TALBOT TALK: MORAVA E-THEORY AND CHANGE-OF-RINGS

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Abstract. These are the notes for a Talbot talk on Morava E-theory and change-of-rings theorem. After a recap added to make the notes more self-contained, we define Morava E-theory. We discuss then change-of-rings theorems in general, after which we sketch the proof of two important particular such theorems, the Miller-Ravenel and Morava change-of-rings theorems.

Contents

1. The big picture

One of the most important open problems in algebraic topology is understanding the stable homotopy groups of spheres. Such understanding either involves being able to do explicit computations or finding deeper structures. The most successful approach to this problem is chromatic homotopy theory, which tries to decompose the stable homotopy groups into different (eventually) periodic phenomena, analogous to the decomposition of light into its composite frequencies.

Let us sketch a very low-brow computational approach to chromatic homotopy theory. We always work at a prime p, since there is essentially no difference between knowing $\pi_*\mathbb{S} \otimes \mathbb{Z}_{(p)}$ for all primes p and knowing them integrally. Here $\mathbb{Z}_{(p)}$ denotes the integers localized at p, obtained by inverting all primes not equal to p. That there is essentially no difference uses several facts about the stable homotopy groups: they are finitely-generated in each degree (using Serre classes) and $\pi_*\mathbb{S} \otimes \mathbb{Q}$ is a single copy of $\mathbb{Q}$ concentrated in degree zero (using rational homotopy theory).

The big picture is this:

(i) The BP_* Adams-Novikov spectral sequence converges to the p-local stable homotopy groups $\pi_*\mathbb{S} \otimes \mathbb{Z}_{(p)}$ and has E^2-page $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$.

(ii) The chromatic spectral sequence converges to the E^2-page of the BP_* Adams-Novikov spectral sequence and has as E^1-page similar Ext-groups $\mathrm{Ext}^{s,t}(BP_*BP)(BP_*, v_n^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty))$.

(iii) Using either n Bockstein spectral sequences or n short exact sequences one computes the E^1-page of the chromatic spectral sequence out of $\mathrm{Ext}^{s,t}(BP_*BP)(BP_*, v_n^{-1}BP_*/I_n)$.

See figure 1 for a diagram containing all these steps. In this note we will explain how to simplify the Ext-groups appearing in steps (ii) and (iii) using change of rings theorems. These theorems identify those Ext-groups in terms of Ext-groups over smaller Hopf algebroids or group cohomology.

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$$\begin{tikzcd}[row sep=large]
\boxed{\pi_*\mathbb{S}_{(p)} = \pi_*(\mathbb{S}) \otimes \mathbb{Z}_{(p)}} \\
\boxed{\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)} \arrow[u, Rightarrow, "\text{BP_*-Adams Novikov spectral sequence}", "\text{Adams resolutions, e.g. a reduced bar construction in spectra}"'] \\
\boxed{\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, v_n^{-1}BP_*/(p^\infty, \ldots, v_1^\infty))} \arrow[u, Rightarrow, "\text{Chromatic spectral sequence}", "\text{Chromatic resolutions of BP_*BP-comodules}"'] \\
\boxed{\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)} \arrow[u, Rightarrow, shift right=12ex, "\text{n Bockstein spectral sequences}"] \arrow[u, phantom, "\text{Short exact sequences relating different chromatic resolutions}"] \arrow[u, shift left=12ex, "\text{n long exact sequences}"']
\end{tikzcd}$$

FIGURE 1. An overview of the basics of the chromatic approach to computing the stable homotopy groups of spheres.

1.1. **Step (i): the Adams-Novikov spectral sequence.** Our approach to computing the homotopy groups of $\mathbb{S}$ localized at p starts with the BP_* Adams-Novikov spectral sequence. This is obtained from an Adams resolution of the sphere $\mathbb{S}$, which for example can be obtained from a reduced cobar construction. Using several facts about BP_*, in particular that BP_*BP is flat over BP_* and that BP_* has π_0 equal to $\mathbb{Z}_{(p)}$ and zero homotopy groups in negative degree, one deduces the following E^2-page and abutment

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Longrightarrow \pi_{t-s}\mathbb{S} \otimes \mathbb{Z}_{(p)}$$

Here $\mathrm{Ext}^{s,t}_{BP_*BP}$ is the derived functor of $\mathrm{Hom}_{BP_*BP}(BP_*, -)$ in the category of left BP_*BP-comodules, with s the homological and t the internal grading. Let us take a closer look at the Adams-Novikov spectral sequence.

Suppose E be any ring spectrum and Y be any other spectrum. One can always construct a so-called E-Adams resolution for Y, for example using the canonical resolution which is nothing but a normalized cobar construction in the world of spectra (a fact that is somewhat obscured in the literature, but a hint of it can be found in [Rav04, proposition 2.2.11]). The result is a pair of sequence of spectra $\{Y_s\}$ and $\{K_s\}$ fitting together by fiber sequences that stitch together in an unfolded exact couple in the world of spectra. Applying homotopy groups to the unfolded exact couple we get a spectral sequence whose E^2-page is $E^2_{s,t} = \pi_{t-s}(K^s)$.

Without more information about E and Y, we don't know whether it converges or what it converges to and we don't know its E^2-term well enough to calculate anything. For both these things we need extra conditions, which are satisfied in many circumstances of interest (though not always, in particular the identification of E^∞ can't always be pushed as far as we'll do it here).

We start with the description of the E^2-page. To state it and really understand it, we need to take a small detour through the world of Hopf algebroids and the dual world of stacks. A Hopf algebroid over a commutative ring K (often $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$ in the cases we are considering) is a pair of K-algebras (A, Γ) with certain maps between them. These maps are such that (A, Γ) with all its maps forms a cogroupoid object in K-algebras. Let's give two rephrasings of this: (i) the functors $R \mapsto \mathrm{Hom}_K(A, R)$ and $R \mapsto \mathrm{Hom}_K(\Gamma, R)$ naturally assign to a K-algebra R the sets of objects and morphisms of a groupoid, (ii) applying Spec to get a groupoid scheme, i.e. a prestack, over $\mathrm{Spec}(K)$. In the latter case one can consider (A, Γ) as presenting an affine stack $\mathcal{M}_{(A,\Gamma)}$ by stackifying the prestack $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$.

A left comodule M over a Hopf algebroid (A, Γ) is a left A-module M together with a coaction $\phi : M \to \Gamma \otimes_A M$ satisfying various compatibilities with the Hopf algebroid structure. Equivalently it is a sheaf over the stack represented by the Hopf algebroid. If Γ is flat over A, as we're assuming in the case (E_*, E_*E), the category of left comodules is an abelian category with enough injectives. We can hence right derive the functor $\mathrm{Hom}_\Gamma(N, -)$ to get Ext groups.

For modules over a ring R we have a close relatoin between Hom_R and tensor product $\otimes_R$ in the form of an adjunction. There is a similar construction for comodules over a Hopf algebroid using the cotensor product, though there is no adjunction. If the tensor product $A \otimes_R B$ is a coequalizer of the two natural maps $A \otimes R \otimes B \to A \otimes B$, then the cotensor product $A \square_\Gamma B$ is the equalizer of the two natural map $A \otimes B \to A \otimes \Gamma \otimes B$ (this construction of $\square_\Gamma$ as a limit should give intuition why there is no adjunction). It turns out that for Hopf algebroids Hom and cotensor product $\square_\Gamma$ are pretty much equivalent: $\mathrm{Hom}_\Gamma(N, M) \cong \mathrm{Hom}_A(N, A) \square_\Gamma M$. In calculations one will therefore sometimes substitute Cotor for Ext.

These comodules correspond to the quasicoherent sheaves on $\mathcal{M}_{(A,\Gamma)}$ and indeed $A \square_\Gamma -$ can interpreted as taking global sections of the sheaf. Writing $\mathcal{F}_M$ for the quasicoherent sheaf on $\mathcal{M}_{(A,\Gamma)}$ corresponding to the Γ-comodule M we thus have that

$$\mathrm{Ext}^*_\Gamma(A, M) \cong R^*\Gamma(\mathcal{M}_{(A,\Gamma)}, \mathcal{F}_M) = H^*(\mathcal{M}_{(A,\Gamma)}, \mathcal{F}_M)$$

It is exactly these groups that appear in good circumstances on the E^2-page, giving an interpretation of the E^2-term of the Adams-Novikov spectral sequence in terms of homological algebra of comodules over Hopf algebroids or equivalent in terms of homological algebra of quasicoherent sheaves.

Identification of E^2-page: One sufficient condition is that E is a commutative ring spectrum and $E_*E = \pi_*(E \wedge E)$ is a flat E_*-module. In this case the pair (E_*, E_*E) is a Hopf algebroid and $E_*(Y)$ a comodule over that Hopf algebroid. We can algebraically identify the E^2-term as

$$E^2_{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y))$$

The other two properties of the Adams-Novikov we have to discuss are more topological:

Convergence: For convergence it suffices that Y is connective and has finitely generated homotopy groups.

Identification of E^∞: In general under the previous conditions the spectral sequence converges to $\pi_*(Y^\wedge_E)$, the homotopy groups of the completion of Y at E, and there are many cases of interest where the identification ends here.

However, in general completion can be an extremely complicated operation (it is defined by the totalization of a cosimplicial spectrum and such totalizations are in general hard to study) and any way of rewriting it would be very useful. If E is connective, finitely generated in each degree and satisfies some condition on π_0, we have that the completion at E is determined by the acyclicity type of E. In particular, if $\pi_0(E) \cong \mathbb{Z}_{(p)}$ it is localization at p and under the assumption that $\pi_*(Y)$ is finitely generated in each degree, $\pi_*(Y^\wedge_p) \cong \pi_*(Y) \otimes \mathbb{Z}_{(p)}$ [Rav04, theorem 2.2.13]. Here $\mathbb{Z}_{(p)}$ denotes the integers localized at p (that is, every prime except p is inverted).

The conclusion of this discussion is that under quite general circumstances we can construct a spectral sequence, called that E Adams-Novikov spectral sequence, described by (if $\pi_0(E) \cong \mathbb{Z}(p)$):

$$E^2_{s,t} = \mathrm{Ext}_{E_*E}(E_*, E_*(Y)) \Rightarrow \pi_*(Y) \otimes \mathbb{Z}_{(p)}$$

We will often set $Y = \mathbb{S}$, so that we can compute the stable homotopy groups of spheres at p, but this doesn't need to be the case. For example, the case $E = H\mathbb{F}_2$ and $Y = MU$ is actually important to determine to $MU_* = \pi_*(MU)$ and goes into Quillen's proof of the identification of MU_* with the Lazard ring L.

