Gross-Hopkins duality and Pic

Saul Glasman feat. Aaron Mazel-Gee

May 15, 2013

1 Introduction

In stable homotopy theory, we generally index π_* on $\mathbb{Z}$. The reason we do this is that the Picard group of the stable homotopy category $\mathcal{S}$ is $\mathbb{Z}$, generated by S^1. For if X is an invertible object, the Künneth formula shows that the homology of X with coefficients in any field, and thus integrally, is the same as that of a sphere, and the Hurewicz theorem shows that X is a sphere.

However, this is no longer true in the category $\mathcal{K}_n$ of $K(n)$-local spectra. (The smash product is now $-\widehat{\wedge}- = L_{K(n)}(-\wedge-)$, and the unit is $L_{K(n)}S$.) $\mathrm{Pic}_n = \mathrm{Pic}\,\mathcal{K}_n$ still has a subgroup $\mathbb{Z}$, generated by $L_{K(n)}S^1$, but there may now be more invertible objects, and for $S^\lambda \in \mathrm{Pic}_n$, we can define $\pi_\lambda X = [S^\lambda, X]$, thus indexing homotopy groups on the Picard group.

2 The Mazel-Gee interruption

In fact, this group is a p-adic Lie group, and the homotopy groups indexed on it exhibit continuity properties. Strickland did this at height 1 for the $K(1)$-local sphere. He defines $\Lambda = \mathbb{Z}_p[[\mathrm{Pic}_1^*]]$, the "holomorphic $\mathbb{Z}_p$-valued functions" on Pic_1, and M_λ the "module of Dirac delta functions," given by $\mathbb{Z}_p$ with action $\beta\cdot x = \lambda^\beta x$.

Recall that $\pi_n L_{K(1)}S = \mathbb{Z}_p/ps$ in degree $n = qs-1$ (s prime to p), $\mathbb{Z}_p$ in degree zero, and 0 otherwise. In fact, $\mathbb{Z}_p/ps$ only depends on the p-adic valuation of s, and we have $\mathbb{Z}_p/ps = \mathbb{Z}_p/p^{1-|s|_p}$. Thus the homotopy groups of $L_{K(1)}S$ are 'continuous in s', and we can define $\pi_n L_{K(1)}S$ for all $n \in \mathbb{Z}_p$ by continuity.

Another way to talk about these groups is in terms of the denominator of the Bernoulli number $B_{2s}/4s$. This is a map $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, and we'd like to extend it to $\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p$. Number theorists have done this: the Bernoulli numbers appear as special values of the Riemann zeta function, and there's a p-adic zeta function that interpolates between them as elements of $\mathbb{Q}_p$.

There are two obvious generalizations we'd like to have: to other $K(1)$-local spectra, and to higher heights. The first is still unclear – there's not even a good construction for the Λ-module associated to a cofiber, since computing its homotopy groups requires solving an extension problem. For the second, Behrens has proved a complicated extension to height 2, and we'd hope to get a map $\mathrm{Pic}_2 \to V \to V/MF_*$, where V is the ring of p-adic modular forms.

3 Pic_1 for $p > 2$

The $K(1)$-local sphere at an odd prime p shows up as the fiber for a map $KU_p^\wedge \to KU_p^\wedge$ given by $\Psi^\gamma - 1$, where γ is a topological generator for $\mathbb{Z}_p^\times$. Note that $(E_1)_*L_{K(1)}S = (E_1)_*$. Now, ψ^λ acts on $KU(S^{2n})$ by λ^n, so if we instead take the fiber of $KU_p^\wedge \xrightarrow{\psi^\gamma-\gamma^n} KU_p^\wedge$, we get $L_{K(1)}S^{2n}$. This gives us half the spheres, and we can in fact extend it to get half of Pic_1. For $\lambda \in \mathbb{Z}_p^\times$, we define $S(\lambda)$ to be the fiber of $KU_p^\wedge \xrightarrow{\psi^\gamma-\lambda} KU_p^\wedge$. This is invertible, with $S(\lambda)\widehat{\wedge}S(\lambda^{-1}) \cong S$, as is shown by the following proposition.

