TAF

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1 Lurie's generalization of the Goerss-Hopkins-Miller theorem

We're looking for E_∞-ring spectra E associated to a 1-dimensional formal group of height n in the formal completion of a commutative group scheme such that $E_{K(n)}$ is a product of copies of $E_n^{\mathrm{h}G}$, where $G \subseteq \mathbb{S}_n$ is a finite subgroup of the Morava stabilizer group. For $n = 1$, this is given by KO; for $n = 2$, it's TMF; TAF will be the analogous thing for $n \geq 3$.

Theorem 1 (Goerss-Hopkins-Miller)**.** *Let $\overline{\mathcal{M}_{\mathrm{ell}}}$ be the moduli stack of generalized elliptic curves (that is, we allow nodal singularities). Given an affine étale open $C : \operatorname{Spec} R \to \overline{\mathcal{M}_{\mathrm{ell}}}$, the problem of realizing an E_∞-ring spectrum E such that*

1. *$\pi_0 E = R$, and*
2. *$\Gamma_E = \operatorname{Spf}(E^0 \mathbb{C}P^\infty) = \widehat{C}$ (the formal completion at the zero section)*

has a solution in the étale topology. That is, there is a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-ring spectra such that $E = \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ is a weakly even periodic spectrum satisfying the two conditions.

Here, a ring spectrum E is **weakly even periodic** if $\pi_* E = 0$ in odd degrees, $\pi_2 E$ is an invertible (rank 1 projective) $\pi_0 E$-module (as opposed to a rank 1 free $\pi_0 E$-module), and the multiplication map $\pi_2 E \otimes_{\pi_0 E} \pi_{2t} E \to \pi_{2+2t} E$ is an isomorphism for each t. In the local case, weakly even periodic is the same as strongly even periodic.

Theorem 2 (Local GHM)**.** *Let k be a perfect field of characteristic $p > 0$, Γ a formal group over k of finite height n. Then there exists a weakly even periodic E_∞-ring spectrum E_Γ such that*

1. *$\pi_0 E_\Gamma = \mathcal{O}_{\mathrm{Def}_\Gamma}$ where Def_Γ is the formal scheme representing the functor of deformations; that is, $\pi_0 E_\Gamma = W(k)[[u_1, \ldots, u_{n-1}]]$, and*
2. *Γ_{E_Γ} is the universal deformation of Γ.*

These allow us to solve our problem at height 2. For higher heights, we need to generalize the GHM theorem, as follows.

Definition 3. A ***p*-divisible group** $\mathbb{G}$ of height n over a scheme X is a sequence of commutative group schemes

$$\{1\} = \mathbb{G}_0 \hookrightarrow \mathbb{G}_1 \hookrightarrow \mathbb{G}_2 \hookrightarrow \cdots$$

over X such that $\mathbb{G}_i$ has rank p^{in} over X, and

$$0 \to \mathbb{G}_i \to \mathbb{G}_{i+1} \xrightarrow{p^i} \mathbb{G}_{i+1}$$

is exact.

The connected component of the identity (that is, the colimit of the connected components of the identity of the $\mathbb{G}_i$) is a formal group $\mathbb{G}_{\mathrm{for}}$ of height between 1 and n.

Theorem 4 (Lurie). *Let $(A, \mathfrak{m})$ be a local ring with perfect residue field k of characteristic p, $\mathfrak{X}$ a locally noetherian separated Deligne-Mumford stack over* $\operatorname{Spec} A$*, and $\mathbb{G}$ a p-divisible group of constant height n and dimension 1 over $\mathfrak{X}$. Then given a formal affine étale open $f : \operatorname{Spf} R \to \mathfrak{X}^{\wedge}_{\mathfrak{m}_A}$, the problem of realizing an E_∞-ring spectrum E such that*

1. *$\pi_0 E = R$, and*

2. *$\Gamma_E = \operatorname{Spf}(E^0 \mathbb{C}P^\infty) = f^* \mathbb{G}_{\text{for}}$ (the formal completion at the zero section)*

has a solution in the étale topology. That is, there is a sheaf $\mathcal{E}_{\mathbb{G}}$ of E_∞-ring spectra such that $E = \mathcal{E}_{\mathbb{G}}(\operatorname{Spf} R)$ is a weakly even periodic spectrum satisfying the two conditions, provided the following condition is satisfied:

- *for some étale cover $\pi : X \to \mathfrak{X}$ for X a scheme, and for every point $x \in X^{\wedge}_{\mathfrak{m}_A}$, the map $X^{\wedge}_x \to \operatorname{Def}_{\pi_* \mathbb{G}_x}$ classifying $(\pi_* \mathbb{G})|_{X^{\wedge}_x}$ of $\pi^* \mathbb{G}_x$ is an isomorphism.*

We'd like to use this theorem to produce TAF. We can achieve our goal if the following two conditions are satisfied:

1. the objects that $\mathfrak{X}$ parametrizes should have canonical p-divisible groups attached to them, and

2. the local geometry of the stack $\mathfrak{X}$ should correspond to the deformations of $\mathbb{G}$ (as expressed in the bulleted condition of Lurie's theorem).