The trick for using the E Adams-Novikov spectral sequence is to make a good choice for E. In general, one want to balance the following two demands: (i) there should not be many differentials, (ii) the E^2-page should be computable, in particular sparse or concentrated in particular bidegrees. Of course one can't have both at the same time, since the complexity of the stable homotopy groups of spheres has to come from somewhere. More specifically, one looks for reasonably complicated spectra – easy enough to compute E_* and E_*E explicitly but hard enough to have a lot happening in the computation of E^2-page – that additionally have nice algebraic properties for their Hopf algebroids.

Since from the nilpotence and periodicity theorems we know that MU controls a lot of the homotopy theory of finite spectra, it should come as no surprise that it is a good candidate for a Adams-Novikov spectral sequence. Because we are interested in a single prime at a time we should look at $MU_{(p)}$ instead. This splits into infinitely many copies of suspensions of a spectrum called BP. This BP plays the role of MU for p-local finite spectra. It is the BP Adams-Novikov spectral sequence that is the main object of our study. It is given by

$$E^2_{s,t} = \mathrm{Ext}_{BP_*BP}(BP_*, BP_*(Y)) \Rightarrow \pi_*(Y) \otimes \mathbb{Z}_{(p)}$$

Let's now look at a few facts about BP. Note that BP depends on the prime p that we have fixed, something a more pedantic person that most algebraic topologists would have included in the notation. We have that $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_i| = 2(p^i - 1)$ and $BP_*BP \cong BP_*[t_1, t_2, \ldots]$ with $|t_i| = 2(p^i - 1)$. There is a more algebraic identification of this Hopf algebroid. Just like (MU_*, MU_*MU) is isomorphic to (L, LB) where L is the Lazard ring corepresenting formal group laws and LB corepresents pairs of formal group laws and an isomorphism between them, (BP_*, BP_*BP) is isomorphic to (V, VT) where V corepresents p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebras and VT corepresents pair of these p-typical formal group laws and an isomorphism between. In particular, one should think of (BP_*, BP_*BP) as being the stack of p-typical formal group laws modulo isomorphisms, or equivalently the stack $\mathcal{M}_{FG}$ of formal groups over $\mathbb{Z}_{(p)}$ (actually the guys admitting a coordinate, but let's not worry about this). In that sense the BP Adams-Novikov spectral sequence's E^2-term consists of cohomology of the quasicoherent sheaf $\mathcal{F}_{BP_*(Y)}$ over the stack of p-typical formal group laws corresponding to $BP_*(Y)$.

Since both BP_* and BP_*BP are concentrated in degrees divisible by $2(p-1)$, the algebraic cobar resolution used for computing Cotor and hence Ext tells us that $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ is concentrated in columns of degrees divisible by $2(p-1)$ as well. This sparseness result tells us that in the beginning the E^2-term pretty much determines E^∞-term. For example, it gives an easy proof that the first p-torsion in the stable homotopy groups of spheres appears in degree $2p-3$. So it makes sense to focus our study on the structure of the E^2-term. In fact, a good understanding of the E^2-term will allow one to get a grip on the differentials in general.

1.2. **Step (ii) and (iii): the chromatic spectral sequence and its friends.** The main method for doing this is the chromatic spectral sequence, which is supposed to isolate the v_n-periodic phenomena in the E^2-term.

To talk about it we need to discuss the invariant prime ideals of BP_*. An invariant prime ideal of a Hopf algebroid (A, Γ) is an ideal I of A which is a sub-Γ-comodule of A or equivalently satisfies $\eta_R(I) \subset I\Gamma$. Landweber proved that the only invariant prime ideal of (BP_*, BP_*BP) are the I_n for $0 \leq n \leq \infty$. Explicitly, these are given by $I_0 = 0$, $I_n = (p, v_1, v_2, \ldots, v_{n-1})$ (where we think of p as v_0) and $I_\infty = BP_*$. Alternatively, one can think of invariant prime ideals as corresponding to the closed substacks of the affine stack represented by (A, Γ). In the stack of p-typical formal group laws, I_n corresponds to the closed substack $\mathcal{M}^{\geq n}_{FG}$ of formal groups of height at least n. One should think of the chromatic spectral sequence as arising from this filtration, but we will not make this precise.

Let $BP_*/(p^\infty)$ by the limit of $BP_*/(p^n)$ and similarly define $BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty)$. There are short exact sequences of BP_*BP-comodules

$$0 \to BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty) \xrightarrow{v_n} v_n^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty) \to BP_*/(p^\infty, v_1^\infty, \ldots, v_n^\infty) \to 0$$

which are collectively called the chromatic resolution of BP_*. More precisely, the middle term is denoted M_n^0 and we have a long exact sequence

$$BP_* \to M_0^0 = p^{-1}BP_* \to M_1^0 = v_1^{-1}BP_*/(p^\infty) \to \dots$$

There exists a similar chromatic resolution of BP_*/I_m where the version of the middle terms is denoted M_n^m. Applying $E^{s,t}(-) := \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, -)$ and splicing the resulting exact sequences together in an unfolded exact couple we get the so-called chromatic spectral sequences

$$E^1_{s,t} = E^{s,t}(v_n^{-1}BP_*/(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty)) \Rightarrow E^{s,t}(BP_*)$$

$$E^1_{s,t} = E^{s,t}(M_m^n) \Rightarrow E^{s,t}(BP_*/I_m)$$

The former is certainly something we're interested in, but the second is also important. This can be seen as follows: there are short exact sequences of BP_*BP-comodules

$$0 \to M_n^{m-1} \longrightarrow M_n^m \xrightarrow{v_m} M_n^m \to 0$$

which gives either m long exact sequences or m Bockstein spectral sequence connecting the Ext-groups $E^{s,t}(M_0^m)$ to $E^{s,t}(M_m^0)$. So a good calculation of $E^{s,t}(M_0^m) = E^{s,t}(v_m^{-1}BP_*/I_m)$ is very helpful in figuring out the E^1-page of the chromatic spectral sequence.

1.3. **Change-of-rings and the goal of this note.** The main goal of this talk is the computation of these parts leading to the E^1-term of the chromatic spectral sequences using change-of-rings theorems. These theorems essentially allow one to replace the relatively big Hopf algebroid (BP_*, BP_*BP) by much smaller ones, under certain conditions of the BP_*BP-comodule. Additionally, one can give a further interpretation in terms of the cohomology of a certain automorphism group of formal group laws. Let's give a short description of this, without defining many of the terms will use. Also see figure 2.

More concretely our goal is to compute $\mathrm{Ext}^{*,*}_{BP_*BP}(BP, M)$ for BP_*BP-comodules M satisfying at least $v_n^{-1}M = M$ or even better $v_n^{-1}M$ and $I_nM = 0$. Note that the $M_n^0 = v_n^{-1}BP_*/(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty)$ is of the first type and $M_0^m = v_m^{-1}BP_*/I_m$ is of the second type.

The condition $v_n^{-1}M = M$ means that M should thought of as a quasicoherent sheaf living over the open substack $\mathcal{M}_{FG}^{\leq n}$ of the moduli stack of formal groups represented of formal group laws of height $\leq n$, and if additionally $I_nM = 0$ it should be thought of as a quasicoherent sheaf living over $\mathcal{M}_{FG}^{=n}$, the relatively closed substack of height exactly n formal group laws, or at least some formal neighborhood of it.

The big picture story is then that (i) under these conditions we can write $\mathrm{Ext}^{*,*}_{BP_*BP}(BP, M)$ as $\mathrm{Ext}^{*,*}_\Gamma(A, A \otimes_{BP_*} M)$ for certain smaller and topologically interesting Hopf algebroids (A, Γ) – for example, we will be considering Hopf algeroids coming from Morava K-theory $K(n)$, Johnson-Wilson theory $E(n)$ or Morava E-theory E_n – and (ii) interpret these Ext-groups as the continuous group cohomology of a certain profinite group known as the Morava stabilizer group S_n.

The first step is mainly computationally interesting, exactly because the Hopf algebroids that appear there are smaller. The second step is mainly structurally interesting, as it allows us to prove structural theorems about the homotopy groups of spheres. We'll say a bit more about this later.

2. Defining E_n and S_n

The goal of this section is give the definitions of Morava E-theory and the nth Morava stabilizer group S_n, as used in subsection 1.3.

2.1. **The wonderful world of p-typical formal group laws.** Since we don't want to spend a large amount of time on background material we will assume the reader is familiar with the following parts of the theory of formal group laws, most of which was already mentioned in subsection 1.1:

(i) The definition of a formal group law over a ring R.
(ii) Morphisms, isomorphisms and strict isomorphisms of formal group laws.

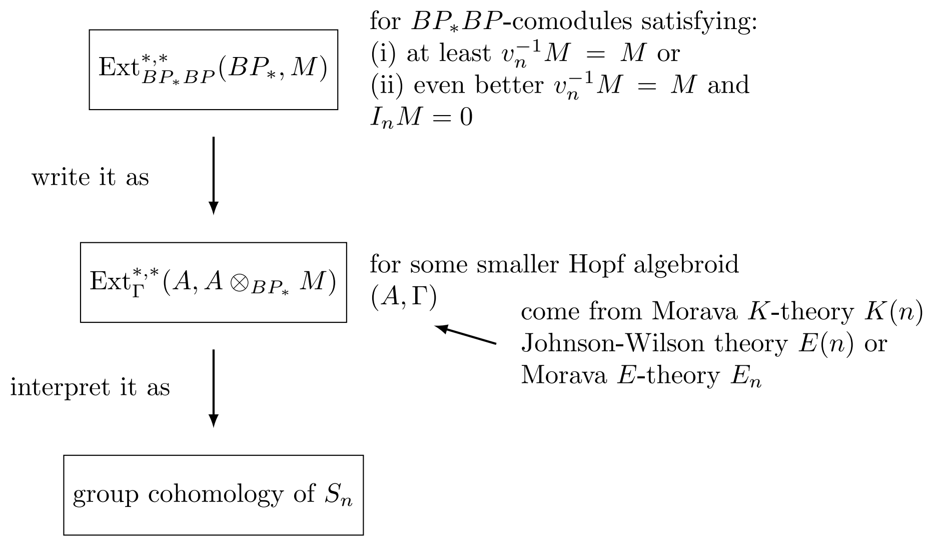

FIGURE 2. An overview of the basics of the applications of the change-of-rings theorems.

