The chromatic spectral sequence

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1 The ANSS and the first Greek letter elements

Primary source: Miller-Ravenel-Wilson 1977. This constructed the CSS as an algebraic tool which converges to the E_2 page of the ANSS. Throughout, let p be any prime; when we start doing computations, we'll take $p > 2$. Recall that

$$BP_* = \mathbb{Z}_{(p)}[v_1, \dots], \quad |v_i| = 2(p^i - 1),$$

$$BP_*BP = BP_*[t_1, \dots], \quad |t_i| = 2(p^i - 1).$$

For every X, BP_*X is a BP_*BP-comodule, though we will take $X = S$ throughout. The ideals $I_n = (p, v_1, \dots, v_{n-1}) \subseteq BP_*$ are the only invariant prime ideals (Landweber).

The E_2 page of the ANSS is then $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) = H^*(\Omega^* BP_*)$, where $\Omega^* BP_*$ is the cobar complex, given by

$$\Omega^t BP_* = BP_* \otimes_{BP_*} \underbrace{BP_*BP \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP}_{t},$$

with differential induced by $d^0(m) = \psi(m) - m \otimes 1 \in BP_* \otimes_{BP_*} BP_*BP$, where ψ is the coaction. We will abbreviate this page as $E_2^* = H^*(BP_*)$.

We now start computing this cohomology in successive degrees (take $p \neq 2$), following the methods of Novikov. For $H^1(BP_*)$, we have short exact sequences

$$0 \to BP_* \stackrel{p^{n+1}}{\to} BP_* \to BP_*/p^{n+1} \to 0$$

which induce maps $\delta : H^0(BP_*/p^{n+1}) \to H^1(BP_*)$. Novikov shows that $v_1^{sp^n} \in H^0(BP_*/p^{n+1})$, for s prime to p, is nontrivial and has nontrivial image in $H^1(BP_*)$. We write $\alpha_{sp^n/n+1} := \delta(v_1^{sp^n})$, with order p^{n+1}. This is the first family of **Greek letter elements**. 'Obviously it's α, which is a Greek letter, and it's an element, so yeah.'

2 The chromatic spectral sequence

For $n \in \mathbb{N}$, let $N_n^0 = BP_*/I_n$ and $M_n^0 = v_n^{-1} BP_*/I_n$. We then inductively construct $N_n^{s+1} = M_n^s / N_n^s$ and $M_n^{s+1} = v_{n+s+1}^{-1} N_n^{s+1}$. Thus, for example,

$$N_n^1 = BP_*/(p, \dots, v_{n-1}, v_n^\infty) = \varinjlim_k BP_*/(p, \dots, v_{n-1}, v_n^k),$$

and more generally,

$$N_n^s = BP_*/(p, \dots, v_{n-1}, v_n^\infty, \dots, v_{n+s-1}^\infty)$$

and

$$M_n^s = v_{n+s}^{-1} BP_*(p, \dots, v_{n-1}, v_n^\infty, \dots, v_{n+s-1}^\infty).$$

The **chromatic filtration** is

$$0 \to BP_*/I_n \to M_n^0 \to M_n^1 \to \cdots .$$

This gives us a spectral sequence as usual, converging to $H^*(BP_*/I_n)$. For instance, we could define the **chromatic cobar complex** as $C_n^u = \bigoplus_{s+t=u} \Omega^s M_n^t$. This is a bigraded complex, where each column comes from the chromatic filtration, and each row is the cobar complex for some M_n^s. Taking the cohomology in the column direction gives us $H^*(BP_*/I_n)$ on $E_1 = E_\infty$, which is what we want. Taking the cohomology in the row direction gives us $E_1^{s,t} = H^s M_n^t$.

To continue, we first generalize the Greek letter construction, which was a map $H^0 N_0^1 \to H^1 BP_*$. Let $v_1^{sp^n}/p^{n+1} \in BP_*/(p^\infty)$ be the image of $v_1^{sp^n} \in BP_*/(p^{n+1})$ (this explains the notation), and generalize this to allow denominators with powers of v_i. The boundary maps on cohomology induced by the short exact sequences $0 \to N_0^{s-1} \to M_0^{s-1} \to N_0^s \to 0$ induce maps

$$H^t N_0^s \to H^{t+1} N_0^{s-1} \to H^{t+2} N_0^{s-2} \to \cdots .$$

The composition of s of these maps is a map $\eta : H^t N_0^s \to H^{t+s} BP_*$. We define

$$\eta\left(\frac{v_s^{up}}{p^{i_0} v_1^{i_1} \cdots v_{s-1}^{i_{s-1}}}\right) = \alpha^{(s)}_{u/i_0,\ldots,i_{s-1}},$$

where by $\alpha^{(s)}$ we mean the sth Greek letter (so $\alpha^{(2)}$ is also written β, and so on). The sth Greek letter elements live in $H^s(BP_*)$. Note that we need to show, in general, that these elements are nonzero – not all of them necessarily exist.

Work of Landweber generalized this to other regular sequences $A = (a_0, a_1, \ldots)$ in BP_*, satisfying certain conditions that give the M_n^s comodule structures and so on. Under these conditions, we can likewise do the work of the above paragraph, and get a map η_A and 'A-Greek letter elements' in $H^*(BP_*)$. Landweber showed that these are always the usual Greek letter elements: the map η_A factors through η.

The last step is to understand $H^0 N_0^s$. Note that there are exact sequences

$$0 \to M_{n+1}^{s-1} \to M_n^s \xrightarrow{v_n} M_n^s \to 0,$$

where the first map just formally divides everything by v_n. Adding these together gives an exact couple and thus a spectral sequence. It's maybe simpler, however, to just say that we're using the long exact sequence on cohomology

$$0 \to H^0 M_{n+1}^{s-1} \to H^0 M_n^s \to \cdots \to H^t M_n^s$$

and inducting on s. It remains to understand

$$M_{n+s}^0 = v_{n+s}^{-1} \frac{BP_*}{(p, \ldots, v_{n+s-1})}.$$

We now use the Morava change of rings theorem, which gives an isomorphism

$$H^* M_n^0 = \mathrm{Ext}_{K(n)_* K(n)}(K(n)_*, K(n)_*).$$

This last group is, in principle, computable. Armed with this information, and after doing substantial algebra, we can find the Greek letter elements in $H^*(BP_*)$. This has been done up to H^2, i.e. for α and β, and this has been the state of knowledge on this problem since 1977. Good luck, young mathematician.