Proposition 1 (Hopkins-Mahowald-Sadofsky)**.** *The following are equivalent for $X \in \mathcal{K}_n$.*

1. $X \in \mathrm{Pic}_n$*;*
2. $K(n)_*X$ *is free of rank* 1 *over* $K(n)_*$*;*

3. $(E_n)_*^\wedge X$ *is free of rank* 1 *over* $(E_n)_*$.

One can check that ψ^γ acts by λ on $(E_1)_*^\wedge S(\lambda)$, so that λ is uniquely determined by $S(\lambda)$ (once we've fixed γ).

Proposition 2 (Hopkins-Mahowald-Sadofsky)**.** *There's a non-split extension*

$$0 \to \mathrm{Pic}_1^0 \to \mathrm{Pic}_1 \to \mathbb{Z}/2 \to 0,$$

where $\mathrm{Pic}_1^0 \cong \mathbb{Z}_p^\times$ *is the group constructed above.*

Since $\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1) \oplus \mathbb{Z}_p$ and p is odd, the only way this extension can be non-split is if $\mathrm{Pic}_1 = \mathbb{Z}/(2(p-1)) \oplus \mathbb{Z}_p$. In particular, there are torsion elements in the Picard group! We can exhibit a $2(p-1)$-torsion element P as follows. If $P = \Sigma^{-1}S(\mu)$ is $2(p-1)$-torsion, then taking $2(p-1)$th smash powers gives $S^{2(p-1)} = \Sigma^{2(p-1)}P^{\wedge(2(p-1))} = S(\mu^{2(p-1)})$. By construction, $S^{2(p-1)} = S(\gamma^{p-1})$, so we must have $\mu = \sqrt{\gamma}$. This exists by Hensel's lemma. In fact, we can define $\gamma = \zeta(p+1)$ where ζ is a primitive $(p-1)$th root of unity. Thus $\gamma^{p-1} = (p-1)^{p+1}$, so we can take $\mu = \sqrt{p+1}$.

4 Large primes

Proposition 3 (Hopkins-Mahowald-Sadofsky)**.** *Suppose* $2p-2 > \max(n^2, 2n+2)$*. Then the functor* $(E_n^\wedge)_0$ *from* Pic_n^0 *to the category of* E_n*-modules with* $\mathbb{S}_n$*-action is injective.*

Sketch of proof. Let $X \in \ker((E_n^\wedge)_0)$. We have an ANSS $H^{*,*}(\mathbb{S}_n, (E_n)_*^\wedge(X)) \Rightarrow W(\mathbb{F}_{p^n}) \otimes \mathbb{Z}_p \pi_* X$, and by sparseness of the ANSS, this collapses. Its E_2 term looks just like the E_2 term for the sphere. There's a class in E_2 for the sphere corresponding to $S \to L_{K(n)}S$, which has to correspond in E_2 for X to a map $S \to X$. One checks that this gives $X = L_{K(n)}S$. $\square$

One important E_n-$\mathbb{S}_n$-module is det, the **unreduced determinant**. $\mathbb{S}_n = \mathcal{O}_D^\times$ acts by linear maps on D, which is a vector space of dimension n^2 over $\mathbb{Q}_p$, giving a determinant map $\mathbb{S}_n \to \mathbb{Z}_p^\times$. The **reduced determinant** is the composition of this with the nth root map.

5 Brown-Comenetz duality

The stable homotopy category has an obvious notion of duality given by Spanier-Whitehead duality. A second natural type of duality is given by Brown-Comenetz duality. For E and X spectra, the **Brown-Comenetz dual** of E is a spectrum IE such that $(IE)_n X \cong (E^n X)^\vee$, where for A a locally compact abelian group, $A^\vee$ is its **Pontryagin dual** $\mathrm{Hom}(A, S^1)$. In particular, $\pi_n IE \cong (\pi_{-n}E)^\vee$. Brown showed this exists, and it in fact suffices to define it for the sphere, since we have $IE \cong F(E, I)$, where $I = IS$.