(iii) The fact that the Lazard ring L corepresents the functor $\mathsf{Rng} \to \mathsf{Set}$

$$R \mapsto \{\text{formal group laws over } R\}$$

(iv) The fact that the Hopf algebroid (L, LT) corepresents the functor $\mathsf{Rng} \to \mathsf{Grpd}$

$$R \mapsto \{\text{formal group laws over } R \text{ and strict isomorphisms}\}$$

(v) Quillen's theorem that we can introduce gradings on (L, LT) such that there is an isomorphism of graded Hopf algebroids $(L, LT) \cong (MU_*, MU_*MU)$.

If we are consider formal group laws over $\mathbb{Z}_{(p)}$-algebras instead of arbitrary rings, every formal group law is canonically isomorphic to a so-called p-typical formal group law. If R is torsion-free then F is p-typical if its logarithm is of the form $\log_F(x) = \sum l_i x^{p^i}$ and in general we say that F is p-typical if

$$[1/q]_F \left(\sum^{F}_{1 \leq i \leq q} \zeta^i x \right) = 0$$

for all primes q not equal to p. Here ζ is a primitive q'th root of unity.

The last three theorems about formal group laws that we stated above have analogues for p-typical formal group laws. In particular, there is a Hopf algebroid (V, VT) over $\mathbb{Z}_{(p)}$ which corepresents the functor $\mathsf{Alg}_{\mathbb{Z}_{(p)}} \to \mathsf{Grpd}$ given by

$$R \mapsto \{p\text{-typical formal group laws over } R \text{ and strict isomorphisms}\}$$

and the Brown-Peterson spectrum BP_* has the property that there exists gradings on (V, VT) (compatible with those on (L, LT)) such that $(V, VT) \cong (BP_*, BP_*BP)$.

A field k of characteristic p is in particular a $\mathbb{Z}_{(p)}$-algebra. The only isomorphism-invariant of p-typical formal group laws over a field of characteristic p is its height. Let's recall the definition, for which we need the following lemma.

Lemma 2.1. *If $f : F \to G$ is a morphism of formal group laws over k and $f \neq 0$, then there exist a unique integer $n \geq 0$ and power series $g \in k[[x]]$ such that $f(x) = g(x^{p^n})$, $g(0) = 0$ and $g'(0) \neq 0$.*

Proof. If $f(F(x, y)) = G(f(x), f(y))$ we can differentiate with respect to y and set $y = 0$ to get

$$f'(F(x, 0))\partial_2 F(x, 0) = \partial_2 G(f(x), 0) f'(0)$$

where ∂_2 denotes the derivative with respect to the second variable. The left hand side is equal to $f'(x)(1 + \text{higher order terms})$, while the right hand side is equal to $(1 + \text{higher order terms})f'(0)$. So if $f'(0) = 0$ then $f'(x) = 0$ and we have that $f(x) = \tilde{g}(x^p)$ for some $\tilde{g} \in k[[x]]$.

Continuing this process until we get a g with $g'(0) \neq 0$, we find our n and g. This works because $\tilde{g}$ is morphism of formal group laws again, from $F(\sigma(-), \sigma(-))$ to G, where σ is the Frobenius, and because the process terminates since $f \neq 0$. $\square$

Now take the p-series of F, given by

$$[p]_F(x) := \overbrace{x +_F \ldots +_F X}^{p}$$

This is a morphism from F to F. If F is not isomorphic to the additive formal group law, then $[p]_F$ is non-zero and by the previous lemma $[p]_F(x) = g(x^{p^n})$ for a unique n.

Definition 2.2. This n is called the height of F and denoted $\mathrm{ht}(F)$.

Note that since $[p]_F$ is compatible with isomorphism of formal group laws the height is an isomorphism-invariant. An alternative description of height is given as follows. Let $f : V \to k$ classify F, then we claim that $\ker(f) = I_n = (p, v_1, \ldots, v_{n-1})$ where $n = \mathrm{ht}(F)$. To see this recall that the Araki generators of BP_* satisfy [Rav04, formula A.2.2.4]

$$[p]_F(x) = px +_F \sum_{i\geq 1}^{F} v_i x^{p^i}$$

This allows for a particularly easy construction of formal group laws of height n.

Definition 2.3. If $k = \mathbb{F}_{p^n}$ then H_n, called the Honda formal group law of height n, is the formal group law classified by the map $V \to \mathbb{F}_{p^n}$ given by

$$v_i \mapsto \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Equivalently H_n is the unique p-typical group law with p-series x^{p^n}.

The choice of Honda formal group law really doesn't matter so much for the constructions we'll do with it later, nor does $k = \mathbb{F}_{p^n}$ really up to some Galois groups. It is just a convenient choice.

2.2. **The Morava stabilizer group.** We can now define the Morava stabilizer group.

Definition 2.4. The nth Morava stabilizer group S_n is given by

$$S_n = \mathrm{Aut}(H_n) = \{\text{not necessarily strict isomorphisms of } H_n\}$$

which is a profinite group, with topology coming from its inclusion in formal power series over $\mathbb{F}_{p^n}$.

There is also a big nth Morava stabilizer group G_n, sometimes also referred to as the nth Morava stabilizer group. It is given by $G_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes S_n$.

To get a bit of an idea of what S_n looks like, we note that it sits inside $\mathrm{End}(H_n)$, the endomorphisms of H_n, as the units. Let's describe some endomorphisms:

(i) Any integer that can be written as $\sum_{i=0}^{k} a_i p^i$ with $0 \leq a_i \leq p-1$ acts on H_n via

$$x \mapsto \sum_{1\leq i\leq k}^{H_n} [a_i p^i]_{H_n}(x)$$

If we take instead of an integer a p-adic number $\sum_{i\geq 0} a_i p^i$ this converges in $\mathrm{End}(H_n)$, so that we get a map $\mathbb{Z}_p \to \mathrm{End}(H_n)$.

(ii) There is the "Frobenius" S (in what sense it's the Frobenius is something we'll see later) given by

$$x \mapsto x^p$$

(iii) The generator of $\mathbb{F}_{p^n}$, i.e. a primitive (p^n-1)st root of unity ω acts via

$$x \mapsto \omega x$$

There are obvious relation between these.

(i) The fact that $[p]_{H_n}(x) = x^{p^n}$ implies that

$$S^n = p$$

(ii) If ω^σ denotes the Frobenius in $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ applied to ω, then obviously we must have

$$S\omega = \omega^\sigma S$$

In some sense these are all endomorphisms and relations between them. This is made precise by the following theorem of Dieudonné and Tate [Rav04, theorem A2.2.17].

Theorem 2.5. *We have that*

$$\mathrm{End}(H_n) = \mathbb{W}(\mathbb{F}_{p^n})\langle S\rangle/(S^n = p,\, Sw = w^\sigma S \text{ for all } w \in \mathbb{W}(\mathbb{F}_{p^n}))$$

where $\mathbb{W}(\mathbb{F}_{p^n})$ denote the Witt vectors (to be defined shortly), $\langle - \rangle$ is adjoining a non-commuting generator and the second relation uses that there exists an extension of the Frobenius to the Witt vectors.

So what are these Witt vectors? Abstractly the Witt vectors $\mathbb{W}(\mathbb{F}_{p^n})$ over $\mathbb{F}_{p^n}$ are the unique (up to isomorphism) complete local ring with residue field $\mathbb{F}_{p^n}$ with the universal property that all for complete local rings B (with maximal ideal $\mathfrak{m}$) and diagrams

$$\begin{tikzcd} \mathbb{W}(\mathbb{F}_{p^n}) \arrow[r, dotted] \arrow[d] & B \arrow[d] \\ \mathbb{F}_{p^n} \arrow[r] & B/\mathfrak{m} \end{tikzcd}$$

there exists a unique dotted lift $\mathbb{W}(\mathbb{F}_{p^n}) \to B$. In particular, taking $B = \mathbb{W}(\mathbb{F}_{p^n})$ and the map $\mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ the Frobenius, we get a lift of the Frobenius to the Witt vectors.

Concretely $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[x]/(q(x))$ where $q(x)$ is any lift to $\mathbb{Z}_p$ of an irreducible factor of $X^{q-1}-1=0$ over $\mathbb{F}_p$. In our ring of endomorphisms of H_n, $\mathbb{Z}_p$ and ω sit inside $\mathbb{W}(\mathbb{F}_{p^n})$ in exactly this way; as a $\mathbb{Z}_p$ and a lift of a solution of $X^{q-1}-1=0$. Also note that $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$.

The Morava stabilizer group has nice cohomological properties, which essentially come from the fact that it sits as a maximal order inside a division algebra of dimension n^2 over $\mathbb{Q}_p$.

2.3. **Morava E-theory.** We will now define Morava E-theory. It is constructed using the Landweber exact functor theorem using a periodic version of the universal deformation ring of the formal group law H_n over $\mathbb{F}_{p^n}$. However, we can just state what E_n is as a graded ring without any fancy language:

$$E_n = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$$

with $|u_i| = 0$ and $|u| = -2$. This definition of course comes completely out of nowhere, so we'll explain an interpretation of this ring. That is, we will make precise what a universal deformation is.

Let's define a functor $\mathrm{Def}_{\mathbb{F}_{p^n}, H_n}$ from the category of complete local rings to the category of groupoids. Given a complete local ring B, the objects of $\mathrm{Def}_{\mathbb{F}_{p^n}, H_n}(B)$ are given by pairs (G, i) of a formal group law G over B and a homomorphism $i : \mathbb{F}_{p^n} \to B/\mathfrak{m}$, such that $i_* H_n = \pi_* G$. Here π is the canonical map $B \to B/\mathfrak{m}$ and π_* means pushforward of the formal group law, i.e. applying π to its coefficients.

