Tate spectra

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1 "Motivation"

Let G be a finite group. A couple weird things happen at the interaction of equivariant and chromatic homotopy theory. One is that $L_{K(n)}S^{\mathrm{h}G} \simeq L_{K(n)}S_{\mathrm{h}G}$ – $K(n)$-locally, homotopy fixed points and homotopy orbits are the same.

Another goes by the name of the **redshift conjecture**. This says that algebraic K-theory shifts chromatic levels. For example:

Example 1.
$$K(H\mathbb{F}_p)^\wedge_p \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p.$$

Example 2.
$$K(H\mathbb{Z}_p)^\wedge_p \simeq \Sigma ku^\wedge_p \times (\mathrm{Im}\, J \times \mathbb{Z}_p) \times B(\mathrm{Im}\, J \times \mathbb{Z}_p).$$

Example 3. $K(ku)$ is still unknown, but it is known to have a v_2-self map.

...and that's basically all the evidence we have for the following conjecture:

Conjecture 4 (Redshift conjecture)**.**
$$V(n) \wedge K(L_{K(n)}A) \xrightarrow{\sim} v_{n+1}^{-1}V(n) \wedge K(L_{K(n)}A)$$
where A is a 'suitably finite' commutative ring spectrum.

Part of the reason this is such a difficult problem is that algebraic K-theory is generally very hard to compute. How do we compute it? Well, there's a map from algebraic K-theory to topological cyclic homology TC, and given a map of ring spectra $A \to B$, these fit into a pullback square

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TC(A) \arrow[d] \\ K(B) \arrow[r] & TC(B). \end{tikzcd}$$

Topological cyclic homology, in turn, comes from a tower of topological Hochschild homology spectra THH. Morally, we say "$TC(A)^\wedge_p = THH(A)^{S^1}$," and so we reduce to computing these S^1-fixed points on THH. In order to do *this*, you have to fit together data about the p-subgroups C_{p^n} of S^1 in a clever way.

Frequently, we get lucky and have
$$THH(A)^{C_{p^{n-1}}} \simeq THH(A)^{\mathrm{t}C_{p^n}},$$
where the t denotes the **Tate construction**, to be discussed below. There's a spectral sequence
$$\widehat{H}(C_{p^n}; THH(A)) \Rightarrow \pi_* THH(A)^{\mathrm{t}C_{p^n}},$$
where $\widehat{H}$ is Tate cohomology.

As an aside, here's the definition of Tate cohomology. We recall that for M a G-module, $H_0(G;M) = M_G$ and $H^0(G;M) = M^G$. There's a norm map $N : M_G \to M^G$ given by $\overline{x} \mapsto \sum_{g \in G} gx$, which one observes is invariant and independent of the class of $x \in M$ mod the action of G. If we take a projective resolution P_* and an injective resolution I_* for M over $\mathbb{Z}[G]$, and truncate them by replacing $P_0 \to M \to 0$ with $H_0(G;M) \to 0$ and $0 \to M \to I_0$ with $0 \to H^0(G;M)$, then the norm map fits them together into a $\mathbb{Z}$-graded complex $P_* \to I_*$. The homology of this is the **Tate cohomology** $\widehat{H}^*(G;M)$. This is a fancy way of saying that

$$\widehat{H}^*(G;M) = \begin{cases} H^*(G;M) & * \geq 1 \\ H_{-*+1}(G;M) & * < -1 \\ \operatorname{coker}(N) & * = 0 \\ \ker(N) & * = 1. \end{cases}$$

2 The Tate construction

In the topological case, if M is a G-spectrum, there are spectral sequences

$$H^*(G;\pi_* M) \Rightarrow \pi_* M^{\mathrm{h}G} \quad \text{and} \quad H_*(G;\pi_* M) \Rightarrow \pi_* M_{\mathrm{h}G},$$

and the Tate construction on M is a spectrum that fits in between the homotopy orbits and the homotopy fixed points, so as to give you the above spectral sequence.

More precisely, consider the cofiber sequence

$$EG_+ \to S^0 \to \widetilde{EG}.$$

Nonequivariantly, $EG_+ \simeq S^0$, so $\widetilde{EG} \simeq *$. On the other hand, $(EG_+)^G \simeq *$, so $\widetilde{EG}^G \simeq S^0$.

For X a G-spectrum, there's a map $X \to F(EG_+, X)$, inducing a diagram

$$\begin{tikzcd} EG_+ \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \widetilde{EG} \wedge X \arrow[d] \\ EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X) =: t_G X. \end{tikzcd}$$

Taking G-fixed points gives us a diagram

$$\begin{tikzcd} X_{\mathrm{h}G} \arrow[r] \arrow[d, "\sim"'] & X^G \arrow[r] \arrow[d] & X^{\Phi G} \arrow[d] \\ X_{\mathrm{h}G} \arrow[r, "N"'] & X^{\mathrm{h}G} \arrow[r] & X^{\mathrm{t}G}, \end{tikzcd}$$

where $X^{\mathrm{t}G}$ is, by definition, the **Tate construction** on X. Often, the norm map is nullhomotopic, and we get an extension $X^{\mathrm{h}G} \to X^{\mathrm{t}G} \to \Sigma X_{\mathrm{h}G}$, as we do algebraically with Tate cohomology.

