Hopkins-Kuhn-Ravenel character theory

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1 Premotivation

Given a complex representation of a finite group, we can associate to it a class function called its **character** by sending each element of the group to the trace of the automorphism associated to it. As it turns out, the character of a complex representation has image in the subring $L = \mathbb{Q}[\mu^\infty]$ of $\mathbb{C}$. The ring generated by the characters is the **representation ring** $R(G) \cong K_G(*)$. This has a map to $K(BG)$ called the **Atiyah-Segal completion map**; Atiyah-Segal showed that $K(BG)$ is the completion of $R(G)$ at its augmentation ideal.

We'd like to repeat this at heights greater than 1. The analogue of $K(BG)$ is $E_n^0(BG)$, but it's less clear what the analogue of $R(G)$ is.

2 Preliminaries

In order to construct this, we need to do some algebraic geometry. Fix a prime p. E_n has a formal group $\mathbb{G}_{E_n}$, which can be viewed as a p-divisible group $\{\mathbb{G}_{E_n}[p^k]\}$. The global sections of the p^k-torsion are

$$\Gamma \mathbb{G}_{E_n}[p^k] = E_n^0(B\mathbb{Z}/p^k) \cong E_n^0[[x]]/[p^k](x) \cong E_n^0[x]/f(x),$$

where $f(x)$ is monic of degree p^{kn} and the second isomorphism is given by a choice of coordinate. In particular, these global sections are a free E_n^0-module of rank p^{kn}.

Given an E_n^0-algebra R, let $R \otimes \mathbb{G}_{E_n}$ be the p-divisible group $\{R \otimes \mathbb{G}_{E_n}[p^k]\} = \{\operatorname{Spec} R \times_{\operatorname{Spec} E_n^0} \mathbb{G}_{E_n}[p^k]\}$.

There's a fully faithful functor from the category of finite groups to the category of group schemes given by $G \mapsto \operatorname{Spec} \prod_G R$, whose image is the **constant** group schemes. There's only one height n constant p-divisible group, given by $(\mathbb{Q}_p/\mathbb{Z}_p)^n = \{(\mathbb{Z}/p)^n\}$.

3 Motivation

Can we approximate height n phenomena in stable homotopy theory by height 0 phenomena? More specifically, can we approximate $E_n^0(X)$ by rational cohomology? This question still isn't specific enough, since we haven't said what we mean by 'approximate' or 'X'.

Here's an easy case. Let X be a finite CW-complex. We have a canonical rationalization map of spectra $E_n \to p^{-1}E_n$, which induces $E_n^0(X) \to (p^{-1}E_n)^0(X)$. Since the coefficients of the codomain are $p^{-1}E_n^0$, we in fact have a map

$$p^{-1}E_n^0 \otimes_{E_n^0} E_n^0(X) \to (p^{-1}E_n)^0(X).$$

But $p^{-1}E_n^0$ is a flat E_n^0-algebra, so the left-hand side is a cohomology theory, and this map is an isomorphism when X is a point, so it's an isomorphism in general.

Here's a harder case. Let $X = B\mathbb{Z}/p^k$. We again have a map

$$p^{-1}E_n^0 \otimes_{E_n^0} E_n^0(B\mathbb{Z}/p^k) \to (p^{-1}E_n)^0(B\mathbb{Z}/p^k).$$

The right-hand side is rational cohomology of a finite group so it's trivial; on the other hand, the left-hand side is a rank 1 $p^{-1}E_n^0$-algebra. Thus, this map is clearly not an isomorphism. What's going on?

We need to understand $p^{-1}E_n^0 \otimes_{E_n^0} E_n^0(X) = \Gamma(p^{-1}E_n^0 \otimes \mathbb{G}_{E_n}[p^k])$. By a theorem of Cartier, the right-hand side is an étale p-divisible group $\mathbb{G}_{et}[p^k]$.