The morphisms of $\mathrm{Def}_{\mathbb{F}_{p^n}, H_n}(B)$, say from (G_1, i_1) to (G_2, i_2), only exist if $i_1 = i_2$ and in that case are given by isomorphisms of formal group laws $f : G_1 \to G_2$ such that $\pi_* f = \mathrm{id}$. Such a map f is called a $\star$-isomorphism

We leave it to the reader to descibe the functoriality in B. The main theorem of Lubin-Tate deformation theory tells us what these groupoids look like and give a corepresenting object for π_0. We state it in a bit more generality, as our construction makes sense for any p-typical formal group law

Γ over a perfect field k of characteristic p, say of height n. The corresponding deformation functor is denoted $\mathrm{Def}_{k,\Gamma}$. A nice sketch of the proof of the following theorem can be found in sections 4 and 5 of [Rez98].

Theorem 2.6 (Lubin-Tate)**.** *First note that* $\mathrm{Def}_{k,\Gamma}(B)$ *splits into a disjoint union of* $\mathrm{Def}_{k,\Gamma}(B)_i$ *with a fixed* $i : k \to B/\mathfrak{m}$*. We have that*

(i) The $\star$*-isomorphism classes of deformations of* $i_*\Gamma$ *are in bijection with* $\mathfrak{m}^{\times(n-1)}$*, i.e.*

$$\pi_0(\mathrm{Def}_{k,\Gamma}(B)_i) \cong m^{\times(n-1)}$$

(ii) The $\star$*-isomorphism classes have no automorphisms (there exists a unique* $\star$*-isomorphism between two representatives of a* $\star$*-isomorphism class), i.e.*

$$\pi_1(\mathrm{Def}_{k,\Gamma}(B)_i) \cong \{e\}$$

Furthermore the functor $\pi_0(\mathrm{Def}_{k,\Gamma}(-))$ *from complete local rings to sets, sending* B *to its set of* $\star$*-isomorphism class of deformations of* Γ*, is corepresented by*

$$\mathbb{W}(k)[[u_1, \ldots, u_{n-1}]]$$

In particular $\mathbb{W}(k)[[u_1, \ldots, u_{n-1}]]$ *carries a formal group law that is a deformation of* H_n*, and is universal as such up to* $\star$*-isomorphism.*

So E_n is just a periodic version of the complete local ring corepresenting $\star$-isomorphism classes of deformations of the Honda formal group law over $\mathbb{F}_{p^n}$. This periodicity can be thought of as encoding a natural circle action, i.e. $\mathbb{G}_m$-action, see the first step in the proof of proposition 5.2.

Let's note some corollaries of this theorem.

Corollary 2.7. *We get actions on* E_n *via the following two actions:*

(i) The nth Morava stabilizer group S_n *acts on* $\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$ *as follows: given a deformation* (G, i) *any automorphism* f *of* H_n *gives new deformation by lifting* $i_*f \in B/\mathfrak{m}[[x]]$ *to* $B[[x]]$ *and applying it to* G*. Up to* $\star$*-isomorphism this is independent of the choice of lift. Thus we get a map from* $\mathrm{Aut}(H_n)$ *to the natural isomorphisms of* $\pi_0(\mathrm{Def}_{\mathbb{F}_{p^n},H_n}(-))$*. This induces an action of* $S_n = \mathrm{Aut}(H_n)$ *on the corepresenting object.*

(ii) The Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ *acts on* $\mathbb{W}(\mathbb{F}_{p^n})$ *or equivalently on*

$$\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$$

by the same construction as in part (i) applied to changing $i : \mathbb{F}_{p^n} \to B/\mathfrak{m}$ *by precomposition with an element of the Galois group.*

(iii) Together these give an action of the nth big Morava stabilizer group $G_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes S_n$ *on* $\mathbb{W}(\mathbb{F})[[u_1, \ldots, u_{n-1}]]$*.*

We'll next explain how to make a cohomology theory and spectrum out of E_n. Note that $E_n = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$ carries a formal group law, which comes from the universal deformation of H_n. This gives us a map $BP_* \to E_n$. We claim, but will not prove, that this gives E_n the structure of BP_*-module where v_i acts as

$$v_i \mapsto \begin{cases} u_i u^{1-p^i} & \text{if } i \leq n-1 \\ u^{1-p^n} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Using this it is straightforward check that E_n satisfies the conditions of the Landweber exact functor theorem (in a complete BP_*-module category), i.e. that $(p, v_1, v_2, \ldots)$ is a regular sequence. As a consequence we obtain that

$$X \mapsto E_n \hat{\otimes}_{BP_*} BP_*(X)$$

is a homology theory, which we denote by $E_n(X)$. It is represented by a spectrum E_n that satisfies $(E_n)_* = E_n$ and $(E_n)_* E_n = (E_n)_* \hat{\otimes}_{BP_*} BP_* BP \hat{\otimes}_{BP_*} (E_n)_*$. Together these form a Hopf algebroid. We will henceforth no longer distinguish between the graded rings E_n and $(E_n)_*$.

From this construction it follows that E_n is homotopy commutative and has a homotopy action of S_n and the Galois group. The world is much kinder, because it turns out that E_n has an essentially unique structure of a E_∞ ring spectrum. Even better, we have that there is a functor from the category of perfect fields of characteristic p and formal group laws of height n over them to the category of E_∞ ring spectra, such that this functor assigns E_n to $(\mathbb{F}_{p^n}, H_n)$. This strictifies the action of the Morava stabilizer stabilizer group and we can now take homotopy fixed points with respect to finite subgroups and do even cooler things. But this is not the talk for that (in fact, the next talk at Talbot was the talk for that).

2.4. **Morava K-theory and Johnson-Wilson theory.** Because we'll see them again in the next couple of sections, we will quickly discuss Morava K-theory and Johnson-Wilson theory.

Johnson-Wilson theory is given by taken the BP_*-module

$$E(n) = BP_*/(v_{n+1}, v_{n+2}, \ldots) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$$

This satisfies the conditions of the Landweber exact functor theorem and hence gives rise to a homology theory and spectrum. This spectrum gives us a Hopf algebroid $(E(n), E(n) \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n))$. Though we won't really talk much about it, a completed cousin of it will play a role in the proof of the Morava's change-of-rings theorem.

Morava K-theory is more important for our purposes than Johnson-Wilson theory and will be subject of the Miller-Ravenel change-of-rings theorem, which is computationally very important and will serve as input for the Morava change-of-rings theorem. One starts with taking the BP_*-module

$$K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$$

with $|v_n| = 2(p^n - 1)$ as always and BP_* acting in the obvious way. This is *not* Landweber-exact. In particular, p does not act in a monic fashion. However, we can still construct a spectrum out of it, either by EKMM $\mathbb{S}$-module techniques or Baas-Sullivan theories of cobordisms with singularities.

Similar to the previous two spectra we considered, $K(n)$ fits into a Hopf algebroid. This is $(K(n), \Sigma(n))$ with $\Sigma(n) = K(n) \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)$, where the latter is not $K(n)_*K(n)$ since we don't have Landweber exactness.

This obvious map $BP_* \to K(n)$ in some sense classifies the version of the Honda formal group law H_n over a periodic version $\mathbb{F}_p$. This is made clearer by passing to the faithfully flat extension $\tilde{K}(n) = \mathbb{F}_{p^n}[u, u^{-1}]$ with $|u| = -2$, which actually carries a periodic version of the Honda formal group law. We will later see this faithfully flat extension gives an equivalence of stacks represented by the Hopf algebroids locally in the faithfully flat topology nd thus doesn't change the computation of Ext's of comodules over them.

3. Overview of change-of-rings theorems

3.1. **The change-of-rings theorems that we will prove.** Let's explain what we want to prove in this section and the following two sections. An overview of this is given in figure 3.

Recall that we are interested in computing $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M)$ for M satisfying particular properties.

M **satisfies** $v_n^{-1}M = M$ **and** $I_nM = 0$**:** We saw before that the conditions that $v_n^{-1}M = M$ and $I_nM = 0$ mean that M in some sense lives over the relatively closed substack $\mathcal{M}_{FG}^{=n}$. This consists of a single point (corresponding to the unique isomorphism class of height n formal group laws) and has automorphism group S_n. The change-of-rings theorem are supposed to exploit this. First of all by replacing the full Hopf algebroid (BP_*, BP_*BP) modelling the entire stack $\mathcal{M}_{FG}$ with a much smaller one $(K(n), \Sigma(n))$. This Hopf algebroid is a periodic version of the one modelling H_n and its automorphisms and as such maps to $\mathcal{M}_{FG}$, in Hopf-algebraic form seen from the quotient map $(BP_*, BP_*BP) \to (K(n), \Sigma(n))$. It is hence not surprising that one can prove that $v_n^{-1}M = M$ and $I_nM = 0$ imply that M is the pushforward of a quasicoherent

M satisfying $v_n^{-1}M = M$ | M satisfying $v_n^{-1}M = M$ and $I_nM = 0$

$$\begin{tikzcd}[column sep=small]
 & \operatorname{Ext}^{*,*}_{BP_*BP}(BP_*,M) \arrow[dl, leftrightarrow, "\cong"'] \arrow[dr, leftrightarrow, "\cong\ \text{Miller-Ravenel}"] & \\
\operatorname{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n),\hat{E}(n)\hat{\otimes} M) \arrow[d, leftrightarrow, "\cong"'] & & \operatorname{Ext}^{*,*}_{\Sigma(n)}(K(n),K(n)\otimes_{BP_*}M) \arrow[d, leftrightarrow, dashed, "\cong\ \text{after tensoring with } \mathbb{F}_p \text{ to kill grading}"] \\
H^{*,*}_c(S_n, E_n\hat{\otimes}_{BP_*}M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} & & H^{*,*}_c(S_n,\mathbb{F}_{p^n}\otimes_{BP_*}M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}
\end{tikzcd}$$

FIGURE 3. An overview of change-of-rings theorems. Here $\hat{E}(n)$ is completed Johnson-Wilson theory. The reference for the left-hand side is [Dev95], the reference for the right-hand side is [MR77] and [Rav04].

sheaf on $(K(n), \Sigma(n))$ and hence we can compute its Ext-groups over (BP_*, BP_*BP) in terms of those over $(K(n), \Sigma(n))$:

$$\operatorname{Ext}^{*,*}_{BP_*BP}(BP_*, M) \cong \operatorname{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n) \otimes M)$$

This can be found in [MR77]. The following further rewriting can be found in [Rav04]. Since $\Sigma(n)$ is essentially a periodic version of the group Hopf algebra of S_n, we get further reduce this to continuous group cohomology after killing of the periodic part (the Galois group shows up since not all automorphisms of H_n exist of $\mathbb{F}_p$, so we need to go up to $\mathbb{F}_{p^n}$ and kill that extension off again):

$$\operatorname{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n) \otimes_{BP_*} M) \otimes_{K(n)} \mathbb{F}_p \cong H^{*,*}_c(S_n, \mathbb{F}_{p^n} \otimes M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$$

M **satisfies** $v_n^{-1}M = 0$**:** If $v_n^{-1}M = M$ then we can go two ways, one leading to the Morava change-of-rings theorem and one leading to a version of the Miller-Ravenel change-of-rings theorem. We will stress the former, though the reader can look up the latter in section 3 of [MR77], it being a straightforward consequence of the Miller-Ravenel change-of-rings theorem.

