Nilpotence

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1 Various forms of nilpotence

The first statement on nilpotence in the homotopy groups of spheres is

Theorem 1 (Nishida)**.** *Every $\alpha \in \pi_n(S^0)$ for $n > 0$ is nilpotent.*

Definition 2. Let E be an associative ring spectrum. If $f : X \to Y$ is such that $1_E \wedge f^{\wedge n} : E \wedge X^{\wedge n} \to E \wedge Y^{\wedge n}$ is trivial for some n, then f is **E-$\wedge$ nilpotent**. If $f : \Sigma^d X \to X$ is a self-map such that $1_E \wedge f^{\circ n}$ is trivial for some n, then f is **E-$\circ$ nilpotent**. (Here by $f^{\circ n}$, we mean the composition $f \circ \Sigma^d f \circ \cdots \circ \Sigma^{(n-1)d} f$.) If R is a (not necessarily associative) ring spectrum, $\alpha \in \pi_* R$, and $1_E \wedge (\mu^n \alpha^n)$ is trivial for any sequence of multiplications $\mu^n : R^n \to R$, then α is **E-Hurewicz nilpotent**.

If $\alpha \in \pi_n R$ for R a ring spectrum, then there's a self-map $\overline{\alpha} : \Sigma^n R \to R \wedge R \to R$, and we define

$$\overline{\alpha}^{-1} R = \operatorname{hocolim}(R \to \Sigma^{-n} R \to \Sigma^{-2n} R \to \cdots).$$

(For example, this could be constructed as a mapping telescope.)

Lemma 3. *$\overline{\alpha}^{-1} R \wedge E \simeq *$ iff α is E-Hurewicz nilpotent.*

2 The main theorem

Theorem 4 (Nilpotence theorem, Devinatz-Hopkins-Smith)**.**

I. Suppose R is a connective associative ring spectrum of finite type. Then if $\alpha \in \pi_n(R)$ is MU-Hurewicz nilpotent, it is nilpotent in $\pi_* R$.

II. The same, where R is any ring spectrum.

III. If $f : F \to X$ for F a finite spectrum is MU-$\wedge$ nilpotent, then f is nilpotent.

IV. If we have a sequence

$$\cdots \to X_{n+1} \overset{f_{n+1}}{\to} X_n \overset{f_n}{\to} X_{n-1} \to \cdots,$$

and f_n is c_n-connected for $c_n \geq mn + b$ for some m, b, with $MU(f_n) = 0$ for all n, then the homotopy colimit of this sequence is contractible.

V. If R is p-local, and $\alpha \in \pi_*(R)$ is BP-Hurewicz nilpotent, then α is nilpotent.

VI. If R is p-local, and $\alpha \in \pi_*(R)$ is $K(n)$-Hurewicz nilpotent for all $0 \leq n \leq \infty$, then α is nilpotent.

Note that Nishida's theorem follows as an immediate corollary – since we know that all elements of $\pi_n S$ for $n > 0$ are torsion and that $\pi_* MU$ is torsion-free, the image of any $\alpha \in \pi_n S$ in $\pi_n MU$ must be zero, so that α is nilpotent by statement III.

Proof of I ⇒ III. Suppose that $\alpha : F \to X$ is MU-$\wedge$ nilpotent; replacing F and X with sufficiently high smash powers, we see that $1 \wedge \alpha : MU \wedge F \to MU \wedge X$ is zero. SInce F is finite, it has a Spanier-Whitehead dual DF, and $\overline{\alpha} : S \to X \wedge DF$ is zero after smashing with MU. Since S is a small object, the image of this map is some finite subspectrum of $X \wedge DF$, and any given nullhomotopy must likewise factor through some finite subspectrum. Let X' be such a subspectrum; after sufficient suspension, we can take X' to be connective. The tensor algebra $T(X') = \bigvee_{n \geq 0} (X')^{\wedge n}$ is then a connective associative ring spectrum of finite type, and if $i : X' \to T(X')$ is the inclusion, then $i_* \overline{\alpha}$ is MU-Hurewicz nilpotent in $T(X')$, so by I, $\overline{\alpha}$ is nilpotent, and thus α is nilpotent. □

III ⇒ II. Suppose $\alpha \in \pi_n(R)$ is MU-Hurewicz nilpotent, so that $MU \wedge S^n \to MU \wedge MU \wedge R \to MU \wedge R$ is nilpotent. It immediately follows that α is nilpotent. □

