The Morava K-theories of Eilenberg-Mac Lane spaces

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1 Motivation

This comes from a paper published in the mid-80's that's quite long, but hopefully we'll be able to reproduce most of the good parts of the computation in this talk. Why should you care? Well, for one, the answer is quite cool. Another reason is that many interesting spaces are given in terms of Postnikov towers. For example, the Postnikov tower of $BO \times \mathbb{Z}$ begins

$$\begin{array}{ccccccccc} BO \times \mathbb{Z} & \longleftarrow & BO & \longleftarrow & BSO & \longleftarrow & BSpin & \longleftarrow & BString \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z} & & B\mathbb{Z}/2 & & K(\mathbb{Z}/2, 2) & & K(\mathbb{Z}, 4). & & \end{array}$$

Lots of people care about the cohomology of $BString$, and we can use this tower to compute it, with, for example, the spectral sequences of Serre, Rothenberg-Steenrod, or (dually) Eilenberg-Moore. If our space is X and $X\langle q\rangle$ is its qth Postnikov section, these spectral sequences take as inputs things like $(H\mathbb{F}_p)_*^{\text{cell}} X\langle q\rangle \otimes K_*$, $K_* \underline{HG}_q$ (the qth space of HG, for G an abelian group), and $K_* X\langle q\rangle$, and output $K_* X\langle q+1\rangle$. (Here K is Morava K-theory.)

2 A simple example

Let G be a finite cyclic group of order p^k, and $q = 1$. The only real things we know about Morava K-theory are

- $K_* = \mathbb{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n - 1)$, and
- $[p](x) = x^{p^n}$.

There's an obvious cofiber sequence

$$\begin{array}{ccccccccccccccc} \underline{H\mathbb{Z}}_0 & \to & \underline{H\mathbb{Z}}_0 & \to & \underline{H\mathbb{Z}/p^k}_0 & \to & \underline{H\mathbb{Z}}_1 & \to & \underline{H\mathbb{Z}}_1 & \to & \underline{H\mathbb{Z}/p^k}_1 & \to & \underline{H\mathbb{Z}}_2 & \to & \underline{H\mathbb{Z}}_2 \\ \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\ \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}/p^k & \to & S^1 & \to & S^1 & \to & B\mathbb{Z}/p^k & \xrightarrow{i} & \mathbb{C}P^\infty & \to & \mathbb{C}P^\infty. \end{array}$$

The fiber sequence $S^1 \to B\mathbb{Z}/p^k \to \mathbb{C}P^\infty$ gives a Gysin sequence

$$\begin{array}{ccc} K_* B\mathbb{Z}/p^k & \longrightarrow & K_*\mathbb{C}P^\infty \\ {\scriptstyle \partial}\nwarrow & & \swarrow{\scriptstyle \Phi = \cdot \cap [p^k](x)} \\ & K_*\mathbb{C}P^\infty & \end{array}$$

where x is the Euler class. (To see that the right-hand map is as described, one notes that $B\mathbb{Z}/p^k$ is the cofiber of $[p^k] : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, which follows from the above fiber sequence.) Of course, $[p^k]x = x^{p^{kn}}$ in Morava K-theory.

Recall that the cap product is defined to be dual to the cup product in cohomology: if σ is a homology class and ω and ω' cohomology classes, then $\langle \sigma \cap \omega, \omega' \rangle = \langle \sigma, \omega \cup \omega' \rangle$. We have $K^*\mathbb{C}P^\infty = K^*[[x]]$ and $K_*\mathbb{C}P^\infty = K_*\{\beta_0, \beta_1, \dots\}$ with β_i dual to x^i. Thus, $\langle \beta_i \cap x^{p^{nk}}, x^j = \langle \beta_i, x^{p^{nk}+j}$, so that $\beta_i \cap x^{p^{kn}} = \beta_{i-p^{nk}}$. Clearly, this implies that the right-hand map in the Gysin sequence is onto. Thus the left-hand map is zero, and the top map is injective, proving that $K_*B\mathbb{Z}/p^k = \ker \Phi$. Dually,

$$K^*B\mathbb{Z}/p^k = K^*\mathbb{C}P^\infty/[p^k](x) = K^*[[x]]/(x^{p^{nk}}).$$

Algebro-geometrically, this is like taking the formal completion of the affine line at a point and enforcing the condition that p^k times everything is zero. That is, this is the p^k torsion in the formal group. We write

$$(B\mathbb{Z}/p^k)_K = \operatorname{Spf} K^*B\mathbb{Z}/p^k \cong \mathbb{C}P^\infty_K[p^k].$$

3 The computation for $q > 1$

The Eilenberg-Mac Lane spaces have the structure of a **Hopf ring**.

- Since they're spaces, they have a comultiplication $\Delta : \underline{H\mathbb{Z}/p^k}_q \to \underline{H\mathbb{Z}/p^k}_q \times \underline{H\mathbb{Z}/p^k}_q$ given by the diagonal map.
- Since they're loop spaces, they have an addition $* : \underline{H\mathbb{Z}/p^k}_q \times \underline{H\mathbb{Z}/p^k}_q \to \underline{H\mathbb{Z}/p^k}_q$, classifying addition of cohomology classes.
- Since they're infinite loop spaces of a spectrum, there's a product $\circ : \underline{H\mathbb{Z}/p^k}_q \times \underline{H\mathbb{Z}/p^k}_{q'} \to \underline{H\mathbb{Z}/p^k}_{q+q'}$.

