# The $K(n)$-local sphere

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## 1 Goals

The $K(n)$-local sphere has a resolution

$$\pi_* L_{K(n)} S \longrightarrow \pi_* E_n \longrightarrow \pi_* L_{K(n)}(E_n \wedge E_n) \rightrightarrows \cdots$$

and by work of Morava, Lubin-Tate theory, and so on, $\pi_* L_{K(n)}(E_n \wedge E_n) \cong \mathrm{Map}_c(\mathbb{G}_n, (E_n)_*)$. Thus, the ANSS for $L_{K(n)}S$ can be rewritten as a spectral sequence $H_c^*(\mathbb{G}_n, (E_n)_*) \Rightarrow \pi_* L_{K(n)} S$. Our first goal is to show that $\mathbb{G}_n$ acts on the spectrum $E_n$, giving a homotopy fixed point spectral sequence $H^*(\mathbb{G}_n, (E_n)_*) \Rightarrow \pi_* E_n^{\mathrm{h}'\mathbb{G}_n}$, the *discrete* homotopy fixed points of the $\mathbb{G}_n$-action on $E_n$. Our second goal is to define continuous homotopy fixed points, replace this spectral sequence with the continuous version, and compare the two spectral sequences giving $\pi_* E_n^{\mathrm{h}\mathbb{G}_n} \cong \pi_* L_{K(n)} S$.

## 2 The $\mathbb{G}_n$-action on $E_n$

We'll in fact construct something more general than this – we'll show that any algebraic map of coefficient rings of Lubin-Tate-type spectra can be realized topologically as an $\mathcal{A}_\infty$ map.

Fix $p$ and let $\mathsf{FG}_{LT}$ be the category of formal groups of finite height over a perfect field of characteristic $p$. Taking the universal deformation by Lubin-Tate theory and applying the LEFT gives a contravariant functor to the category of homology theories.

**Theorem 1** (Hopkins-Miller)**.** *A lift exists in the following diagram:*

$$\begin{tikzcd} & & \mathcal{A}_\infty \arrow[d] \\ \mathsf{FG}_{LT}^{\mathrm{op}} \arrow[r] \arrow[urr, dashed] & \mathsf{HomTh} \arrow[r] & \mathsf{Spec}. \end{tikzcd}$$

(In fact, we can even lift to the category of $E_\infty$ spectra, though we won't investigate that now.)

**Definition 2.** $\mathcal{A}_\infty^{LT}$ is the full subcategory of $\mathcal{A}_\infty$ on the spectra $E$ such that:

1. $E$ is an even periodic, homotopy commutative ring spectrum;
2. $\pi_0 E$ is a complete local ring, with maximal ideal $\mathfrak{m}$, residue field $k = \pi_0 E/\mathfrak{m}$ perfect and positive characteristic, and $E$ is complex oriented by the FGL corresponding to the universal deformation of $k$.
3. $E$ is cofibrant.

If $i : \mathcal{A}_\infty^{LT} \to \mathcal{A}_\infty$ is the inclusion, and $\pi$ is the map $\mathcal{A}_\infty^{LT} \to \mathsf{FG}_{LT}^{\mathrm{op}}$ sending $E$ to the FGL corresponding to the universal deformation of $\pi_0 E$ mod $\mathfrak{m}$, then the homotopy right Kan extension $\mathrm{hRan}(\pi) : \mathsf{FG}_{LT}^{\mathrm{op}} \to \mathcal{A}_\infty$ will be the desired lift, provided we can show that $\pi$ is an equivalence. We do so in two lemmas.

**Lemma 3.** *$\pi$ is fully faithful.*

*Proof.* Let $E,\, F \in \mathcal{A}_\infty^{LT}$ with formal groups $\Gamma_1$ and $\Gamma_2$, and fix an $\mathcal{A}_\infty$-operad $C$. Consider the simplicial resolution

$$\begin{tikzcd} F & CF \arrow[l] & CCF \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

and let $Y^\bullet$ be the cosimplicial object $Y^n = C{-}\mathsf{Alg}(C^{n+1}F, E)$. Then $\operatorname{Tot} Y^\bullet \simeq C{-}\mathsf{Alg}(F, E)$. There's a Bousfield-Kan spectral sequence $\pi^s \pi_t Y^\bullet \Rightarrow \pi_{t-s} \operatorname{Tot} Y^\bullet$, where $\pi^s$ is cohomotopy groups. A map of ring spectra $F \to E$ corresponds to an element of the equalizer of the two maps from $\pi_0 Y^0$ to $\pi_0 Y^1$, that is, an element of $E_2^{0,0}$. The obstructions to realizing this as an $\mathcal{A}_\infty$ map occur in the groups $E_2^{s,s-1}$, that is, along the Adams $(-1)$-line. One does some computations to show that the spectral sequence becomes zero at the $E_2$-page. Thus we have

$$\pi_0 C{-}\mathsf{Alg}(F, E) = E_\infty^{0,0} \cong E_2^{0,0} \cong E_\infty{-}\mathsf{Alg}(E_* F, E_*).$$

