The K(n)-local sphere

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1 Goals

The K(n)-local sphere has a resolution

$$\pi_*L_{K(n)}S \longrightarrow \pi_*E_n \longrightarrow \pi_*L_{K(n)}(E_n \wedge E_n) \xrightarrow{\longrightarrow} \cdots$$

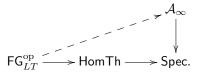
and by work of Morava, Lubin-Tate theory, and so on, $\pi_*L_{K(n)}(E_n \wedge E_n) \cong \operatorname{Map}_c(\mathbb{G}_n, (E_n)_*)$. Thus, the ANSS for $L_{K(n)}S$ can be rewritten as a spectral sequence $H_c^*(\mathbb{G}_n, (E_n)_*) \Rightarrow \pi_*L_{K(n)}S$. Our first goal is to show that \mathbb{G}_n acts on the spectrum E_n , giving a homotopy fixed point spectral sequence $H^*(\mathbb{G}_n, (E_n)_*) \Rightarrow \pi_*E_n^{\mathrm{h}'\mathbb{G}_n}$, the discrete homotopy fixed points of the \mathbb{G}_n -action on E_n . Our second goal is to define continuous homotopy fixed points, replace this spectral sequence with the continuous version, and compare the two spectral sequences giving $\pi_*E_n^{\mathrm{h}\mathbb{G}_n} \cong \pi_*L_{K(n)}S$.

2 The \mathbb{G}_n -action on E_n

We'll in fact construct something more general than this – we'll show that any algebraic map of coefficient rings of Lubin-Tate-type spectra can be realized topologically as an \mathcal{A}_{∞} map.

Fix p and let FG_{LT} be the category of formal groups of finite height over a perfect field of characteristic p. Taking the universal deformation by Lubin-Tate theory and applying the LEFT gives a contravariant functor to the category of homology theories.

Theorem 1 (Hopkins-Miller). A lift exists in the following diagram:



(In fact, we can even lift to the category of E_{∞} spectra, though we won't investigate that now.)

Definition 2. $\mathcal{A}_{\infty}^{LT}$ is the full subcategory of \mathcal{A}_{∞} on the spectra *E* such that:

- 1. E is an even periodic, homotopy commutative ring spectrum;
- 2. $\pi_0 E$ is a complete local ring, with maximal ideal \mathfrak{m} , residue field $k = \pi_0 E/\mathfrak{m}$ perfect and positive characteristic, and E is complex oriented by the FGL corresponding to the universal deformation of k.
- 3. E is cofibrant.

If $i: \mathcal{A}_{\infty}^{LT} \to \mathcal{A}_{\infty}$ is the inclusion, and π is the map $\mathcal{A}_{\infty}^{LT} \to \mathsf{FG}_{LT}^{\mathrm{op}}$ sending E to the FGL corresponding to the universal deformation of $\pi_0 E \mod \mathfrak{m}$, then the homotopy right Kan extension hRan $(\pi): \mathsf{FG}_{LT}^{\mathrm{op}} \to \mathcal{A}_{\infty}$ will be the desired lift, provided we can show that π is an equivalence. We do so in two lemmas.

Lemma 3. π is fully faithful.

Proof. Let $E, F \in \mathcal{A}_{\infty}^{LT}$ with formal groups Γ_1 and Γ_2 , and fix an \mathcal{A}_{∞} -operad C. Consider the simplicial resolution

$$F \longleftarrow CF \xleftarrow{} CCF \xleftarrow{} \cdots$$

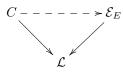
and let Y^{\bullet} be the cosimplicial object $Y^n = C - \operatorname{Alg}(C^{n+1}F, E)$. Then $\operatorname{Tot} Y^{\bullet} \simeq C - \operatorname{Alg}(F, E)$. There's a Bousfield-Kan spectral sequence $\pi^s \pi_t Y^{\bullet} \Rightarrow \pi_{t-s} \operatorname{Tot} Y^{\bullet}$, where π^s is cohomotopy groups. A map of ring spectra $F \to E$ corresponds to an element of the equalizer of the two maps from $\pi_0 Y^0$ to $\pi_0 Y^1$, that is, an element of $E_2^{0,0}$. The obstructions to realizing this as an \mathcal{A}_{∞} map occur in the groups $E_2^{s,s-1}$, that is, along the Adams (-1)-line. One does some computations to show that the spectral sequence becomes zero at the E_2 -page. Thus we have

$$\pi_0 C - \mathsf{Alg}(F, E) = E_{\infty}^{0,0} \cong E_2^{0,0} \cong E_{\infty} - \mathsf{Alg}(E_*F, E_*).$$