An n-dimensional abelian scheme has a p-divisible group $A(p) = \{A[p^i]\}$ of height $2n$ and dimension n, where $A[p^i]$ is the p^i-torsion in A. To achieve condition 1, we need a way to cut down $A(p)$ to have dimension 1, since we only really understand 1-dimensional formal groups. The approach of Gorbunov-Mahowald-Ravenel is to assume that there are endomorphisms of A splitting off a 1-dimensional summand. For condition 2, given a deformation of a 1-dimensional summand, we can complete it uniquely to a deformation of A. Half of this is the Serre-Tate theorem.

2 PEL Shimura varieties

Shimura varieties are higher-dimensional analogues of modular curves, which are moduli spaces of elliptic curves. Thus, they are moduli spaces of other sorts of abelian varieties. To get a nice moduli stack, we need additional structure on the abelian varieties, which we abbreviate by P (for polarization), E (for endomorphism structure), and L (for level structure). It's best to remember these by what they do topologically, rather than by their algebro-geometric definitions. The endomorphism structure provides the desired 1-dimensional summand of $A(p)$; the polarization ensures that its height is n and ensures the summand controls the whole of $A(p)$; and the polarization and the level structure together are used to cut down the number of connected components of the moduli stack to a finite number.

Let F be an imaginary quadratic extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$ such that p splits in F, which means that $\mathcal{O}_F \otimes \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p$. Let e be an idempotent in $\mathcal{O}_F \otimes \mathbb{Z}_p$ not equal to 0 or 1. Complex conjugation takes e to $1 - e$. Consider the functor χ from the category of locally noetherian formal schemes over $\operatorname{Spf} \mathbb{Z}_p$ to the category of groupoids, with the objects of $\chi(X)$ given by the set of quadruples (A, λ, i, η) with

- A an abelian variety over X of dimension n.

- $\lambda : A \to A^\vee = \operatorname{Pic}^0(A)$ a prime-to-p **polarization**, i.e. symmetric isogeny arising from a very ample line bundle. 'Symmetric' means that $\lambda = \lambda^\vee$; 'isogeny' means that λ is a surjection with finite kernel; 'prime-to-p' means that the rank of this kernel is prime to p.

- $i : \mathcal{O}_F \to \operatorname{End}(A)$ is a map such that $\lambda i(\alpha) = i(\overline{\alpha})^\vee \lambda$ for all $\alpha \in \mathcal{O}_F$.

- η is a **level structure**, which we won't discuss in detail, but it's the orbit of something under some subgroup $K^p \leq \prod_{\ell \neq p} \operatorname{Aut}(\mathbb{Z}_\ell^{\oplus 2n})$.

The morphisms of $\chi(X)$ are the isomorphisms of abelian varieties preserving the level structure, commuting with i, and with $f^\vee \lambda_A f = m\lambda_A$ for some $m \in \mathbb{Z}$.

The endomorphism structure is defined as follows. First, $\operatorname{End}(A(p))$ is p-complete, so it's a $\mathbb{Z}_p$-algebra. We get maps

$$\begin{tikzcd} \mathcal{O}_F \arrow[r, "i"] \arrow[dr] & \operatorname{End} A \arrow[r] & \operatorname{End} A(p) \\ & \mathcal{O}_F \otimes \mathbb{Z}_p. \arrow[ur] & \end{tikzcd}$$

Recall that e is a nontrivial idempotent in $\mathcal{O}_F \otimes \mathbb{Z}_p$; this gives a summand $eA(p)$, which Gorbunov-Mahowald-Ravenel assume is 1-dimensional. (Note that we don't know its height yet.)

The polarization is a prime-to-p isogeny, so it induces an isomorphism $A(p) \xrightarrow{\cong} A^\vee(p)$. We have $\lambda e = (1 - e^\vee)\lambda$. Thus λ breaks up into isomorphisms $eA(p) = ((1-e)A(p))^\vee$ and $(1-e)A(p) = (eA(p))^\vee$. It follows that a deformation of $eA(p)$ gives a deformation of $(1-e)A(p)$, and these combine to a deformation of $A(p)$, which by the Serre-Tate theorem gives a deformation of A.

Finally, taking duals preserves height and height is additive, so since $A(p)$ has height $2n$, $eA(p)$ has height n, as desired. (We won't discuss the level structure here.)

Theorem 5. *The functor χ is representable by a stack $\operatorname{Sh}(K^p)$ of dimension $n-1$ over $\operatorname{Spf} \mathbb{Z}_p$. If K^p is sufficiently small, then this stack is actually a quasiprojective scheme.*

We can now apply Lurie's theorem, with $\operatorname{Sh}(K^p)$ as our stack. This gives us a sheaf $\mathcal{E}_{\mathbb{G}}$, and we can make the following definition.

Definition 6. The spectrum TAF is the global sections $\mathcal{E}_{\mathbb{G}}(\operatorname{Sh}(K^p)^\wedge_p)$. Evaluating at an affine open $U = \operatorname{Spf} R$, we have $\pi_k \mathcal{E}_{\mathbb{G}}(U) = 0$ for k odd and $H^0(U, \omega_{f^*\mathbb{G}_{\text{for}}}^{\otimes k/2})$ for k even. There is a similar descent spectral sequence as for TMF.