By algebra and our assumptions, this is just $\operatorname{Hom}(F_*, E_*) \cong \mathsf{FG}_{LT}(\Gamma_1, \Gamma_2)$. $\square$

**Lemma 4.** *$\pi$ is essentially surjective.*

*Proof.* . Let $\Gamma \in \mathsf{FG}_{LT}$ corresponding to the spectrum $E$ via Lubin-Tate; we want to show that $E$ has a $\mathcal{A}_\infty$ structure. It suffices to construct a map

$$\begin{tikzcd} C \arrow[rr, dashed] \arrow[dr] & & \mathcal{E}_E \arrow[dl] \\ & \mathcal{L} & \end{tikzcd}$$

with $C$ an $\mathcal{A}_\infty$-operad, $\mathcal{E}_E[n] = \mathcal{S}(E^{(n)}, E)$ and $\mathcal{L}$ the linear isometries operad. (We're using the LMS construction of spectra, in which the spectrum structure of $E$ is precisely a map from its endomorphism operad to $\mathcal{L}$.) Let $A$ be an $\mathcal{A}_\infty$-operad and $C_* \to \operatorname{Sing} A$ a cofibrant resolution, giving a map $|C_*| \to A$. Define $Y^n = (\mathsf{Operad} \downarrow \mathcal{L})(C_n, \mathcal{E}_E)$. Again we examine the Bousfield-Kan spectral sequence for $\pi_* \operatorname{Tot} Y^\bullet$ and observe that everything vanishes on the $E_2$ page besides $E_2^{0,0}$. By the same argument as in the previous lemma,

$$\pi_0 (\mathsf{Operad} \downarrow \mathcal{L})(C, \mathcal{E}_E) \cong \operatorname{Hom}(\pi_0 C, \pi_0 \mathcal{E}_E),$$

and so the desired map $C \to \mathcal{E}_E$ exists, giving $E$ an $\mathcal{A}_\infty$ structure. This proves the theorem. $\square$

## 3 The $E_\infty$ case

**Definition 5.** The moduli space of a commutative $E_*E$-comodule algebra $A$ (that is, an $E_*E$-comodule with a compatible commutative $E_*$-algebra structure) is $TM(A)$, the classifying space of the category of $E_\infty$-spectra $X$ with $E_*X \cong A$ and $E_*$-isomorphisms.

**Theorem 6** (Goerss-Hopkins)**.** *If $\Gamma$ is a FGL giving the spectrum $E$ by Lubin-Tate, then $TM(E_*E)$ has homotopy type*

$$B \operatorname{Aut}_{E_*{-}\mathsf{Alg}}(E_*E) \simeq B \operatorname{Aut}(\Gamma).$$

This is proved in a similar way as before, though the obstruction theory is harder.

The above allows us to lift the $\mathbb{G}_n$-action on $(E_n)_*$ to one on $E_n$.

## 4 Homotopy fixed points

**Definition 7.** Let $U \subseteq \mathbb{G}_n$ be an open subgroup. Then we define

$$E_n^{\mathrm{h}U} = \operatorname{Tot} \pi^* L_{K(n)}(\pi_{\mathbb{G}_n/U} E_n \wedge E_n^{\wedge \bullet}).$$

Here, for $X : \mathcal{J} \to \mathrm{h}\mathcal{E}$ a diagram, we define

$$\pi^n X = \prod_{[j_n \xrightarrow{\alpha_n} j_{n-1} \to \cdots \xrightarrow{\alpha_1} j_0] \in N(\mathcal{J})_n} F(|\operatorname{Sing} \mathcal{E}(X_{j_n}, X_{j_{n-1}})_{\alpha_n}| \times \cdots \times |\operatorname{Sing} \mathcal{E}(X_{j_1}, X_{j_0})_{\alpha_0}|, X_{j_0}).$$

There's a spectral sequence

$$\pi^s[Z, \pi^* L_{K(n)} C_{\mathbb{G}_n/U}]^t \Rightarrow (E_n^{\mathrm{h}U})^{t+s} Z.$$

With some work, one can show that this $E_2$ page is precisely $H_c^s(U, E_n^t Z)$.