The J-homomorphism

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1 Chromatic level 1

The thesis of this talk is that something very special happens at height 1, without which chromatic homotopy theory could not exist.

Let $F_m(x, y) = x + y + xy$ be the multiplicative formal group law over $\mathbb{F}_p$, and F its p-typification. This has the same p-series as F, $[p]_F(x) = x^p$ – that is, F is the height 1 Honda formal group law. As we saw earlier today, we have a map $\mathbb{Z} \to \mathrm{End}(F)$, $n \mapsto [x \mapsto [n](x)]$, and this extends to $\mathbb{Z}_p \to \mathrm{End}(F)$. As Sander discussed, we have $\mathrm{End}(F) = W(\mathbb{F}_p)\langle S\rangle/(S-p) = W(\mathbb{F}_p) = \mathbb{Z}_p$. Thus $\mathbb{S}_1 = \mathrm{Aut}(F) = \mathbb{Z}_p^\times$. For p odd, this is $\mu_{p-1} \times (1+p\mathbb{Z}_p)$ and is topologically cyclic (it has a dense cyclic subgroup); for $p = 2$, it's $\{\pm 1\} \times (1+4\mathbb{Z}_2)$. We won't talk much about $p = 2$.

The ring of deformations is $E(\mathbb{F}_p, F) = W(\mathbb{F}_p) = \mathbb{Z}_p$, and we can take F as the universal deformation (over $\mathbb{Z}_p$). We'll regrade this by adjoining a generator u of degree -2 and defining $\widetilde{F}(x, y) = u^{-1}F(ux, uy)$, so $(E_1)_* = \mathbb{Z}_p[u, u^{-1}]$.

Given $g(x) \in \mathbb{S}_1$, we get $gF(g^{-1}x, g^{-1}y)$ which is also a deformation of F, so there's a map $\psi : E(\mathbb{F}_p, F) \to E(\mathbb{F}_p, F)$ classifying it. This is a map from $W(\mathbb{F}_p) \to W(\mathbb{F}_p)$ that is the identity on $\mathbb{F}_p$, so by the universal property of the Witt vectors, it's trivial. Thus, the action of g is trivial in degree zero, and the only interesting action is the effect on u, $u^{-1} \mapsto g'(0)u^{-1}$.

Example 1. Let $g(x) = [n]_F(x)$. We have $[n]_{F_m}(x) = (1 + x)^n - 1$, so $g'(0) = n$, and $\psi(u^{-1}) = nu^{-1}$, as with the Adams operations. As we'll later see, this is basically where the Adams operations come from.

2 The classical theory

Let $H(n)$ be the monoid of homotopy self-equivalences of S^n. There are evident inclusions $O(n) \hookrightarrow H(n) \hookrightarrow \Omega^n S^n$, which induce a map $J : \pi_i O(n) \to \pi_{n+i} S^n$. These inclusions commute with the maps increasing n, so we get $J : \pi_i O \to \pi_i S$. We can likewise do this for $U(n)$ and U via the inclusions $U(n) \to SO(2n)$.

Definition 2. The kth **Adams operation** $\psi^k : K(X) \to K(X)$ is the unique natural ring homomorphism of $K(X)$ that takes a line bundle to its kth tensor power. (This is unique because of the splitting principle.)

Unfortunately, these operations aren't stable. If $\beta \in [BU, BU]_2$ is the Bott class, then we have $\psi^k(\beta) = k\beta \neq \beta \circ \Sigma^2\psi^k$. However, if we invert k, then we can define $\widetilde{\psi^k}$ on the $2n$th space of the spectrum K by $\widetilde{\psi^k} = \psi^k/k^n$. This is an honest-to-god map of spectra.

In particular, if we complete at p, then we have ψ^k for all k coprime to p, and we can extend this in an evident way to $k \in \mathbb{Z}_p^\times$. It's easy to see that the ψ^k now give the action of $\mathbb{S}_1$ defined earlier – that is, $K\widehat{\ }_p$ is a model for E_1.