We already saw that $v_n^{-1}M = M$ means that M lives over $\mathcal{M}^{\leq n}_{FG}$. But we can actually do better. There is an algebraic lemma that tells us that $v_n^{-1}M = M$ implies that M is I_n-nil, i.e. every element of M is annihilated by some finite power of I_n. If M is finitely-generated this is equivalent to being I_n-nilpotent. The upshot of this is that in fact M lives over a formal neighborhood of the relatively closed substack $\mathcal{M}^{=n}_{FG}$.

This means we can try to do the same argument as above with the formal neighborhood of the point in $\mathcal{M}^{=n}_{FG}$. This formal neighborhood is a stack modelled by the Hopf algebroid $(E_n, E_n\hat{\otimes}_{BP_*}BP_*BP\hat{\otimes}_{BP_*}E_n)$ so the story should roughly play out as before. We'll go a slightly different route, following [Dev95]. The first is an algebraic change-of-rings theorem:

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \cong \mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*} M)$$

where $(\hat{E}(n), \hat{U}(n))$ is a completed periodic version of the Johnson-Wilson theory Hopf algebroid. For example $\hat{E}(n) = \mathbb{Z}_p[[v_1, \ldots, v_{n-1}]][u, u^{-1}]$. After that we again note that we can essentially dealing with a group Hopf algebra to see

$$\mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*} M) \cong H^{*,*}_c(S_n, E_n\hat{\otimes}_{BP_*} M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_P)}$$

Though this is not obvious, on the right hand side one can take the Galois group inside of the cohomology, replacing S_n by G_n.

3.2. **A general philosophy for change-of-rings theorems.** We will now describe two relevant but different classes change-of-rings theorems. They may very well be the same from the correct viewpoint and there may very well be many more types. However, these are the ones that we will be using in later sections.

We first spend some time looking at functors between categories of comodules over Hopf algebroids. Recall that a Hopf algebroid (A, Γ) is a presentation for a stack. To see this remember that a prestack is a particular type of functor from $\mathsf{Rng} = \mathsf{Aff}^{op}$ to groupoids. Then (A, Γ) gives such a functor by objects given by $\mathrm{Hom}(-, \mathrm{Spec}(A))$ and morphism by $\mathrm{Hom}(-, \mathrm{Spec}(\Gamma))$. We denote the stackification of this by $\mathcal{M}_{A,\Gamma}$.

A quasicoherent sheaf over a stack presented by (A, Γ) is the same as quasicoherent sheaf over $\mathrm{Spec}(A)$ compatible with the two pullbacks to $\mathrm{Spec}(\Gamma)$ along the source and target maps $\mathrm{Spec}(s)$ and $\mathrm{Spec}(t)$. In other words, it is an A-module with an isomorphism $M \otimes_A \Gamma \to \Gamma \otimes_A M$ of Γ-modules. By linearity this is induced by a coaction map $M \to \Gamma \otimes_A M$. The axioms of a Γ-comodule say exactly that the coaction map comes from an isomorphism. We conclude that there is an equivalence of categories between quasicoherent sheaves over the stack presented by (A, Γ) and Γ-comodules. If the latter has the structure of an abelian category, i.e. Γ is flat over A, then this is an equivalence of abelian categories. This allows us to translate between operations on sheaves over stacks and operations on comodules over Hopf algebroids.

$$\begin{aligned} \{\text{quasicoherent sheaves over } \mathcal{M}_{A,\Gamma}\} &\leftrightarrow \{\Gamma\text{-comodules}\} \\ \mathcal{F}_M &\leftrightarrow M \end{aligned}$$

We already saw the relation between global sections and Ext-groups:

$$\{\text{global sections of quasicoherent sheaves}\} \leftrightarrow \{\text{Ext-groups of } \Gamma\text{-comodules}\}$$
$$R^i\Gamma(\mathcal{F}_M) = H^i(\mathcal{M}_{A,\Gamma}, \mathcal{F}_M) \leftrightarrow \mathrm{Ext}^i_\Gamma(A, M)$$

We will now describe the story for pullback and pushforward. We start with a map $f : (A, \Gamma) \to (B, \Sigma)$ of Hopf algebroids. Then $\mathrm{Spec}(f)$ induces a map of stacks $\mathcal{M}_{A,\Gamma} \leftarrow \mathcal{M}_{B,\Sigma}$. Geometrically, on the stacky side we then get the well-known picture

$$\begin{tikzcd} \{\text{quasicoherent sheaves on } \mathcal{M}_{A,\Gamma}\} \arrow[r, bend left, "\mathrm{Spec}(f)^*", "\text{pullback}"'] & \{\text{quasicoherent sheaves on } \mathcal{M}_{B,\Sigma}\} \arrow[l, bend left, "\text{pushforward}"', "\mathrm{Spec}(f)_*"] \end{tikzcd}$$

We get a similar picture on the Hopf algebroid side (where we use opposite notation from Miller-Ravenel to stress the geometric nature of the operations):

$$\begin{tikzcd} \{\Gamma\text{-comodules}\} \arrow[rr, bend left, "f^*", "\text{pullback}"'] & & \{\Sigma\text{-comodules}\} \arrow[ll, bend left, "f_*", "\text{pushforward}"'] \end{tikzcd}$$

Explicitly these functors are given by

$$f^* : M \mapsto B \otimes_A M$$
$$(\Gamma \otimes_A B)\square_\Sigma N \leftarrow\!\shortmid N : f_*$$

Here f^* is left adjoint to f_* and $f^*A = B$, so that we get a natural isomorphism $\mathrm{Ext}_\Gamma(A, f_*N) \to \mathrm{Ext}_\Sigma(B, N)$. We can now discuss the change-of-rings theorems.

Change-of-rings type 1 – Miller-Ravenel "push-pull": The general setup of this type of change-of-rings theorem is as follows. One starts with a map $f : (A, \Gamma) \to (B, \Sigma)$ and a Γ-comodule M, the object of interest being $\mathrm{Ext}_\Gamma(A, M)$. Firstly one assumes a condition on M that implies that $M \cong f_*f^*M$. Secondly one assume a condition on f that implies that the Leray spectral sequence for the composite of pushforwards associated to $\mathcal{M}_{B,\Sigma} \to \mathcal{M}_{A,\Sigma} \to *$ collapses. The result is a pair of isomorphisms:

$$\mathrm{Ext}^*_\Gamma(A, M) \xrightarrow{\cong} \mathrm{Ext}^*_\Gamma(A, f_*f^*M) \xrightarrow{\cong} \mathrm{Ext}^*_\Sigma(B, f^*M)$$

Let us now give a general condition for the second step, due to Miller-Ravenel.

Lemma 3.1. *If $\Gamma \otimes_A B$ is a summand of $X \otimes_B \Sigma$ as a Σ-comodule then $\mathrm{Ext}^*_\Gamma(A, f_*N) \to \mathrm{Ext}^*(B, N)$ is an isomorphism.*

Proof. Let's start at the $\mathrm{Ext}^*(B, N)$ side. This is computed as $H^n\mathrm{Hom}_\Gamma(B, I^*)$ where I^* is a relatively injective resolution of N:

$$0 \to N \to I^0 \to I^1 \to \dots$$

with $I^q = \Sigma_B \otimes Y^q$. Then $f_*I^q = (\Gamma \otimes_A B)\square_\Sigma I^q$ is a summand of $(X \otimes_B \Sigma)\square_\Sigma I^q = X \otimes_B I^q$. This tells us that f_*I^q exact and each terms is of the form $(\Gamma \otimes_A B)\square_\Sigma(\Sigma \otimes_b Z^q) = \Gamma \otimes_A Z^q$. We conclude that

$$\mathrm{Ext}^*_\Gamma(A, f_*N) = H^n\mathrm{Hom}_\Gamma(A, f_*I^q) \xrightarrow{\cong} H^n\mathrm{Hom}_\Sigma(f^*A, I^q) = \mathrm{Ext}_\Sigma(B, N)$$

□

We'll now give two examples of this.

Example 3.2. An easy example is when $I \subset A$ is an invariant ideal and M satisfies $IM = 0$. There is a map of Hopf algebroids $i : (A, \Gamma) \to (A/I, \Gamma/I)$ corresponding to an inclusion of an affine closed substack. Then $M = i_*i^*M$ and the conditions of the lemma are satisfied and thus

$$\mathrm{Ext}^*_\Gamma(A, M) \cong \mathrm{Ext}^*_{\Gamma/I}(A/I, M)$$

An example of this is a BP_*BP-comodule M such that $I_nM = 0$. Then we can compute its Ext-groups in terms of BP_*/I_n. This corresponds to the closed substack $\mathcal{M}^{\geq n}_{FG}$ of height $\geq n$ formal group laws.

Example 3.3. A particular type of groupoid is an action groupoid. Given a group G acting on a set S, the corresponding action groupoid has objects S and morphisms $S \times G$. A split Hopf algebroid is on that corepresents a groupoid-valued functor of the form which is on objects of the form $S(R)$ and on morphisms of the form $S(R) \times G(R)$ with S a set-valued functor and G a group-valued functor. This means that the Hopf algebroid is of the form $(B, B \otimes \Sigma)$ with B any ring and Σ a Hopf algebra.

