The Adams-Novikov spectral sequence and the Landweber exact functor theorem

Gabriel Valenzuela

April 22, 2013

1 The Adams-Novikov spectral sequence

If E is a sufficiently nice ring spectrum, there's a spectral sequence

$$E_2 = \mathrm{Ext}^{s,t}_{E_*E}(\pi_*E, E_*X) \to \pi_{t-s}(\widehat{X}),$$

where $\widehat{X}$ is the E-nilpotent completion of X, and Ext^s is the sth right derived functor of Hom_{E_*E} in the category of comodules over the Hopf algebroid (π_*E, E_*E). $\widehat{X}$ is defined in Bousfield's 1979 paper on localization of spectra: it's an E_*-local object receiving a map from X, which in nice cases is an E_*-isomorphism, forcing $L_EX \simeq \widehat{X}$.

We'll be most interested in the cases $E = H\mathbb{Z}/p$, MU, and BP, the third being the ANSS. All of these have nice properties, such as flatness of the associated Hopf algebroid, connectivity, ...

Theorem 1 (Novikov, 1967)**.** *For any spectrum X, there's a natural spectral sequence $E_r^{s,t}$, with $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ with*

(a) $E_2^{s,t} \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*X)$;

(b) If X is connective, then $E_2^{s,t} \Rightarrow \pi_{t-s}(X_{(p)})$.

The advantage of the ANSS over the classical ASS is that BP_*BP is concentrated in degrees divisible by $2(p-1)$. Thus, for example, if BP_*X is also concentrated in such degrees, then $E_r^{s,t} = 0$ unless $2(p-1)|t$, and so there are fewer differentials. This phenomenon is called **sparseness**.

In order to compute the ANSS E_2 page, we need to understand the Hopf algebroid (BP_*, BP_*BP). Recall that a **Hopf algebroid** (A, Γ) over a commutative ring K is a cogroupoid object in the category of (commutative) K-algebras. Thus, for any ring B, $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are naturally respectively the objects and morphisms of a groupoid.

If E_*E is flat over E_* (we say that 'E is flat' for short), then (E_*, E_*E) is a Hopf algebroid with structure maps

$$\begin{aligned}
\eta_L &= \pi_*(E \wedge S \to E \wedge E) && \text{(source)}\\
\eta_R &= \pi_*(S \wedge E \to E \wedge E) && \text{(target)}\\
\epsilon &= \pi_*(E \wedge E \to E) && \text{(identity)}\\
c &= \pi_*(E \wedge E \xrightarrow{\text{flip}} E \wedge E) && \text{(inverse)}\\
\Delta &= \pi_*(E \wedge S \wedge E \to E \wedge E \wedge E) && \text{(composition).}
\end{aligned}$$

Flatness is used to define the last map, which we should be able to identify as a map $E_*E \to E_*E \otimes_{E_*E} E_*E$.

Definition 2. A (left) **Γ-comodule** M is a left A-module with a coaction map

$$\psi : M \to \Gamma \otimes_A M$$

which is counital and coassociative.

For example, if X is a spectrum and E is as above, then E_*X is an E_*E-comodule with ψ induced by $E \wedge S \wedge X \to E \wedge E \wedge X$.

The structure of (BP_*, BP_*BP) (and likewise (MU_*, MU_*MU)) comes naturally from formal groups. Specifically, let $FGL : \mathsf{CRing} \to \mathsf{Set}$ and $SI : \mathsf{CRing} \to \mathsf{Set}$ send a ring R respectively to its set of formal group laws and its set of strict isomorphisms of formal group laws. These are the objects and morphisms of a groupoid in the obvious way. Now, $FGL = \mathrm{Hom}(L, \cdot)$ where $L = \mathbb{Z}[x_1, x_2, \cdots]$ with $|x_i| = 2i$ (the **Lazard ring**), and $SI = \mathrm{Hom}(LB, \cdot)$ where $LB = L \otimes \mathbb{Z}[b_1, b_2, \cdot]$ with $|b_i| = 2i$. By Quillen's theorem, these are isomorphic to MU_* and MU_*MU respectively.