Conjecture 3 (Adams conjecture). *If $k \in \mathbb{N}$, $x \in K(X)$, then $J((1 - \psi^k)x)$ is annihilated by a power of k.*

In particular, if k is prime to p, this means that $J((1 - \psi_k)x)$ is nullhomotopic, so that $J : U_p \to H_p$ lifts to the homotopy fiber of $1 - \psi^k : BU_p \to BU_p$. This conjecture was proved by Quillen, using algebraic K-theory.

Let $g = (\zeta, 1 + p)$ be a topological generator of $\mathbb{Z}_p^\times$.

Proposition 4. *The homotopy fiber J_p of $(1-\psi^g)$ is $K_p^{h\mathbb{Z}_p^\times} = E_1^{h\mathbb{S}_1} = L_{K(1)}S$.*

Sketch of proof. g generates a cyclic subgroup $\mathbb{Z} \subseteq \mathbb{Z}_p^\times$, and the homotopy fiber of $(1-\psi^g)$ is $K_p^{h\mathbb{Z}}$. There's a map of homotopy fixed point spectral sequences

$$\begin{tikzcd} H_c^*(\mathbb{Z}_p^\times;\mathbb{Z}_p) \arrow[r, Rightarrow] \arrow[d] & K_p^{h\mathbb{Z}_p^\times} \arrow[d] \\ H^*(\mathbb{Z};\mathbb{Z}_p) \arrow[r, Rightarrow] & K_p^{h\mathbb{Z}}. \end{tikzcd}$$

One can compute the groups on the left, and show that the left-hand map is an isomorphism. □

Adams showed that $\pi_n(J_p)$ is a split summand of $\pi_* S_{(p)}$ for $n \geq 0$. So the above proposition identifies the classical picture and the chromatic one.

Moreover, we can use this to compute $\pi_* L_{K(1)}S$. After all, there's a fiber sequence

$$L_{K(1)}S \to K_p \overset{1-\psi^g}{\to} K_p.$$

$\pi_n K_p$ is zero for odd n, and in even n, ψ^g acts by g^k. We thus get

$$\pi_n L_{K(1)}S = \begin{cases} \mathbb{Z}_p & n = 0, -1 \\ \mathbb{Z}/p^{r+1}\mathbb{Z} & n = 2(p-1)p^r\ell,\ p \not| \ell \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5. The Bernoulli numbers are defined via the power series

$$\frac{x}{e^x - 1} = \sum_{t=0}^\infty \beta_t \frac{x^t}{t!}.$$

We have $\beta_1 = \frac{-1}{2}$ and $\beta_{2t+1} = 0$ for $t > 0$. We let $m(2s)$ be the denominator of $\frac{\beta_{2s}}{4s}$.

Then the above $r+1$ is $v_p(m((n+1)/2))$, so that the order of the cyclic group $\pi_n L_{K(1)}S$ is the p-component of the denominator of $\frac{\beta_{(n+1)/2}}{n+1}$.

At height 1, the chromatic fracture square is given by

$$\begin{tikzcd} L_{E(1)}S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(1)}S \arrow[d] \\ L_{E(0)}S \arrow[r] & L_{E(0)}L_{K(1)}S. \end{tikzcd}$$

Of course, $L_{E(0)}$ is just rationalization, i.e. smashing with $H\mathbb{Q} = S\mathbb{Q}$. Thus $L_{E(0)}L_{K(1)}S$ has homotopy groups $\mathbb{Q}_p$ in degrees 0 and -1 and 0 otherwise. We can thus compute the $E(1)$-local sphere:

$$\pi_n L_{E(1)}S = \begin{cases} \pi_n L_{K(1)}S & n \neq 0, -1, -2 \\ \mathbb{Q}_p/\mathbb{Z}_p & n = -2 \\ \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Adams began (in *On the groups $J(X)$ IV*) by defining a map $\pi_{2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$, called the **$e$-invariant**. Let $g : S^{2m-1} \to S^{2n}$ be a representative element of this homotopy group, with cofiber C_g. We get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}(S^{2m}) \arrow[r] \arrow[d] & \widetilde{K}(C_g) \arrow[r] \arrow[d] & \widetilde{K}(S^{2n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{H}^*(S^{2m};\mathbb{Q}) \arrow[r] & \widetilde{H}^*(C_g;\mathbb{Q}) \arrow[r] & \widetilde{H}^*(S^{2n};\mathbb{Q}) \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are the Chern character. If α and a are images of generators along the left-hand horizontal maps, and β and b preimages of generators along the right-hand horizontal maps, we have $ch(\beta) = ra + b$ for some $r \in \mathbb{Q}$, which is well-defined mod $\mathbb{Z}$. This is the e-invariant of g.

Also, for any $f : X \to Y$, we define its **d-invariant** to be $d(f) = f^* \in \mathrm{Hom}(\widetilde{K}(Y), \widetilde{K}(X))$. If $d(f) = d(\Sigma f) = 0$, then there's a short exact sequence $0 \to \widetilde{K}(\Sigma X) \to \widetilde{K}(C_f) \to \widetilde{K}(Y) \to 0$, and we could also define the e-invariant of f as the element of $\mathrm{Ext}^1(\widetilde{K}(Y), \widetilde{K}(X))$ representing this.

Adams showed that if $f \in \pi_{2k-1}U(n)$, then $e(Jf)$ is the denominator of the Bernoulli number given above. If $g = Jf \in \pi_{2n+2k-1}S^{2n}$, then $C_{Jf} \simeq T(E_f)$, the Thom spectrum of E_f and this isomorphism sends β to the Thom class.

3 Constructing a v_1-self map

Theorem 6. *If p is odd, $m = p^f$, and $r = (p-1)p^{f-1}$, then there exists a class $\alpha \in \pi_{2r-1}S$ such that*

(i) $m\alpha = 0$,

(ii) $e(\alpha) = -1/m$, and

(iii) the Toda bracket $\langle m, \alpha, m\rangle$ is 0 mod m.

If we have such a class α, then for q sufficiently large, we have maps

$$S^{2q+2r-2} \xrightarrow{m} S^{2q+2r-2} \xrightarrow{\alpha} S^{2q-1} \xrightarrow{m} S^{2q-1}$$

whose Toda bracket we can form, a map $\langle m, \alpha, m\rangle : S^{2q+2r-1} \to S^{2q-1}$. Since this is zero mod m, it factors through the Moore space Y that is the cofiber of $S^{2q-1} \xrightarrow{m} S^{2q-1}$, giving a map $A : \Sigma^{2r}Y \to Y$. We have a diagram

$$\begin{tikzcd} S^{2q+2r-1} \arrow[r, "\alpha"] \arrow[d, "i"'] & S^{2q} \\ \Sigma^{2r}Y \arrow[r, "A"'] & Y. \arrow[u, "j"'] \end{tikzcd}$$

where i is the inclusion of the bottom cell and j is projection to the top cell.Adams discovered rules for the d- and e-invariants of compositions in terms of the invariants of the individual factors. IN particular,

$$d(j)d(A) = d(jA) = d(i)e(\alpha) = -me(\alpha) = 1,$$

proving that A is an isomorphism on K-theory.

We can in fact iterate this. Let $A^{(s)} = A \circ \Sigma^{2r}A \circ \cdots \circ \Sigma^{2r(s-1)}A$, a map $\Sigma^{2rs}Y \to Y$. Including the bottom cell and projecting to the top cell gives a map $\alpha_s : S^{2q+2rs-1} \to S^{2q}$. As before, you can show that $e(\alpha_s) = -1/m$, so that α_s is in particular nontrivial in homotopy. This was the first infinite family of elements found in the stable homotopy groups of spheres.