Suppose that our base ring is K. Then the inclusion $(K, \Sigma) \to (B, B \otimes \Sigma)$ satisfies the conditions of the lemma and thus:

$$\mathrm{Ext}^*_\Sigma(K, M) \xrightarrow{\cong} \mathrm{Ext}^*_{B\otimes\Sigma}(B, M)$$

This example gives a plausible geometric sketch of proof for the lemma. Geometrically, the story is that a quasicoherent sheaf over $\mathcal{M}_{B,B\otimes\Sigma}$ can be pushed forward to a sheaf over $\mathcal{M}_{K,\Sigma}$. This map is affine and since quasicoherent sheaves have trivial cohomology over affines, the Leray spectral sequence collapses and we get an explanation for isomorphism.

Change-of-rings type 2 – equivalences of Hopf algebroids: The general setup for this type of change-of-rings theorem is again a map $f : (A, \Gamma) \to (B, \Sigma)$. Such a map gives a natural transformation ϕ between the corresponding prestacks (which are the groupoid-valued functors $\mathsf{Rng} \to \mathsf{Grpd}$ corepresented by the Hopf algebroids, as the reader may recall from a couple of pages ago).

Definition 3.4. The map f is an equivalence if there is an inverse natural transformation up to natural equivalence.

If one think of the category of a stacks as a 2-category with only invertible 2-morphisms, there this says there is a 1-morphisms which is inverse up to 2-morphisms. More concretely, a natural equivalence from a natural transformation a to b is a natural collection of maps

$$\tau_S : \mathrm{Ob}(F(S)) \to \mathrm{Mor}(G(S))$$

for rings S, such that for all $f : c_1 \to c_2$ in $F(S)$ we have that the following diagram commutes:

$$\begin{tikzcd} a_S(c_1) \arrow[r, "\tau_S(c_1)"] \arrow[d, "a_S(f)"'] & b_S(c_1) \arrow[d, "b_S(f)"'] \\ a_s(c_2) \arrow[r, "\tau_S(c_2)"'] & b_s(c_2) \end{tikzcd}$$

Indeed, if one fixes S, considers $F(S)$ and $G(S)$ as categories and a and b as functors between these categories, then τ is nothing but a natural transformation between these functors.

If the map $f : (A, \Gamma) \to (B, \Sigma)$ is an equivalence then the two stacks $\mathcal{M}_{A,\Gamma}$ and $\mathcal{M}_{B,\Sigma}$ are equivalent and one should think of (A, Γ) and (B, Σ) as nothing but different presentations of the same stack. An equivalence of stacks implies an equivalence of the categories of quasicoherent sheaves, so we get that $f^* : \mathsf{Comod}_\Gamma \to \mathsf{Comod}_\Sigma$ is an equivalence and more importantly (from our point of view) an isomorphism of Ext-groups

$$\mathrm{Ext}^*_\Gamma(A, M) \xrightarrow{\cong} \mathrm{Ext}^*_\Sigma(B, f^*M)$$

One might like an algebraic definition of a natural equivalence, so that one can check that a map of Hopf algebroids is an equivalence using algebra. In the algebric definition of a natural equivalence we want to be using a (e.g. $f \circ g$) and b (e.g. id) instead of a^* and b^*. By dualizing everything in the definition of a natural equivalence, we see it must be given by a homomorphism $\tau : \Gamma \to B$ such that (i) $\tau \circ \eta_R = a : A \to B$ and $\tau \circ \eta_L = b : A \to B$ (this tells us that τ_S maps objects to arrows with the correct sources and targets) and (ii) the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \Gamma \otimes_A \Gamma \arrow[d, "(b, \eta_R \circ \tau)"] \\ \Gamma \otimes_A \Gamma \arrow[r, "(\eta_L \circ \tau, a)"'] & \Sigma \end{tikzcd}$$

As one may expect, one doesn't really need that the stacks are equivalent to get the change-of-rings theorem. It is a enough that the homological algebra of their sheaves is the same. With this in mind one can prove a more general change-of-rings theorem for f's that are equivalences locally in the faithfully flat topology.

The reader may be wondering why we haven't talked about group cohomology yet, continuous or otherwise. The reason is that going to group cohomology will be nothing else than taking a look at the cobar complex computing Ext and seeing that it is equal to cochain complex that computes group cohomology, coming from the bar resolution.

4. Miller & Ravenel's change-of-rings theorem

Our goal in this section is the following theorem.

Theorem 4.1 (Miller-Ravenel change-of-rings)**.** *Suppose that M is a BP_*BP-comodule such that $v_n^{-1}M = M$ and $I_nM = 0$. Then there is an isomorphism*

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \cong \mathrm{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n) \otimes_{BP_*} M)$$

natural in M.

This is a change-of-rings theorem of the first type. We want to take $(A, \Gamma) = (BP_*, BP_*BP)$ and $(B, \Sigma) = (K(n), \Sigma(n))$. Recall that $K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$ and $\Sigma(n) = K(n) \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)$. Let $\pi : (BP_*, BP_*BP) \to (K(n), \Sigma(n))$ be the quotient map. Suppose we are given a M such that $v_n^{-1}M = M$ and $I_nM = 0$. We want to prove two things

(i) The conditions on M imply that $M \cong \pi_*\pi^*M$.
(ii) $BP_*BP \otimes_{BP_*} K(n)$ is a summand of something of the form $X \otimes_{K(n)} \Sigma(n)$.

Under these conditions we get a pair of isomorphisms:

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \arrow[r, "\cong"] \arrow[d, dotted, "\cong"'] & \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, \pi_*\pi^*M) \arrow[d, "\cong"] \\
\mathrm{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n) \otimes_{BP_*} M) \arrow[r, equal] & \mathrm{Ext}^{*,*}_{\Sigma(n)}(K(n), \pi^*M)
\end{tikzcd}$$

which implies that the dotted map is an isomorphism.

Let's start with proving claim (i). Recall that $\pi^*M = K(n) \otimes BP_*M$ and $\pi_*N = (BP_*BP \otimes_{BP_*} K(n))\square_{\Sigma(n)}N$. Let $B(n)_* = v_n^{-1}BP_*/I_n = \mathbb{Z}_{(p)}[v_n, v_n^{-1}, v_{n+1}, \ldots]$, which is a BP_*-comodule and thus we get a functor $B(n)_* \otimes_{BP_*} - : \mathsf{Comod}_{BP_*BP} \to \mathsf{Comod}_{BP_*BP}$.

Proposition 4.2. *For any BP_*BP-comodule we have that*

$$B(n) \otimes_{BP_*} M \cong \pi_*\pi^*M$$

If M is v_n-local and satisfies $I_nM = 0$, then $B(n) \otimes_{BP_*} M = M$ and we thus have that

$$M \cong \pi_*\pi^*M$$

Sketch of proof. It suffices to prove that

$$K(n) \otimes_{BP_*} BP_*BP \xrightarrow{\cong} \Sigma(n) \otimes_{K(n)} B(n) \tag{1}$$

as $\Sigma(n)$-comodules and $B(n)$-modules. The opposite of this isomorphism would be $BP_*BP \otimes_{BP_*} K(n) \cong B(n) \otimes_{K(n)} \Sigma(n)$. Then $\pi_*\pi^*M = (B(n) \otimes_{K(n)} \Sigma(n))\square_{\Sigma(n)}(K(n) \otimes_{BP_*} M) = B(n) \otimes_{BP_*} M$.

This isomorphism is proven by Miller-Ravenel [MR77, proposition 2.4] using a counting argument over closely related rings $b(n) = \mathbb{Z}_{(p)}[v_n, v_{n+1}, \ldots]$ and $k(n) = \mathbb{F}_p[v_n]$, which more suited for counting arguments. □

Now we prove claim (ii), that is, we make sure that the Miller-Ravenel change-of-rings theorem sketched in the previous section applies:

Lemma 4.3. *$BP_*BP \otimes_{BP_*} K(n)$ is a summand of something of the form $X \otimes_{K(n)} \Sigma(n)$.*

Proof. Isomorphism 1 tells us that $BP_*BP \otimes_{BP_*} K(n) \cong B(n) \otimes_{K(n)} \Sigma(n)$. So instead of even better than being a summand of something of the form $X \otimes_{K(n)} \Sigma(n)$, it is *equal* to something of that form. □

So that was rather easy, though we of course skipped the intricate counting argument, but quite powerful from a computational point of view. The reason for this is that $K(n)$ is significantly smaller than BP_*. Some of this simplicity is deceptive though, because $\Sigma(n)$ is more complicated than BP_*BP. For example, while BP_*BP was polynomial as an algebra, for $\Sigma(n)$ we have the following more complicated description of $\Sigma(n)$ as an algebra

$$\Sigma(n) = K(n)[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i \,|\, i > 0)$$

Using this one can further rewrite $\mathrm{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n) \otimes_{BP_*} M)$. The idea is that the topological dual of $\Sigma(n) \otimes_{K(n)} \mathbb{F}_{p^n}$ is the group ring of S_n over $\mathbb{F}_{p^n}$. This is done in [Rav04, theorem 6.2.3] and the following corollary proves

$$\mathrm{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n) \otimes_{BP_*} M) \otimes_{K(n)} \mathbb{F}_p = H^*_c(S_n, \mathbb{F}_{p^n} \otimes_{BP_*} M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$$

We feel somewhat okay in skipping a proof of this, as we already explained the intuition behind it and will give a detailed proof in the case of E_n in the next section.

5. Morava's change-of-rings theorem

The version of the Morava change-of-rings theorem we will sketch of the proof of is the one proven by Devinatz in [Dev95]. It uses the completed periodic version of Johnson-Wilson theory. This is given by $\hat{E}(n) = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$ and $\hat{U}(n) = \hat{E}(n) \hat{\otimes}_{BP_*} BP_*BP \hat{\otimes}_{BP_*} \hat{E}(n)$.

Theorem 5.1 (Morava change-of-rings)**.** *Suppose that M is a BP_*BP-comodule such that $v_n^{-1}M = M$. Then there are isomorphisms*

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \cong \mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n) \otimes_{BP_*} M) \cong H^{*,*}_c(S_n, E_n \hat{\otimes}_{BP_*} M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$$

natural in M.

The big story of the proof of this change-of-rings isomorphism is given in figure 4. We start with the construction of a map $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \to \mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n) \otimes_{BP_*} M)$, which we will later show to be an isomorphism if $v_n^{-1}M = M$.

