Resolutions of the $K(2)$-local sphere

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We'll discuss the following theorem:

Theorem 1 (Goerss-Henn-Mahowald-Rezk)**.** *Localize at $p = 3$. There is a sequence of maps of spectra*

$$L_{K(2)}S \to E_2^{hG_{24}} \to \Sigma^8 E_2 hSD_{16} \vee E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{40} E_2^{hSD_{16}} \to \Sigma^{40} E_2^{hSD_{16}} \vee \Sigma^{48} E_2^{hG_{24}} \to \Sigma^{48} E_2^{hG_{24}}$$

such that the composition of any two maps is null and all possible Toda brackets are zero mod indeterminacy.

Since all Toda brackets are zero, this refines to a finite tower of fibrations with $L_{K(2)}S$ at the top. Note the periodicity; additionally, $E_2^{hSD_{16}}$ is 16-periodic, so $\Sigma^{40} E_2^{hSD_{16}} \simeq \Sigma_8 E_2^{hSD_{16}}$; we've written it this way to emphasize the duality in the sequence of maps. Finally, this is not an E_2-Adams resolution. In particular, $E_2^{hG_{24}}$ is not E_2 injective.

1 Preliminaries

At height $n = p-1$, $\mathbb{G}_n$ has infinite cohomological dimension: p is a bad prime. However, it's not a really bad prime, and in particular, $\mathbb{G}_n$ has finite virtual cohomological dimension, which means that it has finite-index subgroups that have finite cohomological dimension. We can use these subgroups to build resolutions.

Definition 2. A **Morava module** is an $(E_n)_*$-module with an action of $\mathbb{G}_n$ such that these actions commute.

Suppressing the ns, the prototypical example is $E_*^{\vee}X = \pi_* L_{K(n)}(E \wedge X)$. In fact, we'll generally just write this as E_*X. We'll also be assuming that $K(n)_*X$ is concentrated in even degrees.

Let M be a Morava module and consider $\mathrm{Hom}^c(\mathbb{G}_n, M)$. This is a Morava module with E_* acting by its action on M and with $\mathbb{G}_n$ acting by conjugation.

Theorem 3 (Morava)**.** $E_*E \cong \mathrm{Hom}^c(\mathbb{G}_n, E_*)$ *as Morava modules.*

Theorem 4 (Devinatz-Hopkins)**.** $E_*E_n^{hK} \cong \mathrm{Hom}^c(\mathbb{G}_n/K, E_*)$ *as Morava modules, for K a finite subgroup of $\mathbb{G}_n$.*

Let $M \uparrow_K^{\mathbb{G}_n}$ be the induced module of a $\mathbb{Z}_p[K]$-module M, given by $M \uparrow_K^{\mathbb{G}_n} = \mathbb{Z}_p[[\mathbb{G}_n]] \widehat{\otimes}_{\mathbb{Z}_p[K]} M$. In particular, if $M = \mathbb{Z}_p$ then $\mathbb{Z}_p \uparrow_K^{\mathbb{G}_n} \cong \mathbb{Z}_p[[\mathbb{G}_n/K]]$, and

$$\mathrm{Hom}^c_{\mathbb{Z}_p}(\mathbb{Z}_p \uparrow_K^{\mathbb{G}_n}, E_*) \cong \mathrm{Hom}^c_{\mathbb{Z}_p}(\mathbb{Z}_p[[\mathbb{G}_n/K]], E_*) \cong \mathrm{Hom}^c(\mathbb{G}_n/K, E_*) \cong E_*E^{hK}$$

by the above theorem of Devinatz and Hopkins.

2 The case $n = 1$, $p = 2$

We've talked about height 1 before, but we avoided $p = 2$, which is bad here. We have $\mathbb{G}_1 = \mathbb{S}_1 \cong \mathbb{Z}_2^\times$, E_1 is 2-adic K-theory, and $E_1^{\mathrm{h}C_2} \simeq KO_2^\wedge$. We have a fiber sequence

$$L_{K(1)}S \to KO_2^\wedge \overset{\psi^3-1}{\to} KO_2^\wedge.$$

There's a short exact sequence

$$0 \to \mathbb{Z}_2[[t]] \overset{t}{\to} \mathbb{Z}_2[[t]] \to \mathbb{Z}_2 \to 0,$$