III ⇒ IV. See the DHS paper. □

II ⇒ V. This follows from the splitting of $MU_{(p)}$. □

V ⇒ VI. We have $\langle BP \rangle = \langle K(0) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n+1) \rangle$, where angle brackets denote the Bousfield class and $P(n+1)$ is the cohomology theory with $P(n+1)_* = \mathbb{Z}_{(p)}[v_{n+1}, \ldots]$. Note that $\varinjlim P(n) = H\mathbb{F}_p$. Thus if $f : F \to X$ is $K(n)$-Hurewicz nilpotent for all n, then $F \to \varprojlim P(n) \wedge X$ factors through some $P(n) \wedge X$, and it is therefore BP-Hurewicz nilpotent, thus nilpotent by V. □

The nilpotence theorem resolves the following conjecture:

Conjecture 5 (Ravenel's nilpotence conjecture)**.** *If we have $f : \Sigma^d X \to X$ with $MU_*(f) = 0$, and X is finite, then f is composition nilpotent.*

Proof. The mapping telescope of f is contractible, so some composite of f is zero. □

Outline of proof of the nilpotence theorem. We've reduced the theorem to proving I. We have a sequence of maps

$$* \simeq \Omega SU(1) \to \Omega SU(2) \to \cdots \to \Omega SU \to BU.$$

If $X(n)$ is the Thom spectrum of $\Omega SU(n) \to BU$, then we can write MU as a colimit

$$S^0 = X(0) \to X(1) \to \cdots \to X(\infty) \to MU,$$

where $X(n) \to MU$ is $2n-1$-connected. Thus if $\alpha \in \pi_d(R)$ has zero MU-Hurewicz image, it also has zero $X(n)$-Hurewicz image for sufficiently large n.

We thus reduce to showing that if α is $X(n+1)$-Hurewicz nilpotent, then it is $X(n)$-Hurewicz nilpotent. As usual, it suffices to look locally at each p; this is sketched out below. □

3 Homology of various spaces

Fix $L_0 \in \mathbb{C}P^{n-1}$. We have a map $\Sigma \mathbb{C}P^{n-1} \to SU(n)$ given by $(z, L) \mapsto (z^{-1}\pi_{L_0} + \pi_{L_0^\perp})(z\pi_L + \pi_{L^\perp})$. The adjoint maps $\mathbb{C}P^{n-1} \to \Omega SU(n)$ are compatible with the inclusions $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$, $\Omega SU(n) \hookrightarrow \Omega SU(n+1)$, and thus we get a map $\mathbb{C}P^\infty \to \Omega SU \to BU$.

Let V be the virtual vector bundle $\mathcal{O}(-1) - 1$ over $\mathbb{C}P^{n-1}$. Then the Thom spectrum $(\mathbb{C}P^{n-1})^V$ is $\Sigma^{-2}\mathbb{C}P^n$, and we get complex n-orientations $\Sigma^{-2}\mathbb{C}P^n \to X(n)$ for each n with colimit $\Sigma^{-2}\mathbb{C}P^\infty \to MU$. We have $H_*(\mathbb{C}P^{n-1}) = \mathbb{Z}\{\beta_0, \ldots, \beta_n\}$, and such an identification induces $H_*(\Omega SU(n)) = \mathbb{Z}[\beta_0, \ldots, \beta_n]/(\beta_0 - 1)$. The Thom isomorphism gives $H_*(X(n)) = \mathbb{Z}[b_0, \ldots, b_n]/(b_0 - 1)$, where b_i comes from $\Sigma^{-2}\mathbb{C}P^{n+1}$.

Proposition 6. *Suppose $k \leq n$. Then $X(n)_*\mathbb{C}P^k = X(n)_*\{\beta_1, \ldots, \beta_k\}$, and $X(n)_*X(k) = X(n)_*[b_0, \ldots, b_k]/(b_0 - 1)$.*

Theorem 7. *$X(n)_*X(n)$ is flat over $X(n)$.*

4 Further reductions

There's a pullback square

$$\begin{array}{ccc} F'_{k,n} & \longrightarrow & \Omega SU(n+1) \\ \downarrow & & \downarrow \\ J_k S^{2n} & \longrightarrow & \Omega S^{2n+1}. \end{array}$$

Write $F_{k,n}$ for the Thom spectrum of $F'_{k,n} \to \Omega SU(n+1) \to BU$. We have maps

$$F_{0,n} = X(n) \to F_{1,n} \to \cdots \to X(n+1),$$

and we define $G_{k,n} = (F_{p^{k-1},n})_{(p)}$.

Theorem 8. *if $X(n+1) \wedge \alpha^{-1}R \simeq *$ then $G_{k,n} \wedge \alpha^{-1}R \simeq *$ for large enough k.*

Theorem 9. *$\langle G_{k,n} \rangle = \langle G_{k+1,n} \rangle$ as Bousfield classes.*

Thus, $G_{0,n} \wedge \alpha^{-1}R \simeq *$, and it follows that $X(n)_{(p)} \wedge \alpha^{-1}R \simeq *$, as desired.