Assume that there exists a $p^{-1}E_n^0$-algebra C_0 over which $C_0 \otimes \mathbb{G}_{et}[p^k]$ splits as $(\mathbb{Z}/p^k)^n = \Lambda_k$. Then

$$C_0 \otimes E_n^0(B\mathbb{Z}/p^k) \cong \prod_{\Lambda_k} C_0 \cong C_0^0\left(\coprod_{\Lambda_k} *\right).$$

As it turns out, this will allow us to solve the problem for all spaces of the form $EG \times_G X$, and develop an analogue of the Atiyah-Segal completion theorem.

It's not that surprising that such a thing exists – the desired splitting can be accomplished via the right localizations and algebraic closures – but it's worth trying to find a small one, which we do now.

4 Construction of C_0

The idea is to construct the universal ring over which there is a map $(\mathbb{Q}_p/\mathbb{Z}_p)^n \to \mathbb{G}_{et}$ and then invert so that it's an isomorphism. Let A be a finite abelian group such that $A = A[p^k]$, and let $A^\vee$ be its Pontryagin dual, which is isomorphic to $\mathrm{Hom}(A, \mathbb{Z}/p^k)$.

Lemma 1. *There is a natural isomorphism*

$$E_n^0\text{–}\mathsf{Alg}(E_n^0(BA), R) \cong \mathsf{GpSch}(A^\vee, Rotimes\mathbb{G}_{E_n}[p^k]).$$

Sketch of proof. A map $E_n^0(BA) \to R$ gives $f : R \otimes E_n^0(BA) \to R$, and for $a \in A^\vee$, there's a composition

$$R \otimes E_n^0(B\mathbb{Z}/p^k) \xrightarrow{a} R \otimes E_n^0(BA) \xrightarrow{f} R.$$

These combine to give $R \otimes E_n^0(B\mathbb{Z}/p^k) \to \prod_{A^\vee} R$. □

Corollary 2. *There's a natural isomorphism*

$$p^{-1}E_n^0\text{–}\mathsf{Alg}(p^{-1}E_n^0 \otimes E_n^0(BA), R) \cong \mathsf{GpSch}(A^\vee, R \otimes \mathbb{G}_{et}[p^k]).$$

Let $A = \Lambda_k$ and $R = C'_{0,k} = p^{-1}E_n^0 \otimes E_n^0(B\Lambda_K)$. The lemma gives a canonical map $\Lambda_k^\vee = (\mathbb{Q}_p/\mathbb{Z}_p)^n[p^k] \to R \otimes \mathbb{G}_{et}[p^k]$. Taking global sections gives a map of free modules of the same rank, $\Lambda_k \to \mathbb{G}_{et}[p^k](C'_{0,k})$.

Lemma 3. *This becomes an isomorphism after inverting the determinant.*

We define $C_{0,k} = \det^{-1} C'_{0,k}$.

Lemma 4. *$C_{0,k}$ is a faithfully flat $p^{-1}E_n^0$-algebra.*

We let C_0 be the colimit of the $C_{0,k}$. This is equipped with an isomorphism $(\mathbb{Q}_p/\mathbb{Z}_p)^n \cong C_0 \otimes \mathbb{G}_{et}$.

Exercise 5. It's worth trying this for $n = 1$, where $E_1 = K_p$ – you get $C_0 = L_p$.

5 Character maps

Let X be a finite G-CW-complex. Let

$$G_p^n = \mathrm{Hom}(\mathbb{Z}_p^n, G) = \{(g_1, \ldots, g_n) : [g_i, g_j] = e, g_i^{p^k} = e \text{ for } k \gg 0\}.$$

We define $\mathrm{Fix}_n(X) = \coprod_{\alpha \in G_p^n} X^{\mathrm{im}\,\alpha}$, which is still a G-space in such a way that for $x \in X^{\mathrm{im}\,\alpha}$, $gx \in X^{\mathrm{im}\,g\alpha g^{-1}}$. (Nat thinks we should call this the 'stacky constant loops.')