Proposition 5 (May)**.**

$$X^{\mathrm{t}G} \simeq F(\widetilde{EG}, \Sigma EG_+ \wedge X)^G.$$

If X is finite and has a trivial C_p-action, then

$$X^{\mathrm{t}C_p} \simeq \operatorname{holim}((BC_p)_{-n} \wedge \Sigma X).$$

Here $(BC_p)_{-n}$ is defined via **James periodicity**, which says that for $p^{n-k} | r$,

$$(BC_p)^{2n+r}_{2k+1+r} \simeq \Sigma^r (BC_p)^{2n}_{2k+1},$$

where X^n_m is defined for a CW-spectrum X by crushing out its $(m-1)$-skeleton and restricting to the n-skeleton of the result. Thus, we can crush 'negative-dimensional skeleta' of BC_p by crushing actual skeleta and desuspending. Using this, May's proposition, and the fact that $F(X, Y \wedge Z) \simeq F(X,Y) \wedge Z$ for Z finite, we get the desired statement about $X^{\mathrm{t}C_p}$.

3 $K(n)^{\mathrm{t}G}$ and consequences

Proposition 6. *If G is a finite group acting trivially on $K(n)$, then*

$$K(n)^{\mathrm{t}G} \simeq *.$$

Corollary 7.

$$K(n)_* BG \cong K(n)^* BG.$$

Proof. $K(n)_* BG = \pi_* K(n) \wedge BG = \pi_* K(n)_{\mathrm{h}G}$, and likewise $K(n)^* BG = \pi_* K(n)^{\mathrm{h}G}$; since the Tate construction is trivial, these two spectra are equivalent. $\square$

Lemma 8. *If K is complex-oriented and $K_* BG$ is finitely generated over K_*, then*

$$\operatorname{holim}_{-s} K \wedge BG^{(-s\xi)} \simeq *,$$

where ξ is some complex bundle on BG and $BG^{(-s\xi)}$ the Thom construction on $s\xi$.

Proof. $K_* BG^{(r)} \to K_* BG$ is surjective for sufficiently large r, by the finite generation hypothesis. So there's a diagram

$$\begin{tikzcd} K_*(BG^{(r)})^{-(s+j)\xi} \arrow[rr] \arrow[dr] & & K_* BG^{-s\xi} \\ & K_* BG^{-(s+j)\xi} \arrow[ur] & \end{tikzcd}$$

with the left map surjective. But $(BG^{(r)})^{-(s+j)\xi}$ has a top cell in some finite dimension, and $BG^{-s\xi}$ has a top cell in some finite dimension. For sufficiently large j, we can thus make the top map zero, so that $K_* BG^{-(s+j)\xi} \to K_* BG^{-s\xi}$ is zero as well. $\square$

Lemma 9. *Let V be a finite dimensional G-representation and K a complex-oriented spectrum with $K_* BH$ finitely generated over K for all $H \leq G$. Then $F(S^{\infty V}, K \wedge EG_+)$ is equivariantly contractible.*

Proof.

$$F(S^{\infty V}, K \wedge EG_+) \simeq F(\operatorname{hocolim} S^{nV}, K \wedge EG_+) \simeq \operatorname{holim} F(S^{nV}, K \wedge EG_+) \simeq \operatorname{holim} S^{-nV} \wedge K \wedge EG_+.$$

Now, for any $H \leq G$, V is an H-representation by restriction, and likewise EG_+ is a model for EH_+. Therefore, $(K \wedge EG_+ \wedge S^{-nV})^H \simeq K \wedge BH^{-n\xi}$¡ and for $n \gg 0$, this is contractible since $K_* BH$ is finitely generated, using the previous lemma. $\square$

In particular, $K(n)$ satisfies the above conditions, so $K(n)^{\mathrm{t}G} \simeq *$. As a corollary, if X is type n, then $(L_{K(n)} X)^{\mathrm{t}G} \simeq *$, so $(L_{K(n)} X)^{\mathrm{h}G} \simeq (L_{K(n)} X)_{\mathrm{h}G}$.

Proposition 10. *If K is a p-local v_n-periodic spectrum, with v_n a unit and v_i acting nilpotently for all $0 \leq i \leq n-1$, then $K^{\mathrm{t}G} \simeq *$.*

This allows us to access a phenomenon known as **blueshift**, which says that the Tate construction with respect to trivial finite group actions tends to decrease chromatic levels.

Theorem 11 (Greenlees-Sadofsky)**.** *If K is as above without the nilpotence assumption, then $t_G(K)$ is v_{n-1}-periodic.*

Theorem 12 (Hovey-Sadofsky)**.** *If X is $E(n)$-local and G acts trivially, then $X^{\mathrm{t}G}$ is $E(n-1)$-local, and in fact,*

$$\langle (L_n X)^{\mathrm{t}G} \rangle = \langle L_{n-1} X \rangle$$

where angle brackets denote Bousfield classes.

Work of Ando, Morava, and Sadofsky gives us a weak equivalence

$$(E(n)[w]^{tC_p})^{\wedge}_{I_{n-1}} =: TE \xrightarrow{\sim} HW(\mathbb{F}_p((y))^{\mathrm{sep}})\widehat{\otimes}_{\mathbb{Z}_p} E_{n-1}.$$

Here $E(n)[w]$ is basically $E(n)$ with a $p^{n-1}-1$th root of v_{n-1}, called w, added to its homotopy. We then have $\pi_0 TE = W(\mathbb{F}_p((y))^{\mathrm{sep}})[[w_1, \ldots, w_{n-2}]]$, where w_i is the image of $v_i x^{p^i-1}$.

Proposition 13. *For E complex-oriented, $\pi_*(E^{tC_p}) = E_*((x))/[p](x)$.*