Theorem 3 (Landweber 1967, Novikov 1967, independently!)**.** *These maps give an isomorphism of Hopf algebroids $(L, LB) \to (MU_*, MU_*MU)$.*

Likewise, we have $(V, VT) \overset{\cong}{\to} (BP_*, BP_*BP)$, where $V = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ with $|v_i| = 2(p^i - 1)$, $VT = V \otimes \mathbb{Z}_{(p)}[t_1, t_2, \dots]$ with $|t_i| = 2(p^i - 1)$, and V and VT respectively corepresent the p-typical formal group laws functor and the strict isomorphisms of p-typical formal group laws functor.

2 The Landweber exact functor theorem

If E is a complex-oriented cohomology theory with a fixed orientation class, we can construct a formal group law over E_* in a natural way. We now ask: when can we go back?

Let $F(x, y) \in FGL(R_*)$ for R_* a graded ring; this gives a map $MU_* \to R_*$. Thus there's a functor

$$X \mapsto MU_*X \otimes_{MU_*} R_*.$$

This satisfies all the axioms of a homology theory with the exception of turning cofiber sequences into long exact sequences: tensoring with R_* has destroyed exactness. If R_* were flat over MU_*, we'd be fine, but this is too strong.

The same question can be asked p-typically: if R_* has a p-typical formal group law represented by $BP_* \to R_*$, then when is

$$X \mapsto BP_*X \otimes_{BP_*} R_*$$

a homology theory?

Theorem 4 (Landweber exact functor theorem, Landweber 1976)**.** *For a fixed BP_*-module R_*, the above functor is a homology theory iff for all n, the sequence $(p, v_1, v_2, \dots, v_n)$ is a regular sequence in R_* (that is, each v_n is a non-zero-divisor in $R_*/(p, v_1, \dots, v_{n-1})$).*

The proof is via the study of the category of (finitely presented) BP_*BP-comodules. One arrives at the following theorems:

Theorem 5 (Landweber 1973)**.** *The only prime ideals of BP_* which are also BP_*BP-comodules are the ideals $I_n = (p, v_1, \dots, v_{n-1})$ for $0 \leq n \leq \infty$.*

Theorem 6 (Landweber filtration theorem, Landweber 1973)**.** *Any BP_*BP-comodule M which is finitely presented as a BP_*-module has a filtration by comodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

with $M_{i+1}/M_i \cong BP_*/I_{n_i}$.

(We need to be careful in general about our finiteness conditions when dealing with non-noetherian rings. However, since BP_* is coherent as a ring, a module is finitely presented iff it is coherent.)

Proof of LEFT. Consider the exact sequences

$$0 \to BP_*/I_n \overset{v_n}{\to} BP_*/I_n \to BP_*/I_{n+1} \to 0.$$

The sequences given in the theorem statement are regular in R_* iff tensoring with R_* preserves exactness of these sequences, which is true iff $\mathrm{Tor}^{BP_*}(BP_*/I_n, R_*) = 0$. But by the filtration theorem, this is true iff

$\mathrm{Tor}^{BP_*}(M, R_*) = 0$ for any finitely presented BP_*BP-comodule M, which is true in turn iff tensoring with R_* is exact on the category of finitely presented BP_*BP-comodules. Finally, if X is a finite CW-complex, then BP_*X is finitely presented, so we establish the excision axiom on the finite CW-complexes, which implies it for all spectra.

The only nontrivial step in the converse is showing that enough finitely presented BP_*BP-comodules show up as BP_*X for finite CW-complexes X – we need some analogue of the periodicity theorem. $\square$

Example 7.

- Let $F_a(x, y) = x + y$ over $\mathbb{Q}$. Then $MU_*(X) \otimes \mathbb{Q} \cong H_*(X; \mathbb{Q})$.
- Let $F_m(x, y) = x + y + \beta xy$ over $\mathbb{Z}[\beta, \beta^{-1}]$ with $|\beta| = 2$. Then $MU_*(X) \otimes \mathbb{Z}[\beta, \beta^{-1}] \cong K_*X$.
- For every elliptic curve, there's a natural FGL over a ring representing modular forms with certain Fourier coefficients, and this gives a homology theory $\mathrm{Ell}_*(X)$, called **elliptic homology**.
- The Johnson-Wilson theories $E(n)$, with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, are constructed via the LEFT, and we have to invert v_n in order to achieve Landweber exactness.
- Importantly, Morava K-theory $K(n)$ cannot be built in this way. However, there's a spectral sequence $E_2 = \mathrm{Tor}^{BP_*}(BP_*X, K(n)_*) \Rightarrow K(n)_*(X)$ which arises directly from the failure of Landweber exactness. (It's just a Künneth spectral sequence.)