Theorem 4 (Gross-Hopkins duality)**.** $L_{K(n)}I \in \mathrm{Pic}_n$*, and* $L_{K(n)}I \cong \Sigma^{n^2-n}S(\det)$ *for* $p >> n$*.*

We also have $L_{K(n)}I \cong F(M_nS, I)$, allowing us to show that.

$$\pi_t(M_nX)^\vee \cong [X, L_{K(n)}I]_{-t} \cong [S^{n-n^2-t}(-\det), LK(n)DX] = \pi_{n-n^2-\det -t}L_{K(n)}DX.$$

In particular, this is a formula for the homotopy groups of the $K(n)$-local Spanier-Whitehead dual of X in terms of X, though we have to introduce this strange Picard element $S(\det)$ in order to do so. If $pX = 0$ mod p, then $S(\det)$ behaves like a "p-adic" sphere, which can be accessed by v_n-periodicity. In the following case, we can replace this with an integral sphere.

Theorem 5 (Gross-Hopkins)**.** *If* F *is a finite type* n *spectrum with* $pF = 0$ *with a* v_n*-self map* v *of degree* p^m*, then* $\pi_t(F)^\vee \cong \pi_{\alpha-t}L_{K(n)}DF$*, where*

$$\alpha = \frac{2p^{nm}(p^n-1)}{p-1} + n^2 - n.$$

Gross-Hopkins duality can be thought of as a sort of Serre duality. If X si a smooth proper algebraic variety over a field k, F a coherent sheaf on X, and K is the canonical bundle on X, then Serre duality is an isomorphism

$$H^i(X;F)^\vee \cong H^{n-i}(X;K\otimes F^\vee).$$

Replacing K with I, X with a $K(n)$-local space, cohomology with homotopy groups, the vector space duality on the left by Pontryagin duality, and the sheaf duality on the right with Spanier-Whitehead duality, we recover Gross-Hopkins duality.

Here are a few of the facts that go into this proof. First, $L_{K(n)}DE_n \simeq \Sigma^{-n^2}E_n$, which one can prove via an ANSS. Second, $M_nE_n \simeq \Sigma^{-n}E_n/I_n^\infty$. Third, $\pi_t(E_n/I_n^\infty) \cong \Omega^{n-1}_{(E_n)_0/\mathbb{Z}_{p^n}} \otimes_{(E_n)_0} (E_n)_{-t}$, where Ω means the module of Kähler differentials. Combining these various isomorphisms gives us

$$\begin{aligned}
(E_n)_t^\wedge I &\cong \pi_t(E_n \wedge L_{K(n)}I)\\
&\cong \pi_t(\Sigma^{n^2}L_{K(n)}DE_n \wedge L_{K(n)}I)\\
&\cong \pi_{t-n^2}F(E_n, L_{K(n)}I)\\
&\cong \pi_{n^2-t}(M_nE_n)^\vee\\
&\cong \pi_{n^2+n-t}(E_n/I_n^\infty))^\vee\\
&\cong \Omega^{n-1}_{(E_n)_0/\mathbb{Z}_{p^n}} \otimes_{(E_n)_0} E_{t-n^2-n}.
\end{aligned}$$

We need to prove that $\Omega^{n-1}_{(E_n)_0/\mathbb{Z}_{p^n}} \cong (E_n)_{2n}(\det) \cong \omega^{\otimes n}(\det)$, which Gross and Hopkins prove using a lot of rigid analytic geometry.

6 The Mazel-Gee resurrection

Aaron said some things about the Gross-Hopkins proof and rigid analytic geometry far too fast for me to copy.