We can think of the homotopy fixed points $EG \times_G \mathrm{Fix}_n(X)$ as the space of maps of topological groupoids from $*//\mathbb{Z}_p^n$ to $X//G$; if we restrict from $\mathrm{Fix}_n(X)$ to the subspace indexed by those maps from $\mathbb{Z}_p^n$ to G that factor through $(\mathbb{Z}/p^k)^n$, we get the space of maps of topological groupoids $*//\Lambda_k \to X//G$.

Example 6. Let X be a point. Then $EG \times_G \mathrm{Fix}_n(*) = EG \times_G G_p^n$, where G acts by conjugation on G_p^n; this is just $\coprod_{\alpha \in G_p^n/\sim} BC(\mathrm{im}\,\alpha)$, where C is the centralizer.

When G is a p-group, this is $\mathcal{L}^n BG$.

Example 7. Let $G = \mathbb{Z}/p^k$. Then $E\mathbb{Z}/p^k \times_{\mathbb{Z}/p^k} \Lambda_k$, where Λ_K is given the trivial action, is $\coprod_{\Lambda_k} B\mathbb{Z}/p^k$.

Now, we have $C_0^0(X) = C_0 \otimes_{p^{-1}E_n^0} (p^{-1}E_n)^0(X)$.

Theorem 8 (Hopkins-Kuhn-Ravenel)**.** *The map*

$$\Phi_G : C_0 \otimes_{E_n^0} E_n^0(EG \times_G X) \to C_0^0(EG \times_G \mathrm{Fix}_n(X))$$

is an isomorphism.

Example 9. Let X be a point. Then this map is

$$E_n^0(BG) \to C_0^0\left(\coprod_\alpha BC(\mathrm{im}\,\alpha)\right) \cong \prod_\alpha C_0,$$

which is just the space of class functions from G_p^n to C_0. Thus we recover standard character theory.

When $n = 1$, this is a map $K_p(BG) \to \mathrm{Cl}(G_p^1, L_p)$, which Adams proved was an isomorphism.

Let $G = \mathbb{Z}/p^k$, where X is still a point. Then this is an isomorphism $E_n^0(B\mathbb{Z}/p^k) \to \prod_{\Lambda_k^\vee} C_0$, which is just the global sections of $(\mathbb{Q}_p/\mathbb{Z}_p^n[p] \to \mathbb{G}_{E_n}[p^k])$.

The idea of the proof is as follows. Combining the topological and algebro-geometric stories, we get a square

$$\begin{array}{ccc}
E_n(EG \times_G X) & \longrightarrow & C_0^0(EG \times_G \mathrm{Fix}_n(X)) \\
{\scriptstyle \text{topological}}\downarrow & & \uparrow{\scriptstyle \text{algebraic}} \\
E_n(B\Lambda_k \times EG \times_G \mathrm{Fix}_n(X)) & \xrightarrow{\cong} & E_n^0(B\Lambda_k) \otimes_{E_n^0} E_n^0(EG \times_G \mathrm{Fix}_n X).
\end{array}$$

The topological map is the evaluation map

$$*//\Lambda_k \times \mathrm{Hom}(*//\Lambda_K, X//G) \to X//G.$$

The algebraic map is induced by the canonical map $E_n^0(B\Lambda_k) \to C_0$ given by inverting p. To prove that it's an isomorphism, we reduce to the case where X is a point and $G = \mathbb{Z}/p^k$, and resolve X using spaces with abelian stabilizers. For each of these, we have $EG \times_G G/A \simeq BA$.

6 Applications

Ando's thesis (in *Nature*!), computing the Rezk logarithm, Strickland subgroups. Another is to get algebro-geometric interpretations for $E_n^0(BG)$ by looking at the explicit numbers that pop out when doing this computation. Nat's own work constructs other character theories C_t for $0 \le t < n$.