Proposition 5.2. *There exists a map*

$$\theta_M : \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \to \mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n) \hat{\otimes}_{BP_*} M)$$

natural in M.

Sketch of proof. All the steps of in our proofs happen along the following lines: constructs maps between Hopf algebroids by constructing national transformations between the groupoid-valued functors (i.e. prestacks) they corepresent. There are three such steps.

Incoorporate circle action: Recall that (BP_*, BP_*BP) corepresents the functor $FGL_p(-)$ sending a $\mathbb{Z}_{(p)}$-algebra R to groupoid with objects the p-typical formal group laws F over R and morphisms from F to G the strict isomorphisms of formal group laws from G to F.

We want to compare this to the groupoid-valued functor $FGL_p^\circ(-)$ which sends a $\mathbb{Z}_{(p)}$-algebra R the groupoid with objects pairs (F, a) of a p-typical formal group law F over R and an element $a \in R^\times$, and morphisms from (F, a) to (G, b) those isomorphisms f from G to F satisfying $a = f'(0)b$. It is easy to figure out the corepresenting Hopf algebroid. It is given by (U, US) with $U = BP_*[s_0, s_0^{-1}]$ and $U = BP_*BP[s_0, s_0^{-1}]$.

The reason for doing this is to built in the circle action, i.e. a $\mathbb{G}_M$-action, that keeps track of the grading on BP_*. The reason for secretly working with the opposite is that this makes the identification of (BP_*, BP_*BP) more natural, see [Dev95, remark 2.5].

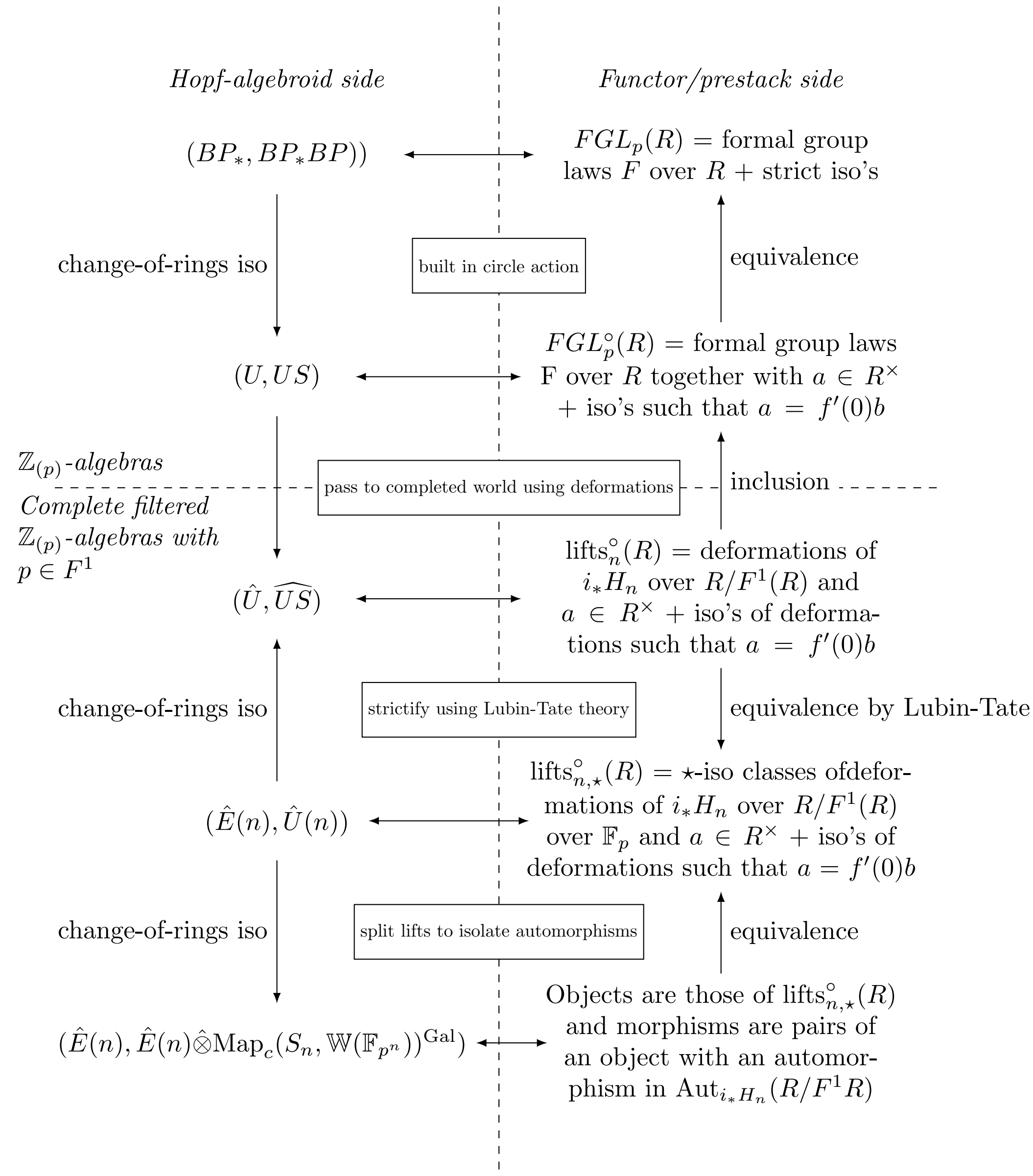

FIGURE 4. An overview of the proof of the Morava change-of-rings theorems as given in [Dev95]. The left-hand side gives the Hopf algebroids and the right hand side the functors they corepresent.

There is a natural transformation from FGL_p° to FGL_p given by sending an object (F, a) to $a^*F = aF(a^{-1}x, a^{-1}y)$ and a morphism $(f, (G, b), (F, a))$ to $(bf(a^{-1}x), b^*G, a^*F)$. This is an equivalence with inverse sending F to $(F, 1)$. Hence there is a map $(BP_*, BP_*BP) \to (U, US)$ and a change-of-rings isomorphism

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \xrightarrow{\cong} \mathrm{Ext}^{*,*}_{US}(U, U \otimes_{BP_*} M)$$

Pass to completed world of deformations: We now want to pass to a completed world of deformations. For general M this will result in just a map on Ext-groups, not necessarily an isomorphism. Also, for our functor to make sense, we need to restrict our attention to what we want to be complete local rings, but for representability purposes are complete filtered $\mathbb{Z}_{(p)}$-algebras such that $p \in F^1$. In this latter category the canonical quotient map $\pi : B \to B/\mathfrak{m}$ is replaced by $\pi : R \to R/F^1(R)$, which necessarily contains $\mathbb{F}_p$ via a canonical inclusion $i : \mathbb{F}_p \to R/F^1(R)$.

We define lifts°_n on a complete filtered $\mathbb{Z}_{(p)}$-algebra R with $p \in F^1(R)$ by setting the objects to be pairs (F, a) of a (p-typical) deformation F over R of $i_* H_n$ (the version of Honda formal group law over $\mathbb{F}_p$) and an element $a \in R^\times$, and morphisms isomorphisms of deformations with effect on the elements of $R^\times$ as before. Since every deformation of $i_* H_n$ is in particular a p-typical formal group law, we get a natural transformation

$$\mathrm{lifts}^\circ_n(-) \Rightarrow FGL^\circ_p(-)$$

This induces a map of a corepresenting Hopf algebroids $(U, US) \to (\hat{U}, \widehat{US})$, where the latter corepresents $\mathrm{lifts}^\circ_n(-)$. It is given by $\hat{U}$ the completion of U with respect to the ideal $I = (p, u_1, u_2, \ldots, u_{n-1}, u_n - 1, u_{n+1}, \ldots)$, which is seen to be correct by trying to corepresent lifts to the every filtration level $F^p(R)$. The morphisms $\widehat{US}$ are similarly given by $\hat{U} \hat{\otimes}_U US \hat{\otimes}_U \hat{U}$. The result is a map of Ext-groups:

$$\mathrm{Ext}^{*,*}_{US}(U, U \otimes_{BP_*} M) \longrightarrow \mathrm{Ext}^{*,*}_{\widehat{US}}(\hat{U}, \hat{U} \hat{\otimes}_{BP_*} M)$$

Strictify using Lubin-Tate theory: As soon as we passed to deformations, the reader must have felt the urge to pass to $\star$-isomorphisms classes of deformations. So let's do exactly that: $\mathrm{lifts}^\circ_{n,\star}(-)$ is the groupoid-valued functor that sends R to the groupoids with objects the pairs of $\star$-isomorphisms of classes of deformations of $i_* H_n$ and elements of $R^\times$ and morphisms the isomorphisms as before. There is a natural transformation

$$\mathrm{lifts}^\circ_n(-) \Rightarrow \mathrm{lifts}^\circ_{n,\star}(-)$$

by taking the quotient on the objects.

Lubin-Tate theory, as in theorem 2.6, implies that this is an equivalence. Indeed, that theorem implies that every deformation is $\star$-isomorphic to one coming from the universal deformation and this is unique up to $\star$-isomorphism. This gives an inverse natural transformation.

This gives us a map $(\hat{U}, \widehat{US}) \leftarrow (\hat{E}(n), \hat{U}(n))$ of Hopf algebroids, where the latter is given by $\hat{E}(n) = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$ using Lubin-Tate theory. There are no Witt vectors yet, as we've fixed the inclusion of $\mathbb{F}_p$. As before $\hat{U}(n)$ is simply $\hat{E}(n) \hat{\otimes}_U US \hat{\otimes}_U \hat{E}(n)$. The result is an isomorphism of Ext-groups:

$$\mathrm{Ext}^{*,*}_{\widehat{US}}(\hat{U}, \hat{U} \hat{\otimes}_{BP_*} M) \xleftarrow{\cong} \mathrm{Ext}^{*,*}_{U\hat{(n)}}(\hat{E}(n), \hat{E}(n) \hat{\otimes}_{BP_*} M)$$

The map θ_M is given by the composite

$$\begin{aligned} \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) &\xrightarrow{\cong} \mathrm{Ext}^{*,*}_{US}(U, U \otimes_{BP_*} M) \\ &\longrightarrow \mathrm{Ext}^{*,*}_{\widehat{US}}(\hat{U}, \hat{U} \hat{\otimes}_{BP_*} M) \\ &\xleftarrow{\cong} \mathrm{Ext}^{*,*}_{U\hat{(n)}}(\hat{E}(n), \hat{E}(n) \hat{\otimes}_{BP_*} M) \end{aligned}$$

□

We now check that this map is an isomorphism in the conditions we're interested in, using the Miller-Ravenel change-of-rings isomorphism and several algebraic lemma's due to Landweber.