and an isomorphism $\mathbb{Z}_2[[t]] \cong \mathbb{Z}_2[[\mathbb{Z}_2]]$ sending t to $\gamma - e$, where γ is an additive topological generator for the inner $\mathbb{Z}_2$ and e is the zero of the inner $\mathbb{Z}_2$. Thus we can write the above short exact sequence as

$$0 \to \mathbb{Z}_2 \uparrow_{C_2}^{\mathbb{G}_1} \to \mathbb{Z}_2 \uparrow_{C_2}^{\mathbb{G}_1} \to \mathbb{Z}_2 \to 0,$$

and apply $\mathrm{Hom}_{\mathbb{Z}_2}^c(-, E_*)$, giving the sequence

$$0 \to E_* \to E_* E^{\mathrm{h}C_2} \to E_* E^{\mathrm{h}C_2} \to 0.$$

This is exact because the domains are projective as $\mathbb{Z}_2$-modules. This is a sort of algebraic version of the fiber sequence defining the $K(1)$-local sphere.

3 The case $n = 2$, $p = 3$

We have an exact sequence

$$0 \to \mathbb{G}_n^1 \to \mathbb{G}_n \overset{\det}{\to} \mathbb{Z}_p \to 0.$$

For n coprime to p, this splits, and we can write $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$.

We now discuss the finite subgroups of the Morava stabilizer group required for the theorem. G_{24} is a subgroup of order 24, isomorphic to $C_3 \rtimes Q_8$. Let ω be a primitive 8th root of unity, and S the noncommuting variable generating $\mathbb{S}_2$ over $W(\mathbb{F}_9)$. Then if $s = -\frac{1}{2}(1 + \omega S)$ and ϕ is the generator of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$, $G_{24} = \langle S, \omega^2, \omega\phi \rangle$.

SD_{16} is the semidirect product $C_8 \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ generated by ω and ϕ, and Q_8 is an index 2 subgroup. Let λ be the pullback of the sign representation on Q_8 to SD_{16}. One can show algebraically that there's an exact sequence of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules

$$0 \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3(\lambda) \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3(\lambda) \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to 0.$$

$\mathbb{Z}_3(\lambda)$, it should be pointed out, is just $\mathbb{Z}_3$, but with ω and ϕ acting by multiplication by -1.

The starting point of the computation is $H^*(S_2^1; \mathbb{F}_3)$, where S_2^1 is the 3-Sylow subgroup of $\mathbb{G}_2$. This was computed by Henn. This helps for several reasons. First, $\mathrm{Ext}^*_{\mathbb{Z}_3[[S_2^1]]}(M; \mathbb{F}_3)$ is $\mathbb{F}_3$-linearly dual to $\mathrm{Tor}_*^{\mathbb{Z}_3[[S_2^1]]}(M; \mathbb{F}_3)$. Second, if G is a finitely generated profinite group and $f : M \to N$ is a morphism of complete $\mathbb{Z}_3[[G]]$-modules such that $\mathbb{F}_3 \otimes f : \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[G]]} M \to \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[G]]} N$ is surjective, then f is also surjective (which can be thought of as a version of Nakayama's lemma).

We start with the augmentation

$$0 \to N_1 \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \to 0.$$

We can compute $\mathrm{Ext}_{\mathbb{Z}_3[[S_2^1]]}(N_1, \mathbb{F}_3)$ using the long exact sequence in Ext. This allows one to define a map $f : \mathbb{Z}_3(\lambda) \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to N_1$ with $f \otimes \mathbb{F}_3$ surjective. Using the 'version of Nakayama's lemma' from above, this means that f is surjective, whose kernel we'll call N_2. Again using the long exact sequence in Ext, we can calculate $\mathrm{Ext}(N_2, \mathbb{F}_3)$, and construct another surjective map $\mathbb{Z}_3(\lambda) \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to N_2$, with kernel N_3.

We want to show $N_3 \cong \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1}$. It turns out that it suffices to show that there's a map $N_3 \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1}$ which is nonzero on Ext groups. This is difficult but can be done. (Note that all these maps past the first augmentation map aren't explicit; Karamanov has constructed explicit approximations to them.)

