## The J-homomorphism

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## 1 Chromatic level 1

The thesis of this talk is that something very special happens at height 1, without which chromatic homotopy theory could not exist.

Let  $F_m(x, y) = x + y + xy$  be the multiplicative formal group law over  $\mathbb{F}_p$ , and F its p-typification. This has the same p-series as F,  $[p]_F(x) = x^p$  – that is, F is the height 1 Honda formal group law. As we saw earlier today, we have a map  $\mathbb{Z} \to \text{End}(F)$ ,  $n \mapsto [x \mapsto [n](x)]$ , and this extends to  $\mathbb{Z}_p \to \text{End}(F)$ . As Sander discussed, we have  $\text{End}(F) = W(\mathbb{F}_p)\langle S \rangle / (S - p) = W(\mathbb{F}_p) = \mathbb{Z}_p$ . Thus  $\mathbb{S}_1 = \text{Aut}(F) = \mathbb{Z}_p^{\times}$ . For p odd, this is  $\mu_{p-1} \times (1 + p\mathbb{Z}_p)$  and is topologically cyclic (it has a dense cyclic subgroup); for p = 2, it's  $\{\pm 1\} \times (1 + 4\mathbb{Z}_2)$ . We won't talk much about p = 2.

The ring of deformations is  $E(\mathbb{F}_p, F) = W(\mathbb{F}_p) = \mathbb{Z}_p$ , and we can take F as the universal deformation (over  $\mathbb{Z}_p$ ). We'll regrade this by adjoining a generator u of degree -2 and defining  $\widetilde{F}(x, y) = u^{-1}F(ux, uy)$ , so  $(E_1)_* = \mathbb{Z}_p[u, u^{-1}]$ .

Given  $g(x) \in S_1$ , we get  $gF(g^{-1}x, g^{-1}y)$  which is also a deformation of F, so there's a map  $\psi : E(\mathbb{F}_p, F) \to E(\mathbb{F}_p, F)$  classifying it. This is a map from  $W(\mathbb{F}_p) \to W(\mathbb{F}_p)$  that is the identity on  $\mathbb{F}_p$ , so by the universal property of the Witt vectors, it's trivial. Thus, the action of g is trivial in degree zero, and the only interesting action is the effect on  $u, u^{-1} \mapsto g'(0)u^{-1}$ .

**Example 1.** Let  $g(x) = [n]_F(x)$ . We have  $[n]_{F_m}(x) = (1+x)^n - 1$ , so g'(0) = n, and  $\psi(u^{-1}) = nu^{-1}$ , as with the Adams operations. As we'll later see, this is basically where the Adams operations come from.

## 2 The classical theory

Let H(n) be the monoid of homotopy self-equivalences of  $S^n$ . There are evident inclusions  $O(n) \hookrightarrow H(n) \hookrightarrow \Omega^n S^n$ , which induce a map  $J : \pi_i O(n) \to \pi_{n+i} S^n$ . These inclusions commute with the maps increasing n, so we get  $J : \pi_i O \to \pi_i S$ . We can likewise do this for U(n) and U via the inclusions  $U(n) \to SO(2n)$ .

**Definition 2.** The kth Adams operation  $\psi^k : K(X) \to K(X)$  is the unique natural ring homomorphism of K(X) that takes a line bundle to its kth tensor power. (This is unique because of the splitting principle.)

Unfortunately, these operations aren't stable. If  $\beta \in [BU, BU]_2$  is the Bott class, then we have  $\psi^k(\beta) = k\beta \neq \beta \circ \Sigma^2 \psi^k$ . However, if we invert k, then we can define  $\widetilde{\psi}^k$  on the 2nth space of the spectrum K by  $\widetilde{\psi}^k = \psi^k/k^n$ . This is an honest-to-god map of spectra.

In particular, if we complete at p, then we have  $\psi^k$  for all k coprime to p, and we can extend this in an evident way to  $k \in \mathbb{Z}_p^{\times}$ . It's easy to see that the  $\psi^k$  now give the action of  $\mathbb{S}_1$  defined earlier – that is,  $\widehat{K_p}$  is a model for  $E_1$ .

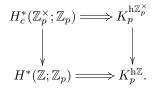
**Conjecture 3** (Adams conjecture). If  $k \in \mathbb{N}$ ,  $x \in K(X)$ , then  $J((1 - \psi^k)x)$  is annihilated by a power of k.

In particular, if k is prime to p, this means that  $J((1 - \psi_k)x)$  is nullhomotopic, so that  $J : U_p \to H_p$ lifts to the homotopy fiber of  $1 - \psi^k : BU_p \to BU_p$ . This conjecture was proved by Quillen, using algebraic K-theory.

Let  $g = (\zeta, 1+p)$  be a topological generator of  $\mathbb{Z}_p^{\times}$ .

**Proposition 4.** The homotopy fiber  $J_p$  of  $(1 - \psi^g)$  is  $K_p^{h\mathbb{Z}_p^{\times}} = E_1^{h\mathbb{S}_1} = L_{K(1)}S$ .

Sketch of proof. g generates a cyclic subgroup  $\mathbb{Z} \subseteq \mathbb{Z}_p^{\times}$ , and the homotopy fiber of  $(1 - \psi^g)$  is  $K_p^{\mathbb{h}\mathbb{Z}}$ . There's a map of homotopy fixed point spectral sequences



One can compute the groups on the left, and show that the left-hand map is an isomorphism.

Adams showed that  $\pi_n(J_p)$  is a split summand of  $\pi_*S_{(p)}$  for  $n \ge 0$ . So the above proposition identifies the classical picture and the chromatic one.