By algebra and our assumptions, this is just $\operatorname{Hom}(F_*, E_*) \cong \operatorname{FG}_{LT}(\Gamma_1, \Gamma_2)$.

Lemma 4. π is essentially surjective.

Proof. Let $\Gamma \in \mathsf{FG}_{LT}$ corresponding to the spectrum E via Lubin-Tate; we want to show that E has a \mathcal{A}_{∞} structure. It suffices to construct a map



with C an \mathcal{A}_{∞} -operad, $\mathcal{E}_E[n] = \mathcal{S}(E^{(n)}, E)$ and \mathcal{L} the linear isometries operad. (We're using the LMS construction of spectra, in which the spectrum structure of E is precisely a map from its endomorphism operad to \mathcal{L} .) Let A be an \mathcal{A}_{∞} -operad and $C_* \to \operatorname{Sing} A$ a cofibrant resolution, giving a map $|C_*| \to A$. Define $Y^n = (\operatorname{Operad} \downarrow \mathcal{L})(C_n, \mathcal{E}_E)$. Again we examine the Bousfield-Kan spectral sequence for $\pi_* \operatorname{Tot} Y^{\bullet}$ and observe that everything vanishes on the E_2 page besides $E_2^{0,0}$. By the same argument as in the previous lemma,

$$\pi_0(\mathsf{Operad} \downarrow \mathcal{L})(C, \mathcal{E}_E) \cong \mathrm{Hom}(\pi_0 C, \pi_0 \mathcal{E}_E),$$

and so the desired map $C \to \mathcal{E}_E$ exists, giving E an \mathcal{A}_{∞} structure. This proves the theorem.

3 The E_{∞} case

Definition 5. The moduli space of a commutative E_*E -comodule algebra A (that is, an E_*E -comodule with a compatible commutative E_* -algebra structure) is TM(A), the classifying space of the category of E_{∞} -spectra X with $E_*X \cong A$ and E_* -isomorphisms.

Theorem 6 (Goerss-Hopkins). If Γ is a FGL giving the spectrum E by Lubin-Tate, then $TM(E_*E)$ has homotopy type

$$B\operatorname{Aut}_{E_*-\operatorname{Alg}}(E_*E) \simeq B\operatorname{Aut}(\Gamma).$$

This is proved in a similar way as before, though the obstruction theory is harder. The above allows us to lift the \mathbb{G}_n -action on $(E_n)_*$ to one on E_n .

4 Homotopy fixed points

Definition 7. Let $U \subseteq \mathbb{G}_n$ be an open subgroup. Then we define

$$E_n^{\mathsf{h}U} = \operatorname{Tot} \pi^* L_{K(n)}(\pi_{\mathbb{G}_n/U} E_n \wedge E_n^{\wedge \bullet})$$

Here, for $X : \mathcal{J} \to h\mathcal{E}$ a diagram, we define

$$\pi^n X = \prod_{\substack{[j_n \stackrel{\alpha_n}{\rightarrow} j_{n-1} \to \dots \stackrel{\alpha_1}{\rightarrow} j_0] \in N(\mathcal{J})_n}} F(|\operatorname{Sing} \mathcal{E}(X_{j_n}, X_{j_{n-1}})_{\alpha_n}| \times \dots \times |\operatorname{Sing} \mathcal{E}(X_{j_1}, X_{j_0})_{\alpha_0}|, X_{j_0}).$$

4. HOMOTOPY FIXED POINTS

There's a spectral sequence

$$\pi^s[Z, \pi^* L_{K(n)} C_{\mathbb{G}_n/U}]^t \Rightarrow (E_n^{\mathrm{h}U})^{t+s} Z.$$

With some work, one can show that this E_2 page is precisely $H_c^s(U, E_n^t Z)$.