We can then tensor this up to $\mathbb{G}_2$ itself. It remains to represent these groups by actual spectra. Some of this can be done via the Devinatz-Hopkins theorem above: for example, $\mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2} \cong E_*E^{\mathrm{h}G_{24}}$. For SD_{16}, let e_λ be the idempotent

$$e_\lambda = \frac{1}{16} \sum_{g \in SD_{16}} \lambda(g) g^{-1}$$

in $\mathbb{Z}_3[SD_{16}]$. We can form the telescope $E^\lambda = \mathrm{colim}(E \xrightarrow{e_\lambda} E \xrightarrow{e_\lambda} \cdots)$, and we have

$$\begin{aligned} E_*E^\lambda &\cong \mathrm{Hom}_{\mathbb{Z}_3[[SD_{16}]]}(\mathbb{Z}_3(\lambda), E_*E) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_3[[SD_{16}]]}(\mathbb{Z}_3(\lambda), \mathrm{Hom}^c(\mathbb{G}_2, E_*)) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2]]}(\mathbb{Z}_3(\lambda) \uparrow_{SD_{16}}^{\mathbb{G}_2}, \mathrm{Hom}^c(\mathbb{G}_2, E_*)) \\ &\cong \mathrm{Hom}^c_{\mathbb{Z}_3}(\mathbb{Z}_3(\lambda) \uparrow_{SD_{16}}^{\mathbb{G}_2}, E_*). \end{aligned}$$

It turns out that $E_*E^\lambda \cong E_*(\Sigma^8 E^{\mathrm{h}SD_{16}})$. We take the algebraic sequence and apply $\mathrm{Hom}^c_{\mathbb{Z}_3}(-, E_*)$, giving us

$$\begin{aligned} 0 \to E_* \to E_*E^{\mathrm{h}G_{24}} \to E_*\Sigma^8 E_{\mathrm{h}SD_{16}} \oplus E_*E^{\mathrm{h}G_{24}} \to E_*\Sigma^8 E^{\mathrm{h}SD_{16}} \oplus E_*\Sigma^{40} E^{\mathrm{h}SD_{16}} \\ \to E_*\Sigma^{40} E^{\mathrm{h}SD_{16}} \oplus E_*E^{\mathrm{h}G_{24}} \to E_*E^{\mathrm{h}G_{24}} \to 0. \end{aligned}$$

Here $E_*E^{\mathrm{h}G_{24}} \cong E_*(\Sigma^{48} E^{\mathrm{h}G_{24}})$, as mentioned earlier. It remains to realize this topologically.

Proposition 5 (Proposition 2.7 of GHMR)**.** *Let H_1, H_2 be closed subgroups of $\mathbb{G}_n$, with H_2 finite. Then there is a commutative diagram*

$$\begin{tikzcd} \pi_* E_n[[\mathbb{G}_n/H_1]]^{\mathrm{h}H_2} \arrow[r] \arrow[d, "\cong"'] & (E_n)_*[[\mathbb{G}_n/H_1]]^{H_2} \arrow[d, "\cong"] \\ \pi_* F(E_2^{\mathrm{h}H_1}, E_2^{\mathrm{h}H_2}) \arrow[r] & \mathrm{Hom}_{\mathrm{Morava}}(E_*E^{\mathrm{h}H_1}, E_*E^{\mathrm{h}H_2}), \end{tikzcd}$$

where the top map is the edge homomorphism in the HFPSS.

(Here if E is a spectrum and $X = \lim X_i$ a profinite group, then $E[[X]] = \mathrm{holim}_i(E \wedge X_i)$.)

Realizing maps topologically means lifting them along the bottom map in the square of the proposition, and so it suffices to show that the top map is surjective. In most cases, the HFPSS collapses, allowing us to do this. The Toda bracket calculations work in a similar way.

4 Applications

Several applications have been done by combinations of Goerss, Henn, Mahowald, Rezk, Shimimura, Karamanov (and Agnès's thesis!). These include calcuation of $\pi_* L_{K(2)} V(0)$, finding exotic elements in the Picard group, a possible disproof of the chromatic splitting conjecture, the Brown-Comenetz dual of $L_{K(2)}S$, and the rational homotopy of $L_{K(2)}S$. We also have $L_{K(2)} tmf = E_2^{\mathrm{h}G_{24}}$, and we can hopefully use this to compute $L_{K(2)}S$ entirely.