Moreover, we can use this to compute  $\pi_* L_{K(1)}S$ . After all, there's a fiber sequence

$$L_{K(1)}S \to K_p \stackrel{1-\psi^g}{\to} K_p.$$

 $\pi_n K_p$  is zero for odd n, and in even n,  $\psi^g$  acts by  $g^k$ . We thus get

$$\pi_n L_{K(1)} S = \begin{cases} \mathbb{Z}_p & n = 0, -1 \\ \mathbb{Z}/p^{r+1} \mathbb{Z} & n = 2(p-1)p^r \ell, p \not\mid \ell \ell \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5. The Bernoulli numbers are defined via the power series

$$\frac{x}{e^x-1} = \sum_{t=0}^\infty \beta_t \frac{x^t}{t!}$$

We have  $\beta_1 = \frac{-1}{2}$  and  $\beta_{2t+1} = 0$  for t > 0. We let m(2s) be the denominator of  $\frac{\beta_{2s}}{4s}$ .

Then the above r+1 is  $v_p(m((n+1)/2))$ , so that the order of the cyclic group  $\pi_n L_{K(1)}S$  is the *p*-component of the denominator of  $\frac{\beta_{(n+1)/2}}{n+1}$ .

At height 1, the chromatic fracture square is given by

Of course,  $L_{E(0)}$  is just rationalization, i.e. smashing with  $H\mathbb{Q} = S\mathbb{Q}$ . Thus  $L_{E(0)}L_{K(1)}S$  has homotopy groups  $\mathbb{Q}_p$  in degrees 0 and -1 and 0 otherwise. We can thus compute the E(1)-local sphere:

$$\pi_n L_{E(1)} S = \begin{cases} \pi_n L_{K(1)} S & n \neq 0, -1, -2 \\ \mathbb{Q}_p / \mathbb{Z}_p & n = -2 \\ \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Adams began (in On the groups J(X) IV) by defining a map  $\pi_{2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$ , called the *e*-invariant. Let  $g: S^{2m-1} \to S^{2n}$  be a representative element of this homotopy group, with cofiber  $C_g$ . We get a diagram

3

where the vertical maps are the Chern character. If  $\alpha$  and a are images of generators along the lefthand horizontal maps, and  $\beta$  and b preimages of generators along the right-hand horizontal maps, we have  $ch(\beta) = ra + b$  for some  $r \in \mathbb{Q}$ , which is well-defined mod  $\mathbb{Z}$ . This is the *e*-invariant of *g*.

Also, for any  $f: X \to Y$ , we define its *d*-invariant to be  $d(f) = f^* \in \text{Hom}(\widetilde{K}(Y), \widetilde{K}(X))$ . If  $d(f) = d(\Sigma f) = 0$ , then there's a short exact sequence  $0 \to \widetilde{K}(\Sigma X) \to \widetilde{K}(C_f) \to \widetilde{K}(Y) \to 0$ , and we could also define the *e*-invariant of f as the element of  $\text{Ext}^1(\widetilde{K}(Y), \widetilde{K}(X))$  representing this.

Adams showed that if  $f \in \pi_{2k-1}U(n)$ , then e(Jf) is the denominator of the Bernoulli number given above. If  $g = Jf \in \pi_{2n+2k-1}S^{2n}$ , then  $C_{Jf} \simeq T(E_f)$ , the Thom spectrum of  $E_f$  and this isomorphism sends  $\beta$  to the Thom class.

## 3 Constructing a $v_1$ -self map

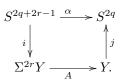
**Theorem 6.** If p is odd,  $m = p^{f}$ , and  $r = (p-1)p^{f-1}$ , then there exists a class  $\alpha \in \pi_{2r-1}S$  such that

- (i)  $m\alpha = 0$ ,
- (ii)  $e(\alpha) = -1/m$ , and
- (iii) the Toda bracket  $\langle m, \alpha, m \rangle$  is 0 mod m.

If we have such a class  $\alpha$ , then for q sufficiently large, we have maps

$$S^{2q+2r-2} \xrightarrow{m} S^{2q+2r-2} \xrightarrow{\alpha} S^{2q-1} \xrightarrow{m} S^{2q-1}$$

whose Toda bracket we can form, a map  $\langle m, \alpha, m \rangle : S^{2q+2r-1} \to S^{2q-1}$ . Since this is zero mod m, it factors through the Moore space Y that is the cofiber of  $S^{2q-1} \xrightarrow{m} S^{2q-1}$ , giving a map  $A : \Sigma^{2r}Y \to Y$ . We have a diagram



where i is the inclusion of the bottom cell and j is projection to the top cell.Adams discovered rules for the d- and e-invariants of compositions in terms of the invariants of the individual factors. IN particular,

$$d(j)d(A) = d(jA) = d(i)e(\alpha) = -me(\alpha) = 1,$$

proving that A is an isomorphism on K-theory.

We can in fact iterate this. Let  $A^{(s)} = A \circ \Sigma^{2r} A \circ \cdots \circ \Sigma^{2r(s-1)} A$ , a map  $\Sigma^{2rs} Y \to Y$ . Including the bottom cell and projecting to the top cell gives a map  $\alpha_s : S^{2q+2rs-1} \to S^{2q}$ . As before, you can show that  $e(\alpha_s) = -1/m$ , so that  $\alpha_s$  is in particular nontrivial in homotopy. This was the first infinite family of elements found in the stable homotopy groups of spheres.