These maps fit together of various diagrams, the most complicated of which says that if $\Delta(x) = \sum x' \otimes x''$, then

$$x \circ (y * z) = \Sigma(x' \circ y) * (x'' \circ z),$$

which one should think of as a form of distributivity. The quickest way to describe these sorts of objects is that they're ring objects in the category of K_*-coalgebras.

In addition, $H\mathbb{Z}/p^k$ is a connective spectrum, and so

$$\underline{H\mathbb{Z}/p^k}_{q+1} = |B(\underline{H\mathbb{Z}/p^k}_q)|.$$

(We're thinking of this classifying space as a simplicial space coming from the 2-sided bar construction.) The skeletal filtration on the bar construction induces the Rothenberg-Steenrod spectral sequence

$$E^2_{*,*} = \operatorname{Tor}_{*,*}^{K_*\underline{H\mathbb{Z}/p^k}_q}(K_*, K_*) \Rightarrow K_*\underline{H\mathbb{Z}/p^k}_{q+1}.$$

These two bits of structure fit together quite nicely, since the Hopf ring structure maps respect the skeletal filtrations. This implies that the above RSSS is a spectral sequence of Hopf algebras. Moreover, the map $\circ$ induces a pairing on the spectral sequences for various q, which respects the differentials in the sense that $d(x \circ y) = x \circ d(y)$ where $x \in K_*\underline{H\mathbb{Z}/p^k}_q$ and y is an element in one of these spectral sequences. The upshot of this is that we can use differentials in earlier spectral sequences to compute differentials in later ones.

We'd like to do this inductively. In order to do this, we need to do the base case $\operatorname{Tor}_{*,*}^{K_*\underline{H\mathbb{Z}/p^k}_0} \Rightarrow K_*\underline{H\mathbb{Z}/p^k}_1$, and then produce a $\circ$-decomposition formula for lots of classes in the higher spectral sequences.

4 The computation

We have $K_*\underline{H\mathbb{Z}/p^k}_0 = K_*\mathbb{Z}/p^k = K_*\{[0], \ldots, [p^k-1]\}$, where the $*$-product is defined by $[i]*[j] = [i+j]$. We can write this as

$$K_*[\,[1]\,]/([1]^{p^k} - [0]) = K_*[\,[1]\,]/([1]^{p^k} - [0]^{p^k}) = K_*[[1]-[0]]/([1]-[0])^{p^k}.$$

If we write $a_\emptyset = [1]-[0]$, we finally have

$$\mathrm{Tor}_{*,*}^{K_*[a_\emptyset]/a_\emptyset^{p^k}} = \Lambda[\sigma(a_\emptyset)] \otimes \Gamma[\phi(a_\emptyset)].$$

The notation is from the paper, Λ is an exterior algebra, Γ is a divided power algebra, σ is suspension in the bar complex, and ϕ is a higher sort of homological suspension called the 'transpotence.' The differentials are forced to be $d(\phi(a_\emptyset)^{[p^{nk}]}) = \sigma(a_\emptyset)$ and $d(\phi(a_\emptyset)^{[p^{nk}+i]} = \sigma(a_\emptyset)\cdot\phi(a_\emptyset)^{[i]}$. (Here brackets denote divided powers.)

Theorem 1 (Ravenel-Wilson)**.**

(i) $K_*\underline{H\mathbb{Z}/p^k}_q = \Lambda^q K_*\underline{H\mathbb{Z}/p^k}_1$ *under the* $\circ$*-product.*

(ii) The E_2*-pages of the spectral sequences are* $\mathrm{Tor}_{*,*}^{K_*\underline{H\mathbb{Z}/p^k}_q} = \bigotimes(\Lambda[\sigma(a_I)] \otimes \Gamma[\phi(a_I)]$*, where the tensor product ranges over length* q *multi-indices* I*.*

(iii) There's a rewriting formula $\phi(a_I)^{[p^j]} = \phi(a_{(I_1,\ldots,I_{q-1})})^{[p^j]} \circ a_{(I_q+j)}$ *mod* $*$*-decomposables.*

(iv) This fully determines the differentials and the multiplicative extension problems in each spectral sequence using the previous one.

(v) Just enough classes are killed to prove (i) for $q+1$*.*

5 Conclusions

First, for $q > n$, the exterior product vanishes and so the Hopf algebra is just K_*. Thus, for example, at low chromatic levels, $BSpin$ and $BString$ appear the same.

Second, Spf $K^*\underline{H\mathbb{Z}/p^\infty}_q \cong \widehat{\mathbb{A}}^{\binom{n-1}{q-1}}$ is a p-divisible group, and $K^*\underline{H\mathbb{Z}/p^\infty}_q \cong K^*[[x_1, \ldots, x_{\binom{n-1}{q-1}}]]$. When $q = n$, this p-divisible group is height 1 – in fact, when n is even, it's just $\widehat{G_m}$, and when n is odd, it becomes $\widehat{G_m}$ after adjoining ζ_{2p-2}.

Third, this all works if we replace K with E. The only necessary facts are that $B\mathbb{Z}/p^k_E = \mathbb{C}P^\infty_E[p^k]$ (Hopkins-Kuhn-Ravenel) and that $E_*^\vee\underline{H\mathbb{Z}/p^k}_q = \Lambda^q E_*^\vee\underline{H\mathbb{Z}/p^k}_1$ (Peterson).

If you're curious about this, you can use this method to understand $(H\mathbb{F}_p)_*(\underline{H\mathbb{F}_p}_*)$ and thus get a nice description of the unstable cooperations on mod p cohomology.