb{A}^1$*-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999), 45–143.
[20] Naumann, Niko; Spitzweck, Markus, *Brown representability in* $\mathbb{A}^1$*-homotopy theory.* J. K-Theory 7 (2011), no. 3, 527–539.
[21] Naumann, Niko; Spitzweck, Markus; Østvær, Paul Arne, *Chern classes, K-theory and Landweber exactness over nonregular base schemes.* Motives and algebraic cycles, 307–317, Fields Inst. Commun., 56, Amer. Math. Soc., Providence, RI, 2009.
[22] Naumann, Niko; Spitzweck, Markus; Østvær, Paul Arne, *Motivic Landweber exactness.* Doc. Math. 14 (2009), 551–593.
[23] Naumann, Niko, *The stack of formal groups in stable homotopy theory.* Adv. Math. 215 (2007), no. 2, 569–600.
[24] Neeman, Amnon, *On a theorem of Brown and Adams.* Topology 36 (1997), no. 3, 619–645.
[25] Panin, I., *Homotopy invariance of the sheaf* $W_{Nis}$ *and of its cohomology.* Quadratic forms, linear algebraic groups, and cohomology, 325–335, Dev. Math., 18, Springer, New York, 2010.
[26] Panin, Ivan, *Oriented cohomology theories of algebraic varieties. II* (After I. Panin and A. Smirnov). Homology, Homotopy Appl. 11 (2009), no. 1, 349–405.
[27] Panin, I., Pimenov, K. and Röndigs, O., *A universality theorem for Voevodsky's algebraic cobordism spectrum.* Homology, Homotopy Appl. **10** (2008), no. 2, 211226.
[28] Röndigs, Oliver; Østvær, Paul Arne, *Modules over motivic cohomology.* Adv. Math. 219 (2008), no. 2, 689–727.
[29] Spitzweck, Markus, *Relations between slices and quotients of the algebraic cobordism spectrum.* Homology, Homotopy Appl. **12** (2010), no. 2, 335–351.
[30] Spitzweck, Markus, *Slices of motivic Landweber spectra.* J. K-Theory 9 (2012), no. 1, 103–117.
[31] Voevodsky, V., *On the zero slice of the sphere spectrum.* Proc. Steklov Inst. Math. 2004, no. 3 (246), 93–102.
[32] Voevodsky, V., *Open problems in the motivic stable homotopy theory. I.* Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), 3–34, Int. Press Lect. Ser., **3, I** Int. Press, Somerville, MA, 2002.
[33] Voevodsky, V., *A possible new approach to the motivic spectral sequence for algebraic* $K$*-theory.* Recent progress in homotopy theory (Baltimore, MD, 2000) 371–379, Contemp. Math., **293** Amer. Math. Soc., Providence, RI, 2002.
[34] Voevodsky, V., $\mathbb{A}^1$*-homotopy theory*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 579–604.
[35] Voevodsky, Vladimir, *Reduced power operations in motivic cohomology.* Publ. Math. Inst. Hautes Études Sci. No. 98 (2003), 1–57.
[36] Voevodsky, Vladimir, *Triangulated categories of motives over a field.* Cycles, transfers, and motivic homology theories, 188–238, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.
[Ayo07] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.
[Bar10] Clark Barwick. On left and right model categories and left and right Bousfield localizations. *Homology, Homotopy and Applications*, 12(2):245–320, 2010.
[Bek00] Tibor Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.
[BV73] John Michael Boardman and Rainer M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin, 1973.

[CD09] Denis-Charles Cisinski and Frédéric Déglise. Local and stable homological algebra in Grothendieck abelian categories. *Homology, Homotopy and Applications*, 11(1):219–260, 2009.

[DLØ07] B. I. Dundas, M. Levine, P. A. Østvær, O. Röndigs, and V. Voevodsky. *Motivic homotopy theory*. Universitext. Springer-Verlag, Berlin, 2007. Lectures from the Summer School held in Nordfjordeid, August 2002.

[DS95] William G. Dwyer and Jan Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[Gro10] Moritz Groth. A short course on $\infty$-categories. `http://arxiv.org/abs/1007.2925`, 2010.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Hov01] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[Joy08] André Joyal. Notes on quasi-categories. `http://www.math.uchicago.edu/ may/IMA/Joyal.pdf`, 2008.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur12] Jacob Lurie. Higher Algebra. `http://www.math.harvard.edu/ lurie/papers/higheralgebra.pdf`, August 2012.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Rob13] Marco Robalo. Noncommutative Motives I: A Universal Characterization of the Motivic Stable Homotopy Theory of Schemes, June 2013. `http://arxiv.org/pdf/1206.3645.pdf`.

[Spi01] Markus Spitzweck. Operads, algebras and modules in general model categories. `http://arxiv.org/abs/math/0101102`, 2001.

[SS00] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.