Proposition 5.3. *If M satisfies $v_n^{-1} M = M$, then θ_M is an isomorphism.*

Sketch of proof. We use two tricks due to Landweber; (i) every BP_*BP-comodule is a direct limit of finitely presented BP_*BP-comodules, (ii) every finitely presented BP_*BP-comodule has a finite filtration with successive quotients suspensions of BP_*/I_k for some $0 \leq k \leq \infty$.

We can write $M = v_n^{-1}M'$. By (i) we can assume that M' is finitely presented, since Ext's commute with direct colimits in the second entry. So we can apply (ii) and prove the result by induction on the filtration length of M'. This reduces us to proving the result for $v_n^{-1}BP_*/I_k$. But this is non-zero if and only if $k = n$ and hence we need to prove just this single case! This case will be consequence of the Miller-Ravenel change-of-rings theorem. So we're interested in:

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \xrightarrow{\theta} \mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*} v_n^{-1}BP_*/I_n)$$

To the left hand side we can apply the Miller-Ravenel change-of-rings theorem. To the right hand side we want to apply the quotient Hopf algebroid change-of-rings theorem. The idea is to note that $\hat{I}_n = p\mathbb{Z}_p[[u_1, \ldots, u_{n_1}]]$ is an invariant ideal of $\hat{E}(n)$ and $\hat{I}_n$ kills $\hat{E}(n)\hat{\otimes}_{BP_*} v_n^{-1}BP_*/I_n$. The result is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \arrow[r, "\theta"] \arrow[d, "\cong"] & \mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*} v_n^{-1}BP_*/I_n) \arrow[d, "\cong"'] \\
\mathrm{Ext}^{*,*}_{\Sigma(n)}(K(n), K(n)) \arrow[r, "\cong"'] & \mathrm{Ext}^{*,*}_{\hat{U}(n)/\hat{I}_n}(\mathbb{F}_p[u, u^{-1}], \mathbb{F}_p[u, u^{-1}])
\end{tikzcd}$$

where the bottom map is induced by the inclusion map $\mathbb{F}_p[v_n, v_n^{-1}] \to \mathbb{F}_p[u, u]$. This is an equivalence locally in the faithfully flat topology, so an isomorphism. We remark that Devinatz gives a more computational argument in [Dev95, proof of 6.6]. $\square$

This completes the sketch of the first part of theorem 5.1. We now turn to the second part of the theorem, which identifies some Ext-groups with continuous cohomology groups.

Proposition 5.4. *If $v_n^{-1}M = M$ there is an isomorphism*

$$\mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*} M) \cong H^{*,*}_c(S_n, E_n\hat{\otimes}_{BP_*} M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$$

natural in M.

Proof. There are two steps in this proof. By the same argument due to Landweber as before, it suffices to prove this in the case that M is finitely presented, which will be used once in the second step.

Split the functor $\mathrm{lifts}^{\circ}_{n,\star}$**:** Recall that the Hopf algebroid $(\hat{E}(n), \hat{U}(n))$ corepresents the functor $\mathrm{lifts}^{\circ}_{n,\star}(-)$. We want to prove that this functor is equivalent to a split functor. Recall that this means that the groupoid it assigns to a complete filtered $\mathbb{Z}_{(p)}$-algebra with $p \in F^1$ is an action groupoid: the objects are a set S and the morphisms $S \times G$ for a group G.

In our case we want to prove that $\mathrm{lifts}^{\circ}_{n,\star}$ is equivalent to the functor that assigns to R the groupoid with objects those of $\mathrm{lifts}^{\circ}_{n,\star}$ but with morphisms pairs of an object (G, a) of $\mathrm{lifts}^{\circ}_{n,\star}$ and f an automorphism of i_*H_n over $R/F^1(R)$. There is a natural transformation from this functor to $\mathrm{lifts}^{\circ}_{n,\star}$ given by the identity on objects and on morphisms by $((F, a), f) \mapsto (\hat{f}(x), [(F, a)], [(\hat{f}_*F, \hat{f}'(0)^{-1}a)])$, where $\hat{f}$ is any lift of f to $R[[x]]$. This is an equivalence by Lubin-Tate theory.

As a consequence we get a map of Hopf algebroids from $(\hat{E}(n), \hat{U}(n))$ to the Hopf algebroid corepresenting the split version of $\mathrm{lifts}^{\circ}_{n,\star}$ which induces a change-of-rings isomorphism of Ext-groups. What is the corepresenting Hopf algebroid of the latter split functor? For objects the corepresenting algebra is of course still $\hat{E}(n)$, but for objects it is now $\hat{E}(n)\hat{\otimes}D$, where D is the Hopf algebra corepresenting the automorphisms $\mathrm{Aut}_{i_*H_n}(R/F^1(R))$.

To identify D, we start by remarking that [Rav04, proposition A2.2.20] says that we have that $\sum_{i=0}^{\infty} a_i x^i$ is an automorphism of H_n if and only if $a_i^{p^n} = a_i$ for all $i \geq 0$. Let $D^m = \mathbb{F}_p[c_0, c_0^{-1}, c_1, \ldots, c_m]/(c_0^{p_n} - c_0, \ldots, c_m^{p^n} - c_m)$ then D is the direct limit of the D_m's with profinite topology. The map $D^m \to \mathrm{Map}(\mathrm{Hom}_{\mathbb{F}_p}(D^m, \mathbb{F}_{p^n}), \mathbb{F}_{p^n})^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$ given by sending d to the homomorphism $f \mapsto f(d)$ is an isomorphism by noting that D^m is a product of finite field extensions of $\mathbb{F}_p$ and elementary Galois theory.

Next set $S_n = \lim_m \mathrm{Hom}_{\mathbb{F}_p}(D^m, \mathbb{F}_{p^n}) = \mathrm{Map}_c(S_n, \mathbb{F}_{p^n})$ as a profinite group. This is equal to the Morava stabilizer group S_n by definition, hence the notation. Note that the previous description D shows why we used $\mathbb{F}_{p^n}$ and not $\mathbb{F}_p$ or $\bar{\mathbb{F}}_p$ in the definition of S_n: $\mathbb{F}_p$ doesn't see all automorphisms yet and $\bar{\mathbb{F}}_p$ is unnecessarily big, but $\mathbb{F}_{p^n}$ is exactly right.

To corepresent the functor $R \mapsto \mathrm{Aut}_{i_* H_n}(R/F^1(R))$ we need to find the universal complete filtered $\mathbb{Z}_{(p)}$-algebra A with $p \in F^1$ such that the quotient $A/F^1(A)$ is $\mathrm{Map}_c(S_n, \mathbb{F}_{p^n})^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$. But $\mathrm{Map}_c(S_n, \mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$ has this property. We conclude that the split functor is corepresented by the Hopf algebroid $(\hat{E}(n), \hat{E}(n)\hat{\otimes}\mathrm{Map}_c(S_n, \mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)})$ and there is a change-of-rings isomorphism

$$\mathrm{Ext}^{*,*}_{\hat{U}(n)}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*}M) \xrightarrow{\cong} \mathrm{Ext}^{*,*}_{\hat{E}(n)\hat{\otimes}\mathrm{Map}_c(S_n,\mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}}}(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*}M)$$

Identify the result with group cohomology: Next we want to identify the Ext-group that appear on the right-hand side of the previous equation. By the split Hopf algebroid change-of-rings theorem we conclude that

$$\begin{aligned}\mathrm{Ext}^{*,*}_{\hat{E}(n)\hat{\otimes}\mathrm{Map}_c(S_n,\mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}}}&(\hat{E}(n), \hat{E}(n)\hat{\otimes}_{BP_*}M) \\ &\cong \mathrm{Ext}^{*,*}_{\mathrm{Map}_c(S_n,\mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}}}(\mathbb{Z}_p, \mathbb{W}(\mathbb{F}_{p^n})\hat{\otimes}_{\mathbb{Z}_p}M)\end{aligned}$$

To compute the latter we write down the $\mathrm{Map}_c(S_n, \mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}}$-cobar complex for N. We will rewrite this complex slightly to become the complex computing group cohomology coming from the bar resolution. This will involve two plausible sounding technicalities, which we won't prove. First of all, for discrete N, Devinatz proves there is an isomorphism [Dev95, lemma 5.15]

$$\begin{aligned}\mathrm{Map}_c(S_n, \mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}}\hat{\otimes}\ldots\hat{\otimes}&\mathrm{Map}_c(S_n, \mathbb{W}(\mathbb{F}_{p^n}))^{\mathrm{Gal}}\hat{\otimes}N \\ &\cong \mathrm{Map}_c(S_n \times \ldots \times S_n; \mathbb{W}(\mathbb{F}_{p^n})\hat{\otimes}_{BP_*}N)^{\mathrm{Gal}}\end{aligned}$$

The latter computes $H^*_{\mathrm{Gal}}(S_n; \mathbb{W}(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} N)$, but secondly Devinatz proves that this isomorphic to $H^*_c(S_n; \mathbb{W}(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} N)^{\mathrm{Gal}}$, using that for the type of "complete filtered Galois equivariant S_n-modules" that appear we have $\mathbb{W}(\mathbb{F}_{p^n})\hat{\otimes}_{\mathbb{Z}_p}N^{\mathrm{Gal}} \cong N$ [Dev95, lemma 5.4].

We want to set $N = \hat{E}(n)\hat{\otimes}_{BP_*}M$. Remark that since M is assumed to be finitely presented and $v_n^{-1}M = M$ implies that every element of M of annihilated by some power of I_n, there exists a k such that $I_n^k M = 0$. Hence $\hat{E}(n)\hat{\otimes}_{BP_*}M$ is discrete. The previous arguments apply and we get $H^*_c(S_n; E_n \otimes_{\mathbb{Z}_p} N)^{\mathrm{Gal}}$ since $E_n = \mathbb{W}(\mathbb{F}_{p^n})\hat{\otimes}_{\mathbb{Z}_p}\hat{E}(n)$.

□

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